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Research Article

Differentiable Solutions of Equations Involving Iterated Functional Series

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The nonmonotonic differentiable solutions of equations involving iterated functional series are investigated. Conditions for the existence, uniqueness, and stability of such solutions are given. These extend earlier results due to Murugan and Subrahmanyam.

1. Introduction

Let f be a self-mapping on a topological space X and f^m denote the mth iterate of f, that is, $f^m = f \circ f^{m-1}$, $f^0 = id$, $m = 1, 2, \ldots$. Let $C(X, X)$ be the set of all continuous self-mappings on X. Equations having iteration as their main operation, that is, including iterates of the unknown mapping, are called iterative equations. It is one of the most interesting classes of functional equations [1–4], because it concludes the problem of iterative roots [1, 5, 6], that is, finding $f \in C(X, X)$ such that f^n is identical to a given $F \in C(X, X)$. As a natural generalization of the problem of iterative roots, a class of iterative equations named as polynomial-like iterative equation

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = F(x), \quad x \in I = [a, b], \tag{1.1}$$

had fascinated many scholars, such as Dhombres [7], Zhao [8], Mukherjea and Ratti [9]. Despite their nice constructive proofs, the classical methods prevented them from obtaining more fruitful results. In 1986, Zhang [10] constructed an interesting operator called "structural operator" for (1.1) and used the fixed point theory in Banach space to get the solutions of (1.1). Hence he overcame the difficulties encountered by the formers. By means of this method, Zhang and Si made a series of work concerning these qualitative problems, such as [11–15]. In 2002, Kulczycki and Tabor [16] improved Zhang's method and investigated the existence of Lipschitzian solutions of the iterative functional equation

$$\sum_{n=1}^{\infty}\lambda_n f^n(x) = F(x), \quad x \in B, \tag{1.2}$$

where B is a compact convex subset of $\mathbb{R}^n$ and $F : B \to B$ are a given Lipschitz function. It is easy to see that (1.1) is the special case of (1.2) with $\lambda_i = 0$, $i = n+1,\ldots$ and $B = [a,b]$.

Recently Zhang et al. [17] and Xu et al. [18] developed this method and they have got the nonmonotonic, convex, and decreasing continuous solutions of (1.1). In fact they have answered the open problem 2 which was proposed by J. Zhang et al. [19].

The problem of differentiable solutions of iterative equation had also fascinated many scholars' attentions. In Zhang [12] and Si [15], the C^1 and C^2 solutions of (1.1) are considered. In Wang and Si [20] the differentiable solutions of the below equation

$$H\left(x, \phi^{n_1}(x), \ldots, \phi^{n_i}(x)\right) = F(x), \quad x \in I = [a,b] \tag{1.3}$$

are considered. Murugan and Subrahmanyam [21, 22] discussed the existence and uniqueness of C^1 solutions of the more general equations

$$\sum_{i=1}^{\infty}\lambda_i H_i\left(f^i(x)\right) = F(x), \quad x \in I = [a,b], \tag{1.4}$$

$$\sum_{i=1}^{\infty}\lambda_i H_i\left(x, \phi^{a_{i1}}(x), \ldots, \phi^{a_{in_i}}(x)\right) = F(x), \quad x \in I = [a,b], \tag{1.5}$$

which involve iterated functional series. All the above references only got the increasing differentiable solutions for the above equations because they only considered the case that F is increasing. Li and Deng [23] considered the C^1 solutions of the (1.2). In [24] C^1 solutions of the equation

$$\sum_{n=1}^{\infty}\lambda_n(x) f^n(x) = F(x), \quad x \in B, \tag{1.6}$$

where B is a compact convex subset of $\mathbb{R}^n$ and $\lambda_n(x) : B \to R$ are discussed. Li and Deng [23] and Li [24] work in higher dimensional case, they do not require monotonicity. It should be pointed out that Mai and Liu [25] made an important contribution to C^m solutions of iterative equations. Mai and Liu proved the existence, uniqueness of C^m solutions of a relatively general kind of iterative equations

$$G\left(x, f(x), \ldots, f^n(x)\right) = 0, \quad x \in J, \tag{1.7}$$

where J is a connected closed subset of $\mathbb{R}$ and $G \in C^m(J^{n+1}, \mathbb{R})$, $n \geq 2$. Here $C^m(J^{n+1}, \mathbb{R})$ denotes the set of all C^m mappings from J^{n+1} to $\mathbb{R}$.

Inspired by the above work, we will investigate (1.4) and extend earlier results due to Murugan and Subrahmanyam in two directions. In [21] the authors only get increasing solutions of (1.4), so the present paper will investigate the nonmonotonic differentiable solutions of (1.4) and give conditions for the existence, uniqueness, and stability of such solutions. In [21] the authors require not only all coefficients are nonnegative but also $H_i,\ i = 2,3,\dots$ are all increasing, but we will find that those conditions are not necessary.

2. Preliminaries

Let $I = [a,b]$ and $J = [c,d]$ be two compact intervals. Let $C^1(I,J)$ be the set of all continuously differentiable functions from I to J. Then $C^1(I,J)$ is a closed subset of the Banach space $C^1(I,\mathbb{R})$ consisting of all continuously differentiable functions from I to $\mathbb{R}$ with norm $\|\cdot\|_{c^1}$. Here the norm $\|\cdot\|_{c^1}$ is defined by $\|\varphi\|_{c^1} = \|\varphi\|_{c^0} + \|\varphi'\|_{c^0}$, $\varphi \in C^1(I,\mathbb{R})$, where $\|\varphi\|_{c^0} = \max_{x\in I}|\varphi(x)|$ and φ' is the derivative of φ. Following Zhang [12], we define the families of functions

$$\begin{aligned}
\mathfrak{A}(I,J,m,M,N) = \{&\varphi \in C^1(I,J) : \varphi(a) = c,\ \varphi(b) = d, m \le |\varphi'(x)| \le M,\\
&|\varphi'(x_1) - \varphi'(x_2)| \le N|x_1 - x_2|,\ \forall x, x_1, x_2 \in I\},\\
\mathfrak{A}'(I,J,m,M,N) = \{&\varphi \in C^1(I,J) : \varphi(a) = d,\ \varphi(b) = c, m \le |\varphi'(x)| \le M,\\
&|\varphi'(x_1) - \varphi'(x_2)| \le N|x_1 - x_2|,\ \forall x, x_1, x_2 \in I\},
\end{aligned} \tag{2.1}$$

where $0 \le m < M$, $N > 0$ are all constants.

Lemma 2.1. *Both $\mathfrak{A}(I,J,m,M,N)$ and $\mathfrak{A}'(I,J,m,M,N)$ are compact convex subsets of $C^1(I,J)$.*

The Lemma above can be proved by a method which is contained in the proof of Theorem 3.1 in [12].

Lemma 2.2 (see [12]). *Suppose that $\varphi,\phi \in \mathfrak{A}(I,J,m,M,N)$ (or $\varphi,\phi \in \mathfrak{A}'(I,J,m,M,N)$). Then for $n = 1,2,\dots,$*

$$\begin{aligned}
&|(\varphi^n)'(x)| \le M^n, \quad \forall x \in I,\\
&|(\varphi^n)'(x_1) - (\varphi^n)'(x_2)| \le N\left(\sum_{i=n-1}^{2n-2} M^i\right)|x_1 - x_2|, \quad \forall x_1, x_2 \in I,\\
&\|\varphi^n - \phi^n\|_{c^0} \le \left(\sum_{i=1}^{n} M^{i-1}\right)\|\varphi - \phi\|_{c^0},\\
&\|(\varphi^n)' - (\phi^n)'\|_{c^0} \le nM^{n-1}\|\varphi' - \phi'\|_{c^0} + Q(n)N\left(\sum_{i=1}^{n-1}(n-i)M^{n+i-2}\right)\|\varphi - \phi\|_{c^0},
\end{aligned} \tag{2.2}$$

where $Q(1) = 0$, $Q(m) = 1$ as $m = 2,3,\dots$ and $(\varphi^n)'$ denotes $d\varphi^n/dx$.

We can get the following Lemma from [12]. In [12] the author proved that the lemma is valid for $C^1(I,I)$, but we find it is also valid for $C^1(I,J)$.

Lemma 2.3 (see [12]). *Suppose that $f \in C^1(I,J)$ satisfies that*

$$\begin{gathered} 0 < \delta \leq f'(x), \quad \forall x \in I, \\ \left|f'(x_1) - f'(x_2)\right| \leq M^* \left|x_1 - x_2\right|, \quad \forall x_1, x_2 \in I, \end{gathered} \tag{2.3}$$

where δ, M^* are positive constants. Then

$$\left|(f^{-1})'(y_1) - (f^{-1})'(y_2)\right| \leq \frac{M^*}{\delta^3}\left|y_1 - y_2\right|, \quad \forall y_1, y_2 \in J. \tag{2.4}$$

Lemma 2.4 (see [18]). *If both $f_i : I \to J$, $i = 1,2$ are homeomorphisms from I to J such that*

$$\left|f_i(x_1) - f_i(x_2)\right| \leq K\left|x_1 - x_2\right|, \quad \forall x_1, x_2 \in I, \tag{2.5}$$

where K is a positive constant. Then

$$\left\|f_1 - f_2\right\|_{c^0} \leq K\left\|f_1^{-1} - f_2^{-1}\right\|_{c^0}. \tag{2.6}$$

3. Differentiable solutions of (1.4)

3.1. Existence of solutions

Let $\{\lambda_i\}_{i=1}^{\infty}$ be coefficients of (1.4) and $I = [a,b]$. For any $f \in \mathfrak{A}(I,I,0,M,N)$ (or $\mathfrak{A}'(I,I,0,M,N)$) define $A = \sum_{i=1}^{\infty}\lambda_i H_i(f^{i-1}(a))$ and $B = \sum_{i=1}^{\infty}\lambda_i H_i(f^{i-1}(b))$. It is easy to see that both the convergence and the value of A, B have nothing to do with the choice of f.

Theorem 3.1. *Suppose $M > 1$, L are positive constants and $H_1 \in \mathfrak{A}(I,I,m_1,M_1,N_1)$, $H_i \in \mathfrak{A}(I,I,0,M_i,N_i)$ or $H_i \in \mathfrak{A}'(I,I,0,M_i,N_i)$ for $i = 2,3,\ldots$, where $m_1 > 0$ and $M_i \geq 1, N_i$ are positive constants for $i = 1,2,\ldots$. Assume further the following conditions:*

$$\begin{gathered} K_0 = \lambda_1 m_1 - \sum_{i=2}^{\infty}\left|\lambda_i\right| M_i M^{i-1} > 0, \\ K_1 = \frac{1}{(M-1)}\sum_{i=1}^{\infty}\left|\lambda_{i+1}\right| M_{i+1} M^{i-1}(M^i - 1) < \infty, \\ K_0 - K_1 M^2 > 0, \\ K_2 = \sum_{i=1}^{\infty}\left|\lambda_i\right| N_i M^{i-1}(M^i - 1) < \infty, \\ -\infty < A < B < \infty, \end{gathered} \tag{3.1}$$

hold. Then for any given $F \in \mathfrak{A}(I,J,0,K_0 M,L)$ (or $\mathfrak{A}'(I,J,0,K_0 M,L)$), (1.4) has a solution $f \in \mathfrak{A}(I,I,0,M,M^*)$ (or $\mathfrak{A}'(I,I,0,M,M^*)$), where $M^* \geq (L + K_4 M^2)/(K_0 - K_1 M^2)$, $K_4 = \sum_{i=1}^{\infty}|\lambda_i| N_i M^{2(i-1)}$ and $I = [a,b]$, $J = [A,B]$.

As in [22], firstly we give the following three Lemmas which lead directly to the proof of Theorem 3.1. In the sequel we denote

$$N = \frac{\left(L + K_4 M^2\right)}{\left(K_0 - K_1 M^2\right)}. \tag{3.2}$$

Lemma 3.2. *Under the assumptions of Theorem 3.1 the following series:*

$$\begin{aligned}
K_3 &= \lambda_1 M_1 + \sum_{i=2}^{\infty} \left|\lambda_i\right| M_i M^{i-1}, \\
K_4 &= \sum_{i=1}^{\infty} \left|\lambda_i\right| N_i M^{2(i-1)}, \\
K_5 &= \sum_{k=1}^{\infty} \left|\lambda_{k+1}\right| M_{k+1} \left(\sum_{i=1}^{k} \left(M^{i-1}\right) \right), \\
K_6 &= \sum_{k=1}^{\infty} \left|\lambda_{k+1}\right| M_{k+1} k M^{k-1}, \\
K_7 &= \sum_{k=1}^{\infty} \left|\lambda_{k+1}\right| N_{k+1} M^{k} \left(\sum_{i=1}^{k} M^{i-1} \right), \\
K_8 &= \sum_{K=2}^{\infty} \left|\lambda_{k+1}\right| M_{k+1} \sum_{j=1}^{k-1} (k-j) M^{k+j-2},
\end{aligned} \tag{3.3}$$

are all convergent.

Proof. The convergence of K_3, K_4, K_5, K_6, and K_7 is easy to be verified. As mentioned in [21], the equality

$$\sum_{i=1}^{n-1} (n-i) x^{n+i-2} = x^{n-1} \left(\frac{x^n - 1}{(x-1)^2} - \frac{n}{x-1} \right), \quad x \neq 1 \tag{3.4}$$

holds. We get that

$$K_8 = \sum_{K=2}^{\infty} |\lambda_{k+1}| M_{k+1} \left(M^{k-1} \left(\frac{M^k - 1}{(M-1)^2} - \frac{k}{M-1} \right) \right). \tag{3.5}$$

By the convergence of K_1 and K_6, K_8 is also convergent. □

Lemma 3.3. *Under the assumptions of Theorem 3.1, for each $f \in \mathfrak{A}(I, I, 0, M, N)$ (or $\mathfrak{A}'(I, I, 0, M, N)$) the mapping $L_f : I \to \mathbb{R}$ defined by*

$$L_f(x) = \sum_{i=1}^{\infty} \lambda_i H_i\left(f^{i-1}(x)\right), \quad x \in I \tag{3.6}$$

has the following properties:

(i) $L_f \in \mathfrak{A}(I, J, K_0, K_3, K_1N + K_4)$;

(ii) $L_f^{-1} \in \mathfrak{A}(J, I, 1/K_3, 1/K_0, (K_1N + K_4)/(K_0)^3)$,

where $I = [a, b]$ *and* $J = [A, B]$.

Proof. For any $f \in \mathfrak{A}(I, I, 0, M, N)$ (or $\mathfrak{A}'(I, I, 0, M, N)$), we have

$$L_f(a) = \sum_{i=1}^{\infty} \lambda_i H_i\left(f^{i-1}(a)\right) < L_f(b) = \sum_{i=1}^{\infty} \lambda_i H_i\left(f^{i-1}(b)\right). \tag{3.7}$$

It is easy to see that for any $x \in I$

$$0 < \lambda_1 m_1 - \sum_{i=2}^{\infty} |\lambda_i| M_i M^{i-1} \le \sum_{i=1}^{\infty} \lambda_i H_i'\left(f^{i-1}(x)\right)\left(f^{i-1}\right)'(x) \le \lambda_1 M_1 + \sum_{i=2}^{\infty} |\lambda_i| M_i M^{i-1}. \tag{3.8}$$

We have for any $x \in I$

$$0 < K_0 \le L'_f(x) \le K_3, \tag{3.9}$$

and for any $y \in J$

$$0 < \frac{1}{K_3} \le \left(L_f^{-1}\right)'(y) \le \frac{1}{K_0}. \tag{3.10}$$

Thus $L_f : I \to J$ is an orientation-preserving diffeomorphism. □

By Lemma 2.2 we can see that for any $x_1, x_2 \in I$,

$$\begin{aligned}
\left|L'_f(x_1) - L'_f(x_2)\right| &= \left|\sum_{i=1}^{\infty} \lambda_i H_i'\left(f^{i-1}(x_1)\right)\left(f^{i-1}\right)'(x_1) - \sum_{i=1}^{\infty} \lambda_i H_i'\left(f^{i-1}(x_2)\right)\left(f^{i-1}\right)'(x_2)\right| \\
&\le \sum_{i=1}^{\infty} |\lambda_i| \left\{\left|H_i'\left(f^{i-1}(x_1)\right)\right| \cdot \left|\left(f^{i-1}\right)'(x_1) - \left(f^{i-1}\right)'(x_2)\right|\right. \\
&\qquad \left. + \left|H_i'\left(f^{i-1}(x_1)\right) - H_i'\left(f^{i-1}(x_2)\right)\right| \cdot \left|\left(f^{i-1}\right)'(x_2)\right|\right\} \\
&\le \left\{\sum_{i=2}^{\infty} |\lambda_i| M_i N \left(\sum_{j=i-2}^{2(i-2)} M^j\right) + \sum_{i=1}^{\infty} |\lambda_i| N_i M^{2(i-1)}\right\} |x_1 - x_2| \\
&= (K_1 N + K_4) |x_1 - x_2|.
\end{aligned} \tag{3.11}$$

By (3.9), (3.11), and Lemma 2.3 we get for any $y_1, y_2 \in J$:

$$\left|\left(L_f^{-1}\right)'(y_1) - \left(L_f^{-1}\right)'(y_2)\right| \le \frac{K_1 N + K_4}{(K_0)^3} |y_1 - y_2|. \tag{3.12}$$

Lemma 3.4. *Under the assumptions of Theorem 3.1, for each $f_1, f_2 \in \mathfrak{A}(I, I, 0, M, N)$ (or $\mathfrak{A}'(I, I, 0, M, N)$) the mappings $L_{f_1}, L_{f_2} : I \to J$ satisfy the following inequalities:*

(i) $\|L_{f_1}^{-1} - L_{f_2}^{-1}\|_{c^0} \le K_5/K_0 \|f_1 - f_2\|_{c^0}$;

(ii) $\|(L_{f_1}^{-1})' - (L_{f_2}^{-1})'\|_{c^0} \le (((K_1 N + K_4)K_5 + (K_7 + NK_8)K_0)/(K_0)^3)\|f_1 - f_2\|_{c^0} + K_6/(K_0)^2 \|f_1' - f_2'\|_{c^0}$.

Proof. Firstly we have

$$\begin{aligned} \|L_{f_1} - L_{f_2}\|_{c^0} &\le \sum_{k=1}^{\infty} |\lambda_{k+1}| \cdot \|H_{k+1} \circ f_1^k - H_{k+1} \circ f_2^k\|_{c^0} \\ &\overset{(2.2)}{\le} \sum_{k=1}^{\infty} |\lambda_{k+1}| M_{k+1} \left(\sum_{i=1}^{k} M^{i-1} \right) \cdot \|f_1 - f_2\|_{c^0}, \end{aligned} \tag{3.13}$$

and thus

$$\|L_{f_1}^{-1} - L_{f_2}^{-1}\|_{c^0} \overset{(\text{Lemma } (2.4))}{\le} \frac{1}{K_0} \|L_{f_1} - L_{f_2}\|_{c^0} \le \frac{K_5}{K_0} \|f_1 - f_2\|_{c^0}. \tag{3.14}$$

Secondly we get

$$\begin{aligned} \|(L_{f_1}^{-1})' - (L_{f_2}^{-1})'\|_{c^0} &= \max_{y \in J} \left\{ \left| \frac{1}{L'_{f_1}(L_{f_1}^{-1}(y))} - \frac{1}{L'_{f_2}(L_{f_2}^{-1}(y))} \right| \right\} \\ &\overset{(3.9)}{\le} \max_{y \in J} \left\{ \frac{|L'_{f_2}(L_{f_2}^{-1}(y)) - L'_{f_1}(L_{f_1}^{-1}(y))|}{(K_0)^2} \right\} \\ &\le \frac{1}{(K_0)^2} \cdot \max_{y \in J} \{ |L'_{f_2}(L_{f_2}^{-1}(y)) - L'_{f_2}(L_{f_1}^{-1}(y))| \} \\ &\quad + \frac{1}{(K_0)^2} \cdot \max_{y \in J} \{ |L'_{f_2}(L_{f_1}^{-1}(y)) - L'_{f_1}(L_{f_1}^{-1}(y))| \} \\ &\overset{(3.11)}{\le} \frac{K_1 N + K_4}{(K_0)^2} \cdot \max_{y \in J} \{ |L_{f_2}^{-1}(y) - L_{f_1}^{-1}(y)| \} + \frac{1}{(K_0)^2} \cdot \max_{x \in I} \{ |L'_{f_2}(x) - L'_{f_1}(x)| \} \\ &= \frac{K_1 N + K_4}{(K_0)^2} \cdot \|L_{f_2}^{-1} - L_{f_1}^{-1}\|_{c^0} + \frac{1}{(K_0)^2} \cdot \|L'_{f_2} - L'_{f_1}\|_{c^0}. \end{aligned} \tag{3.15}$$

Notice that

$$\|L'_{f_2} - L'_{f_1}\|_{c^0} \le \sum_{k=1}^{\infty} |\lambda_{k+1}| \cdot \|(H_{k+1} \circ f_2^k)' - (H_{k+1} \circ f_1^k)'\|_{c^0} \tag{3.16}$$

and for $k = 1, 2, \ldots,$

$$\begin{aligned}
\left\|\left(H_{k+1}\circ f_2^k\right)' - \left(H_{k+1}\circ f_1^k\right)'\right\|_{c^0} &= \left\|\left(H'_{k+1}\circ f_2^k\right)\cdot\left(f_2^k\right)' - \left(H'_{k+1}\circ f_1^k\right)\cdot\left(f_1^k\right)'\right\|_{c^0} \\
&\leq \left\|H'_{k+1}\circ f_2^k - H'_{k+1}\circ f_1^k\right\|_{c^0}\cdot\left\|\left(f_2^k\right)'\right\|_{c^0} \\
&\quad + \left\|H'_{k+1}\circ f_1^k\right\|_{c^0}\left\|\left(f_2^k\right)' - \left(f_1^k\right)'\right\|_{c^0} \\
&\leq M^k N_{k+1}\sum_{i=1}^{k} M^{i-1}\left\|f_2 - f_1\right\|_{c^0} + M_{k+1}kM^{k-1}\left\|f_2' - f_1'\right\|_{c^0} \\
&\quad + M_{k+1}Q(k)N\sum_{i=1}^{k-1}(k-i)M^{k+i-2}\left\|f_2 - f_1\right\|_{c^0},
\end{aligned} \tag{3.17}$$

then

$$\left\|L'_{f_2} - L'_{f_1}\right\|_{c^0} \leq \left(K_7 + NK_8\right)\left\|f_2 - f_1\right\|_{c^0} + K_6\left\|f_2' - f_1'\right\|_{c^0}. \tag{3.18}$$

Finally we get

$$\left\|\left(L_{f_1}^{-1}\right)' - \left(L_{f_2}^{-1}\right)'\right\|_{c^0} \leq \left(\frac{\left(K_1N + K_4\right)K_5 + \left(K_7 + NK_8\right)K_0}{\left(K_0\right)^3}\right)\left\|f_1 - f_2\right\|_{c^0} + \frac{K_6}{\left(K_0\right)^2}\left\|f_1' - f_2'\right\|_{c^0}. \tag{3.19}$$

□

Proof of Theorem 3.1. For any $f \in \mathfrak{A}(I, I, 0, M, N)$ (or $\mathfrak{A}'(I, I, 0, M, N)$) we define $\Theta(f)$ as follows:

$$\Theta(f) = L_f^{-1}\circ F, \tag{3.20}$$

and denote $\Theta(f) = g$ for convenience. Clearly $g \in C^1(I, I)$, $g(a) = a$, $g(b) = b$, (or $g(a) = b,\ g(b) = a$), and (3.10) yields that for any $x \in I$,

$$\left|g'(x)\right| = \left|\left(L_f^{-1}\right)'(F(x))\cdot F'(x)\right| \leq \frac{K_0M}{K_0} = M. \tag{3.21}$$

Furthermore by (3.12) we get that for any $x_1, x_2 \in I$,

$$\begin{aligned}
\left|g'(x_1) - g'(x_2)\right| &\leq \left|\left(L_f^{-1}\right)'\left(F(x_1)\right)\right|\cdot\left|F'(x_1) - F'(x_2)\right| \\
&\quad + \left|\left(L_f^{-1}\right)'\left(F(x_1)\right) - \left(L_f^{-1}\right)'\left(F(x_2)\right)\right|\cdot\left|F'(x_2)\right| \\
&\leq \frac{1}{K_0}L\left|x_1 - x_2\right| + K_0M\cdot\frac{K_1N + K_4}{\left(K_0\right)^3}\cdot K_0M\left|x_1 - x_2\right| \\
&= N\left|x_1 - x_2\right|.
\end{aligned} \tag{3.22}$$

So $g \in \mathfrak{A}(I, I, 0, M, N)$ (or $\mathfrak{A}'(I, I, 0, M, N)$), which means that $\Theta(\mathfrak{A}(I, I, 0, M, N)) \subset \mathfrak{A}(I, I, 0, M, N)$ (or $\Theta(\mathfrak{A}'(I, I, 0, M, N)) \subset \mathfrak{A}'(I, I, 0, M, N)$).

Secondly we prove that

$$\begin{gathered}\Theta : \mathfrak{A}(I, I, 0, M, N) \longrightarrow \mathfrak{A}(I, I, 0, M, N)\\ (\text{or } \Theta : \mathfrak{A}'(I, I, 0, M, N) \longrightarrow \mathfrak{A}'(I, I, 0, M, N))\end{gathered} \tag{3.23}$$

is continuous. For any $f_1, f_2 \in \mathfrak{A}(I, I, 0, M, N)$ (or $\mathfrak{A}'(I, I, 0, M, N)$) we denote $g_i = \Theta(f_i)$, $i = 1, 2$. It is easy to see that

$$\left\|g_1 - g_2\right\|_{c^0} = \left\|L_{f_1}^{-1} \circ F - L_{f_2}^{-1} \circ F\right\|_{c^0} \le \left\|L_{f_1}^{-1} - L_{f_2}^{-1}\right\|_{c^0} \le \frac{K_5}{K_0}\left\|f_1 - f_2\right\|_{c^0}. \tag{3.24}$$

By Lemma 3.4 we get

$$\begin{aligned}\left\|g'_1 - g'_2\right\|_{c^0} &= \left\|\left(\left(L_{f_1}^{-1}\right)' \circ F\right) \cdot F' - \left(\left(L_{f_2}^{-1}\right)' \circ F\right) \cdot F'\right\|_{c^0}\\ &\le K_0 M \cdot \left\|\left(\left(L_{f_1}^{-1}\right)' - \left(L_{f_2}^{-1}\right)'\right)\right\|_{c^0}\\ &\le \left(\frac{(K_1 N + K_4) K_5 M + (K_7 + N K_8) K_0 M}{(K_0)^2}\right)\left\|f_1 - f_2\right\|_{c^0} + \frac{M K_6}{K_0}\left\|f'_1 - f'_2\right\|_{c^0}.\end{aligned} \tag{3.25}$$

By the discussion above we get

$$\begin{aligned}\|g_1 - g_2\|_{c^1} &= \left\|g_1 - g_2\right\|_{c^0} + \left\|g'_1 - g'_2\right\|_{c^0}\\ &\le \left\{\frac{K_5}{K_0} + \frac{(K_1 N + K_4) K_5 M + (K_7 + N K_8) K_0 M}{(K_0)^2}\right\}\left\|f_2 - f_1\right\|_{c^0} + \frac{M K_6}{K_0}\left\|f'_2 - f'_1\right\|_{c^0}\\ &\le E\left\|f_1 - f_2\right\|_{c^1},\end{aligned} \tag{3.26}$$

where

$$E = \max\left\{\frac{K_5}{K_0} + \frac{(K_1 N + K_4) K_5 M + (K_7 + N K_8) K_0 M}{(K_0)^2}, \frac{M K_6}{K_0}\right\}. \tag{3.27}$$

□

Hence $\Theta : \mathfrak{A}(I, I, 0, M, N) \to \mathfrak{A}(I, I, 0, M, N)$ (or $\Theta : \mathfrak{A}'(I, I, 0, M, N) \to \mathfrak{A}'(I, I, 0, M, N)$) is continuous. By Schauder fixed point theorem, there exists a function $f \in \mathfrak{A}(I, I, 0, M, N)$ (or $\mathfrak{A}'(I, I, 0, M, N)$) such that

$$f = \Theta(f) = L_f^{-1} \circ F. \tag{3.28}$$

That means f is a solution of (1.4) in $\mathfrak{A}(I, I, 0, M, N)$ (or $\mathfrak{A}'(I, I, 0, M, N)$).

4. Uniqueness and stability of solutions

Theorem 4.1. *Let* $\{\lambda_i\}_{i=1}^{\infty}$ *be coefficient of* (1.4) *and* $M > 1$, $N > 0$ *positive constants. Suppose that the conditions in* (3.1) *are valid. Further one assumes that*

$$E = \max\left\{\frac{K_5}{K_0} + \frac{(K_1N + K_4)K_5M + (K_7 + NK_8)K_0M}{(K_0)^2},\ \frac{MK_6}{K_0}\right\} < 1. \tag{4.1}$$

Then for any $F \in \mathfrak{A}(I, J, 0, K_0M, L)$ *(or* $\mathfrak{A}'(I, J, 0, K_0M, L)$*), there exists a unique function* $f \in \mathfrak{A}(I, I, 0, M, M^*)$ *(or* $\mathfrak{A}'(I, I, 0, M, M^*)$*) satisfying* (1.4)*, where* $M^* \geq (L + K_4M^2)/(K_0 - K_1M^2)$*. Furthermore the solution* f *depends continuously on the given function* F*.*

Proof. If $E < 1$, then by (3.26) the map Θ defined in Theorem 3.1 becomes a strict contraction. The fix point of Θ, which is a solution of (1.4), is unique by Banach's contraction principle.

Let f_1, f_2 be the solutions of (1.4) for the corresponding functions F_1, F_2. First, since

$$f_i = L_{f_i}^{-1} \circ F_i, \quad i = 1, 2, \tag{4.2}$$

we get

$$\begin{aligned} \|f_1 - f_2\|_{c^0} &= \|L_{f_1}^{-1} \circ F_1 - L_{f_2}^{-1} \circ F_2\|_{c^0} \\ &\leq \|L_{f_1}^{-1} \circ F_1 - L_{f_1}^{-1} \circ F_2\|_{c^0} + \|L_{f_1}^{-1} \circ F_2 - L_{f_2}^{-1} \circ F_2\|_{c^0} \\ &\overset{\text{(Lemma (3.4))}}{\leq} \frac{1}{K_0}\|F_1 - F_2\|_{c^0} + \frac{K_5}{K_0}\|f_1 - f_2\|_{c^0}. \end{aligned} \tag{4.3}$$

Second, we have

$$\begin{aligned} \|f'_1 - f'_2\|_{c^0} &= \|((L_{f_1}^{-1})' \circ F_1) \cdot F'_1 - ((L_{f_2}^{-1})' \circ F_2) \cdot F'_2\|_{c^0} \\ &\leq \|((L_{f_1}^{-1})' \circ F_1) \cdot F'_1 - ((L_{f_1}^{-1})' \circ F_1) \cdot F'_2\|_{c^0} \\ &\quad + \|((L_{f_1}^{-1})' \circ F_1) \cdot F'_2 - ((L_{f_1}^{-1})' \circ F_2) \cdot F'_2\|_{c^0} \\ &\quad + \|((L_{f_1}^{-1})' \circ F_2) \cdot F'_2 - ((L_{f_2}^{-1})' \circ F_2) \cdot F'_2\|_{c^0} \\ &\overset{(3.12)}{\leq} \frac{1}{K_0}\|F'_1 - F'_2\|_{c^0} + \frac{M(K_1N + K_4)}{(K_0)^2}\|F_1 - F_2\|_{c^0} \\ &\quad + K_0M\|(L_{f_1}^{-1})' - (L_{f_2}^{-1})'\|_{c^0} \\ &\overset{\text{(Lemma (3.4))}}{\leq} \frac{1}{K_0}\|F'_1 - F'_2\|_{c^0} + \frac{M(K_1N + K_4)}{(K_0)^2}\|F_1 - F_2\|_{c^0} \\ &\quad + \left\{\frac{(K_1N + K_4)K_5M + (K_7 + NK_8)K_0M}{(K_0)^2}\right\}\|f_1 - f_2\|_{c^0} \\ &\quad + \frac{MK_6}{K_0}\|f'_2 - f'_1\|_{c^0}. \end{aligned} \tag{4.4}$$

By the above discussion we get

$$\begin{aligned}\|f_1 - f_2\|_{c^1} &= \|f_1 - f_2\|_{c^0} + \|f'_1 - f'_2\|_{c^0}\\ &\le E\|f_1 - f_2\|_{c^1} + \left\{\frac{M(K_1N + K_4)}{(K_0)^2} + \frac{1}{K_0}\right\}\|F_1 - F_2\|_{c^1},\end{aligned} \tag{4.5}$$

which means that

$$\|f_1 - f_2\|_{c^1} \le \frac{1}{1-E}\left\{\frac{M(K_1N + K_4)}{(K_0)^2} + \frac{1}{K_0}\right\}\|F_1 - F_2\|_{c^1}. \tag{4.6}$$

So the solution f depends continuously on the given function F. □

4.1. Examples

Example 4.2. Let $M = 10$, $L = 20$ and $I = [0,1]$. The equation

$$\frac{1001(e^{f(x)} - 1)}{1000(e-1)} - \frac{(f^2(x))^2}{1000} = x + \frac{1}{2}\sin(2\pi x), \tag{4.7}$$

where $x \in [0,1]$, has a unique solution $f \in \mathfrak{A}([0,1],[0,1],0,10,900)$.

Proof. It is easy to see that $H_1(x) = (e^x - 1)/(e-1) \in \mathfrak{A}(I,I,1/2,2,2)$, $H_2(x) = x^2 \in \mathfrak{A}(I,I,0,2,2)$ and $F(x) = x + 1/2\sin(2\pi x) \in \mathfrak{A}(I,I,0,4.5,20)$. By simple calculation we get that

$$\begin{gathered} K_0 = \frac{961}{2000}, \qquad K_1 = \frac{1}{500}, \qquad K_2 = \frac{9999}{500}, \qquad K_3 = \frac{2022}{1000}, \qquad K_4 = \frac{1101}{500},\\ K_5 = K_6 = \frac{1}{500}, \qquad K_7 = \frac{1}{50}, \qquad K_8 = 0, \qquad K_0M = \frac{961}{200},\\ K_0 - K_1M^2 = \frac{561}{2000}, \qquad A = 0 < B = 1,\\ N = \frac{(L + K_4M^2)}{(K_0 - K_1M^2)} = \frac{480400}{561} < 900, \end{gathered} \tag{4.8}$$

by Theorem 3.1 the equation has a solution $f \in \mathfrak{A}([0,1],[0,1],0,10,900)$. Further we get that

$$\begin{gathered} \frac{K_5}{K_0} + \frac{(K_1N + K_4)K_5M + (K_7 + NK_8)K_0M}{(K_0)^2} < \frac{804}{961},\\ \frac{MK_6}{K_0} = \frac{40}{961}, \end{gathered} \tag{4.9}$$

this means $E < 1$. By Theorem 4.1 the solution is unique. □

By similar discussion we have the following example.

Example 4.3. Let $M = 10$, $L = 20$, and $I = [0,1]$. The equation

$$\frac{1001\left(e^{f(x)}-1\right)}{1000(e-1)} - \frac{1-\left(f^2(x)\right)^2}{1000} = 1 - x - \frac{1}{2}\sin(2\pi x), \tag{4.10}$$

where $x \in [0,1]$ has a unique solution $f \in \mathfrak{A}'([0,1],[0,1],0,10,900)$.

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