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Research Article

On the Symmetries of the q-Bernoulli Polynomials

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Kupershmidt and Tuenter have introduced reflection symmetries for the q-Bernoulli numbers and the Bernoulli polynomials in (2005), (2001), respectively. However, they have not dealt with congruence properties for these numbers entirely. Kupershmidt gave a quantization of the reflection symmetry for the classical Bernoulli polynomials. Tuenter derived a symmetry of power sum polynomials and the classical Bernoulli numbers. In this paper, we study the new symmetries of the q-Bernoulli numbers and polynomials, which are different from Kupershmidt's and Tuenter's results. By using our symmetries for the q-Bernoulli polynomials, we can obtain some interesting relationships between q-Bernoulli numbers and polynomials.

1. Introduction

Let p be a fixed prime. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integer, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p < p^{-1}$. Let q be variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, we assume that $|1-q|_p < 1$. We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and we denote this property by $f \in \mathrm{UD}(\mathbb{Z}_p)$ if the difference quotients,

$$F_f : \mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow \mathbb{Z}_p \qquad \text{by } F_f(x,y) = \frac{f(x)-f(y)}{x-y}, \tag{1.1}$$

have a limit $l = f'(a)$ as $(x,y) \to (a,a)$. The p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x)dx = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \tag{1.2}$$

[1–22]. From this integral, we derive several further interesting properties of symmetry for the q-Bernoulli numbers and polynomials in this paper. Kupershmidt [14] and Tuenter [20] have introduced reflection symmetries for the q-Bernoulli numbers and the Bernoulli polynomials. However, they have not dealt with congruence properties for these numbers entirely. Kupershmidt gave a quantization of the reflection symmetry for the classical Bernoulli polynomials. Tuenter derived a symmetry of power sum polynomials and the classical Bernoulli numbers. In this paper, we study the new symmetries of the q-Bernoulli numbers and polynomials, which are different from Kupershmidt's and Tuenter's results. By using our symmetries for the q-Bernoulli polynomials, we can obtain some interesting relationships between q-Bernoulli numbers and polynomials.

2. On the symmetries of the q-Bernoulli polynomials

For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined as

$$I(f) = \int_{\mathbb{Z}_p} f(x)dx = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \tag{2.1}$$

Let $f_1(x)$ be a translation with $f_1(x) = f(x+1)$. Then, we have

$$I(f_1) = I(f) + f'(0). \tag{2.2}$$

From (2.2), we can also derive

$$I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i), \quad f'(i) = \frac{df(i)}{dx}. \tag{2.3}$$

Let $f(x) = q^x e^{tx}$, then we have

$$\int_{\mathbb{Z}_p} q^x e^{tx}\, dx = \frac{t + \log q}{qe^t - 1}. \tag{2.4}$$

It is known that the q-Bernoulli polynomials are defined as

$$\frac{t + \log q}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!} \tag{2.5}$$

[17, 19]. Now we define an integral representation for the q-extension of Bernoulli numbers as follows:

$$\int_{\mathbb{Z}_p} q^x e^{tx}\, dx = \frac{\log q + t}{qe^t - 1} = \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{n!}. \tag{2.6}$$

From (2.3), (2.4), and (2.6), we can derive

$$\int_{\mathbb{Z}_p} q^y (x+y)^n dy = B_{n,q}(x), \qquad \int_{\mathbb{Z}_p} q^x x^n \, dx = B_{n,q}. \tag{2.7}$$

By (2.3), we easily see that

$$\begin{aligned}\frac{1}{\log q + t}\left(\int_{\mathbb{Z}_p} q^{n+x} e^{(n+x)t} \, dx - \int_{\mathbb{Z}_p} q^x e^{xt} \, dx\right) &= \frac{q^n e^{nt} - 1}{t + \log q} \int_{\mathbb{Z}_p} q^x e^{xt} \, dx = \frac{q^n e^{nt} - 1}{qe^t - 1} \\ &= \sum_{i=0}^{n-1} q^i e^{it} = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n-1} i^k q^i\right) \frac{t^k}{k!}.\end{aligned} \tag{2.8}$$

In (2.2), it is not difficult to show that

$$\frac{1}{\log q + t}\left(\int_{\mathbb{Z}_p} q^{n+x} e^{(n+x)t} \, dx - \int_{\mathbb{Z}_p} q^x e^{xt} \, dx\right) = \frac{n \int_{\mathbb{Z}_p} e^{xt} q^x \, dx}{\int_{\mathbb{Z}_p} e^{nxt} q^{nx} \, dx}. \tag{2.9}$$

For each integer $k \geq 0$, let

$$S_{k,q}(n) = 0^k + 1^k q + 2^k q^2 + \cdots + q^n n^k. \tag{2.10}$$

From (2.8) and (2.9), we derive

$$\frac{1}{\log q + t}\left(\int_{\mathbb{Z}_p} q^{n+x} e^{(n+x)t} \, dx - \int_{\mathbb{Z}_p} q^x e^{xt} \, dx\right) = \frac{n \int_{\mathbb{Z}_p} e^{xt} q^x \, dx}{\int_{\mathbb{Z}_p} e^{nxt} q^{nx} \, dx} = \sum_{k=0}^{\infty} S_{k,q}(n-1) \frac{t^k}{k!}. \tag{2.11}$$

From (2.11), we note that

$$B_{k,q}(n) - B_{k,q} = k S_{k-1,q}(n-1) + \log q S_{k,q}(n-1), \quad \text{where } k, n \in \mathbb{N}. \tag{2.12}$$

Let $w_1, w_2 \in \mathbb{N}$, then we have

$$\frac{\int\int_{\mathbb{Z}_p} e^{(w_1 x_1 + w_2 x_2)t} q^{w_1 x_1 + w_2 x_2} \, dx_1 \, dx_2}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} q^{w_1 w_2 x} \, dx} = (t + \log q) \frac{q^{w_1 w_2} e^{w_1 w_2 t} - 1}{(q^{w_1} e^{w_1 t} - 1)(q^{w_2} e^{w_2 t} - 1)}. \tag{2.13}$$

By (2.11), we see that

$$\frac{w_1 \int_{\mathbb{Z}_p} e^{xt} q^x \, dx}{\int_{\mathbb{Z}_p} q^{w_1 x} e^{w_1 x t} \, dx} = \sum_{l=0}^{\infty} \left(\sum_{k=0}^{w_1 - 1} k^l q^k\right) \frac{t^l}{l!} = \sum_{l=0}^{\infty} S_{l,q}(w_1 - 1) \frac{t^l}{l!}. \tag{2.14}$$

Let

$$T(w_1, w_2; x, t) = \frac{\int\int_{\mathbb{Z}_p} q^{w_1x_1+w_2x_2} e^{(w_1x_1+w_2x_2+w_1w_2x)t}\, dx_1\, dx_2}{\int_{\mathbb{Z}_p} e^{w_1w_2x_3t} q^{w_1w_2x_3}\, dx_3}, \tag{2.15}$$

then we have

$$T(w_1, w_2; x, t) = \frac{(t + \log q) e^{w_1w_2xt}\left(q^{w_1w_2} e^{w_1w_2t} - 1\right)}{\left(q^{w_1} e^{w_1t} - 1\right)\left(q^{w_2} e^{w_2t} - 1\right)}. \tag{2.16}$$

From (2.15) we derive

$$T(w_1, w_2; x, t) = \left(\frac{1}{w_1}\int_{\mathbb{Z}_p} e^{w_1(x_1+w_2x)t} q^{w_1x_1}\, dx_1\right)\left(\frac{w_1\int_{\mathbb{Z}_p} e^{w_2x_2t} q^{w_2x_2}\, dx_2}{\int_{\mathbb{Z}_p} e^{w_1w_2xt} q^{w_1w_2x}\, dx}\right). \tag{2.17}$$

By (2.5), (2.14), and (2.17), we see that

$$\begin{aligned} T(w_1, w_2; x, t) &= \frac{1}{w_1}\left(\sum_{i=0}^{\infty} B_{i,q^{w_1}}(w_2x)\frac{w_1^i t^i}{i!}\right)\left(\sum_{l=0}^{\infty} S_{l,q^{w_2}}(w_1-1)\frac{w_2^l t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i} B_{i,q^{w_1}}(w_2x) S_{n-i,q^{w_2}}(w_1-1) w_1^{i-1} w_2^{n-i}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.18}$$

By the symmetry of p-adic invariant integral on $\mathbb{Z}_p$, we also see that

$$\begin{aligned} T(w_1, w_2; x, t) &= \left(\frac{1}{w_2}\int_{\mathbb{Z}_p} e^{w_2(x_2+w_1x)t} q^{w_2x_2}\, dx_2\right)\left(\frac{w_2\int_{\mathbb{Z}_p} e^{w_1x_1t} q^{w_1x_1}\, dx_1}{\int_{\mathbb{Z}_p} e^{w_1w_2xt} q^{w_1w_2x}\, dx}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i} B_{i,q^{w_2}}(w_1x) S_{n-i,q^{w_1}}(w_2-1) w_2^{i-1} w_1^{n-i}\right)\frac{t^n}{n!}. \end{aligned} \tag{2.19}$$

By comparing the coefficients $t^n/n!$ on the both sides of (2.18) and (2.19), we obtain the following theorem.

Theorem 2.1. *For all $w_1, w_2 (\in \mathbb{N})$, we have*

$$\sum_{i=0}^{n}\binom{n}{i} B_{i,q^{w_1}}(w_2x) S_{n-i,q^{w_2}}(w_1-1) w_1^{i-1} w_2^{n-i} = \sum_{i=0}^{n}\binom{n}{i} B_{i,q^{w_2}}(w_1x) S_{n-i,q^{w_1}}(w_2-1) w_2^{i-1} w_1^{n-i}, \tag{2.20}$$

where $\binom{n}{i}$ is the binomial coefficient.

If we take $w_2 = 1$ in Theorem 2.1, then we have

$$B_{n,q}(w_1 x) = \sum_{i=0}^{n} \binom{n}{i} B_{i,q^{w_1}}(x) S_{n-i,q}(w_1 - 1) w_1^{i-1}. \tag{2.21}$$

Therefore, we obtain the following corollary.

Corollary 2.2. *For $n \geq 0$, we have*

$$B_{n,q}(w_1 x) = \sum_{i=0}^{n} \binom{n}{i} B_{i,q^{w_1}}(x) S_{n-i,q}(w_1 - 1) w_1^{i-1}. \tag{2.22}$$

By (2.17), (2.18), and (2.19), we also see that

$$\begin{aligned} T(w_1, w_2; x, t) &= \left(\frac{e^{w_1 w_2 x t}}{w_1} \int_{\mathbb{Z}_p} e^{w_1 x_1 t} q^{w_1 x_1} \, dx_1 \right) \left(\frac{w_1 \int_{\mathbb{Z}_p} e^{w_2 x_2 t} q^{w_2 x_2} \, dx_2}{\int_{\mathbb{Z}_p} e^{w_1 w_2 x t} q^{w_1 w_2 x} \, dx} \right) \\ &= \left(\frac{e^{w_1 w_2 x t}}{w_1} \int_{\mathbb{Z}_p} e^{w_1 x_1 t} q^{w_1 x_1} \, dx_1 \right) \left(\sum_{i=0}^{w_1 - 1} q^{w_2 i} e^{w_2 i t} \right) \\ &= \frac{1}{w_1} \sum_{i=0}^{w_1 - 1} q^{w_2 i} \int_{\mathbb{Z}_p} e^{(x_1 + w_2 x + (w_2/w_1) i) t w_1} q^{x w_1} \, dx_1 \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{w_1 - 1} B_{n,q^{w_1}} \left(w_2 x + \frac{w_2}{w_1} i \right) w_1^{n-1} q^{w_2 i} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.23}$$

From the symmetry of $T(w_1, w_2; x, t)$, we can also derive

$$T(w_1, w_2; x, t) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{w_2 - 1} B_{n,q^{w_2}} \left(w_1 x + \frac{w_1}{w_2} i \right) w_2^{n-1} q^{w_1 i} \right) \frac{t^n}{n!}. \tag{2.24}$$

By comparing the coefficients $t^n/n!$ on the both sides of (2.23) and (2.24), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_+$, $w_1, w_2 \in \mathbb{N}$, we have*

$$\sum_{i=0}^{w_1 - 1} B_{n,q^{w_1}} \left(w_2 x + \frac{w_2}{w_1} i \right) w_1^{n-1} q^{w_2 i} = \sum_{i=0}^{w_2 - 1} B_{n,q^{w_2}} \left(w_1 x + \frac{w_1}{w_2} i \right) w_2^{n-1} q^{w_1 i}. \tag{2.25}$$

Remark 2.4. Setting $w_2 = 1$ in Theorem 2.3, we get the multiplication theorem for the q-Bernoulli polynomials as follows:

$$B_{n,q}(w_1 x) = w_1^{n-1} \sum_{i=0}^{w_1 - 1} B_{n,q^{w_1}} \left(x + \frac{i}{w_1} \right) q^i. \tag{2.26}$$

I cannot obtain the extended formulae of Theorems 2.1 and 2.3 related to the Carlitz's q-Bernoulli numbers and polynomials. So, we suggest the following two questions.

Question 1. Find the extended formulae of Theorems 2.1 and 2.3, which are related to the Carlitz's q-Bernoulli numbers and polynomials.

Question 2. Find the twisted formulae of Theorems 2.1 and 2.3, which are related to the twisted Carlitz's q-Bernoulli polynomials.

Remark 2.5. In [12], q-Volkenborn integral is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x. \tag{2.27}$$

Thus, we note that Carlitz's q-Bernoulli numbers can be written by

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [x]_q^n \, d\mu_q(x), \quad \text{Witt's type formula.} \tag{2.28}$$

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