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Research Article

Existence and Multiplicity of Positive Solutions for Dirichlet Problems in Unbounded Domains

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We consider the elliptic problem $-\Delta u + u = b(x)|u|^{p-2}u + h(x)$ in Ω, $u \in H_0^1(\Omega)$, where $2 < p < (2N/(N-2))$ $(N \geq 3)$, $2 < p < \infty$ $(N = 2)$, Ω is a smooth unbounded domain in $\mathbb{R}^N$, $b(x) \in C(\Omega)$, and $h(x) \in H^{-1}(\Omega)$. We use the shape of domain Ω to prove that the above elliptic problem has a ground-state solution if the coefficient $b(x)$ satisfies $b(x) \to b^\infty > 0$ as $|x| \to \infty$ and $b(x) \geq c$ for some suitable constants $c \in (0, b^\infty)$, and $h(x) \equiv 0$. Furthermore, we prove that the above elliptic problem has multiple positive solutions if the coefficient $b(x)$ also satisfies the above conditions, $h(x) \geq 0$ and $0 < \|h\|_{H^{-1}} < (p-2)(1/(p-1))^{(p-1)/(p-2)}[b_{\sup} S^p(\Omega)]^{1/(2-p)}$, where $S(\Omega)$ is the best Sobolev constant of subcritical operator in $H_0^1(\Omega)$ and $b_{\sup} = \sup_{x \in \Omega} b(x)$.

1. Introduction

In this paper, we are concerned with the existence and multiplicity of positive solutions of the following elliptic problems:

$$\begin{gathered} -\Delta u + u = b(x)|u|^{p-2}u + h(x) \quad \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{gathered} \tag{1.1}$$

where $2 < p < (2N/(N-2))$ $(N \geq 3)$, $2 < p < \infty$ $(N = 2)$, and Ω is a smooth unbounded domain in $\mathbb{R}^N$. We assume that $b(x) \in C(\Omega) \cap L^\infty(\Omega)$ satisfies

$$b(x) > 0, \quad \forall x \in \Omega, \tag{1.2}$$

and $h(x)$ satisfies

$$h(x) \in H^{-1}(\Omega), \quad h(x) \geq 0. \tag{1.3}$$

Associated with (1.1), we consider the energy functional J_h^b in the Sobolev space $H_0^1(\Omega)$:

$$J_h^b(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_\Omega b(x)|u|^p - \int_\Omega h(x)u, \tag{1.4}$$

where $\|u\|_{H^1} = (\int_\Omega |\nabla u|^2 + u^2)^{1/2}$. By Rabinowitz [1, Proposition B.10], $J_h^b \in C^1(H_0^1(\Omega), \mathbb{R})$. It is well known that the solutions of (1.1) are the critical points of the energy functional J_h^b in $H_0^1(\Omega)$.

Under the assumption (1.3) and $h(x) \not\equiv 0$, (1.1) can be regarded as a perturbation problem of the following homogeneous elliptic equation:

$$\begin{gathered} -\Delta u + u = b(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{gathered} \tag{1.5}$$

A typical approach for solving a problem of this kind is to use the minimax method:

$$\alpha_\Gamma^b(\Omega) = \inf_{\gamma \in \Gamma(\Omega)} \max_{t \in [0,1]} J_0^b(\gamma(t)), \tag{1.6}$$

where

$$\Gamma(\Omega) = \{\gamma \in C([0,1], H_0^1(\Omega)) \mid \gamma(0) = 0,\ \gamma(1) = e\}, \tag{1.7}$$

$J_0^b(e) = 0$, and $e \neq 0$. By the mountain pass lemma due to Ambrosetti and Rabinowitz [2], we called the nonzero critical point $u \in H_0^1(\Omega)$ of J_0^b is as ground-state solution of (1.5) in Ω if $J_0^b(u) = \alpha_\Gamma^b(\Omega)$. We note that the ground-state solutions of (1.5) in Ω can also be obtained by the Nehari minimization problem

$$\alpha_0^b(\Omega) = \inf_{v \in \mathbf{M}_0^b(\Omega)} J_0^b(v), \tag{1.8}$$

where $\mathbf{M}_0^b(\Omega) = \{u \in H_0^1(\Omega) \backslash \{0\} \mid \|u\|_{H^1}^2 = \int_\Omega b(x)|u|^p\}$. Note that $\mathbf{M}_0^b(\Omega)$ contains every nonzero solution of (1.5) in Ω, $\alpha_\Gamma^b(\Omega) = \alpha^b(\Omega) > 0$ (see Willem [3] and Wang and Wu [4]), and if $b(x) \equiv b^\infty > 0$ is a constant, then J_0^b and $\alpha_0^b(\Omega)$ are replaced by J_0^∞ and $\alpha_0^\infty(\Omega)$, respectively.

That the existence of ground-state solutions of (1.5) is affected by the shape of the domain Ω and $b(x)$ that satisfies some suitable conditions has been the focus of a great deal of research in recent years. By the Rellich compactness theorem and the minimax method, it is easy to obtain a ground-state solution for (1.5) in bounded domains. When Ω is an unbounded domain and $b(x) \equiv b^\infty$, the existence of ground-state solutions has been established by several authors under various conditions. We mention, in particular, results by Berestycki and Lions [5], Lien et al. [6], Chen and Wang [7], and Del Pino and Felmer [8, 9]. In [5], $\Omega = \mathbb{R}^N$. Actually, Kwong [10] proved that the positive solution of (1.5) in $\mathbb{R}^N$ is unique. In [6], Ω is a periodic domain. In [7, 6], the domain Ω is required to satisfy that

($\Omega 1$) $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1, Ω_2 are domains in $\mathbb{R}^N$ and $\Omega_1 \cap \Omega_2$ is bounded;

($\Omega 2$) $\alpha_0^\infty(\Omega) < \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2)\}$.

In [8, 9], for $1 \le l \le N-1$, $\mathbb{R}^N = \mathbb{R}^l \times \mathbb{R}^{N-l}$. For a point $x \in \mathbb{R}^N$, we have $x = (y,z)$, where $y \in \mathbb{R}^l$ and $z \in \mathbb{R}^{N-l}$. Let $y \in \mathbb{R}^l$, we denote by $\Omega^y \subset \mathbb{R}^{N-l}$ the projection of Ω onto $\mathbb{R}^{N-l}$, that is,

$$\Omega^y = \{z \in \mathbb{R}^{N-l} \mid (y,z) \in \Omega\}. \tag{1.9}$$

The domain Ω is required to satisfy that

($\Omega 3$) Ω is a smooth subset of $\mathbb{R}^N$ and the projections Ω^y are bounded uniformly in $y \in \mathbb{R}^l$;

($\Omega 4$) there exists a nonempty closed set $F \subset \mathbb{R}^{N-l}$ such that $F \subset \Omega^y$ for all $y \in \mathbb{R}^l$;

($\Omega 5$) for each $\delta > 0$, there exists $K > 0$ such that

$$\Omega^y \subset \{z \in \mathbb{R}^{N-l} \mid \operatorname{dist}(z,F) < \delta\} \tag{1.10}$$

for all $|y| \ge K$.

Moreover, when $\Omega = \mathbb{R}^N \backslash \omega$ is an exterior domain, where ω is a bounded domain. It is well known that (1.5) in $\mathbb{R}^N \backslash \omega$ does not admit any ground-state solution (see Benci and Cerami [12]). However, Bahri and Lions [11] and Benci and Cerami [12] asserted that (1.5) in $\mathbb{R}^N \backslash \omega$ has a higher-energy positive solution. As Ω is an Esteban-Lions domain, (1.5) in Ω does not admit any nontrivial solution (see Esteban and Lions [13]), where the definition of Esteban-Lions domain is as follows: for a proper unbounded domain Ω in $\mathbb{R}^N$, there exists $\chi \in \mathbb{R}^N$, $\|\chi\| = 1$ such that $n(x) \cdot \chi \ge 0$ and $n(x) \cdot \chi \not\equiv 0$ on $\partial\Omega$, where $n(x)$ is the unit outward normal vector to $\partial\Omega$ at the point x.

When $b(x) \not\equiv b^\infty$, which satisfies the condition (1.2), the existence of ground-state solutions of (1.5) has been established by the condition $b(x) \ge b^\infty$ and the existence of ground-state solutions of limit equation

$$\begin{gathered} -\Delta u + u = b^\infty |u|^{p-2} u \quad \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{gathered} \tag{1.11}$$

On the other hand, for $\Omega = \mathbb{R}^N$ and $b(x) \le b^\infty$ on $\mathbb{R}^N$ with a strict inequality on a set of positive measures, (1.5) in $\mathbb{R}^N$ does not admit any ground-state solution. However, Bahri and Lions [11], Cao [14], and Bahri and Li [15] asserted that (1.5) in $\mathbb{R}^N$ has a higher-energy positive solution under the coefficient $b(x)$ which satisfies conditions $b(x) \ge (1/2)^{(p-2)/2} b^\infty$ and $b(x) \to b^\infty$ as $|x| \to \infty$ such that the functional J_0^b in $H_0^1(\Omega)$ satisfies the Palais-Smale condition for energy level β with

$$\alpha_0^\infty(\mathbb{R}^N) < \beta < \alpha_0^\infty(\mathbb{R}^N) + \alpha_0^b(\mathbb{R}^N). \tag{1.12}$$

The first result of our paper is relaxing the condition $b(x) \geq b^\infty$ to show the existence of ground-state solution of (1.5) by the shape of domain Ω. First, we consider the following assumptions:

($\Omega 1'$) given $k \geq 0$ and $1 \leq m \leq k$, the domain $\Omega = \bigcup_{i=1}^k \Omega_i$, where $\Omega_i \cap \Omega_j$ is bounded for all $i \neq j$ and Ω_j is unbounded domain for all $j = 1,2,\ldots,m$;

($\Omega 2'$) the functional J_0^∞ in $H_0^1(\Omega)$ satisfies the Palais-Smale condition for energy level $\alpha_0^\infty(\Omega)$;

($b1$) $b(x) \geq (\alpha_0^\infty(\Omega)/\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2),\ldots,\alpha_0^\infty(\Omega_m)\})^{(p-2)/2} b^\infty$ and $b(x) \to b^\infty$ as $|x| \to \infty$.

Then we have the following result.

Theorem 1.1. *If the domain Ω satisfies the conditions ($\Omega 1'$)-($\Omega 2'$) and $b(x)$ satisfies the condition ($b1$), then (1.5) in Ω has a ground-state solution.*

Remark 1.2. If the domain Ω satisfies the conditions ($\Omega 1$)-($\Omega 2$), then the functional J_0^∞ in $H_0^1(\Omega)$ satisfies the Palais-Smale condition for energy level $\alpha_0^\infty(\Omega)$, and we have

$$0 < \alpha_0^\infty(\Omega) < \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2),\ldots,\alpha_0^\infty(\Omega_m)\} \tag{1.13}$$

(see Lien et al. [6] and Chen and Wang [7]). Thus,

$$0 < \left(\frac{\alpha_0^\infty(\Omega)}{\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2),\ldots,\alpha_0^\infty(\Omega_m)\}}\right)^{(p-2)/2} < 1. \tag{1.14}$$

It is known that the general unbounded domains in $\mathbb{R}^N$ can be classified into three kinds. If Ω is an unbounded domain in $\mathbb{R}^N$, then it satisfies one of the following conditions:

(1) J_0^∞ in $H_0^1(\Omega)$ satisfies the Palais-Smale condition for energy level $\alpha_0^\infty(\Omega)$. In particular, (1.11) in Ω has a ground-state solution u_0 such that $J_0^\infty(u_0) = \alpha_0^\infty(\Omega)$;

(2) J_0^∞ in $H_0^1(\Omega)$ does not satisfy the Palais-Smale condition for energy level $\alpha_0^\infty(\Omega)$, but (1.11) in Ω has a ground-state solution u_0 such that $J_0^\infty(u_0) = \alpha_0^\infty(\Omega)$;

(3) equation (1.11) in Ω does not admit any ground-state solution.

In this motivation, consider a general unbounded domain Ω and its exterior domain $\Omega^c(r) = \Omega \backslash \overline{B^N(0;r)}$, and the following assumptions:

($\Omega 3'$) equation (1.11) in Ω has a ground state solution u_0 such that $J_0^\infty(u_0) = \alpha_0^\infty(\Omega)$.

($b2$) $b(x) \geq (\alpha_0^\infty(\Omega)/\lim_{r\to\infty} \alpha_0^\infty(\Omega^c(r)))^{(p-2)/2} b^\infty$ and $b(x) \to b^\infty$ as $|x| \to \infty$.

Then we have the following result.

Theorem 1.3. *If the unbounded domain Ω satisfies the condition ($\Omega 3'$) and $b(x)$ satisfies the condition ($b2$), then (1.5) in Ω has a ground-state solution.*

Remark 1.4. (1) If the domain Ω satisfies the conditions ($\Omega 3$)–($\Omega 5$), J_0^∞ in $H_0^1(\Omega)$ satisfies the Palais-Smale condition for energy level $\alpha_0^\infty(\Omega)$. Then $\alpha_0^\infty(\Omega) < \alpha_0^\infty(\Omega^c(r))$ for all $r > 0$ (see Del Pino and Felmer [8, 9] or Wu [16]). Since $\alpha_0^\infty(\Omega^c(r))$ is nondecreasing as r is increasing, we have

$$0 \leq \left(\frac{\alpha_0^\infty(\Omega)}{\lim_{r\to\infty} \alpha_0^\infty(\Omega^c(r))} \right)^{(p-2)/2} < 1. \tag{1.15}$$

(2) If Ω is a periodic domain, then J_0^∞ in $H_0^1(\Omega)$ does not satisfy the Palais-Smale condition for energy level $\alpha_0^\infty(\Omega)$, but (1.11) in Ω has a ground-state solution u_0 such that $J_0^\infty(u_0) = \alpha_0^\infty(\Omega)$. Then $\alpha_0^\infty(\Omega) = \alpha_0^\infty(\Omega^c(r))$ for all $r > 0$ (see Lien et al. [6]). Thus,

$$\left(\frac{\alpha_0^\infty(\Omega)}{\lim_{r\to\infty} \alpha_0^\infty(\Omega^c(r))} \right)^{(p-2)/2} \equiv 1. \tag{1.16}$$

Remark 1.5. If the domain $\Omega = \mathbb{R}^N$, coefficient $b(x)$ satisfies the condition (1.2) and $b(x) \leq b^\infty$ with a strict inequality on a set of positive measures, then (1.5) in $\mathbb{R}^N$ does not admit any ground-state solution and $\alpha_0^\infty(\mathbb{R}^N) = \alpha_0^b(\mathbb{R}^N)$. However, if the domain Ω satisfies the conditions (Ω1)-(Ω2) (or (Ω3)–(Ω5)), $b(x)$ satisfies the condition (b1) (or (b2)) and $b(x) \leq b^\infty$ with a strict inequality on a set of positive measure, then from Theorem 1.1 (or Theorem 1.3), we can conclude that (1.5) has a ground-state solution. Moreover, $\alpha_0^\infty(\Omega) < \alpha_0^b(\Omega)$.

Finally, we consider (1.1). For $\Omega = \mathbb{R}^N$, several authors have shown the existence of at least two positive solutions of (1.1) in $\mathbb{R}^N$ under some suitable conditions. In [17] by Zhu for $b(x) = b^\infty$, $h(x)$ is exponential decay and $\|h\|_{L^2}$ is sufficiently small. By Cao and Zhou in [18] and Jeanjean [19], for $b(x) \geq b^\infty$ and $\|h\|_{H^{-1}}$ sufficiently small. By Adachi and Tanaka in [20], for $b(x) \geq b^\infty - Ce^{-\lambda|x|}$ for some $C, \lambda > 0$ and $\|h\|_{H^{-1}}$ sufficiently small. Moreover, Adachi and Tanaka [21] used that (1.5) in $\mathbb{R}^N$ does not admit any ground-state solution for the condition $b(x) \leq b^\infty$ with a strict inequality on a set of positive measures, to show that (1.1) in $\mathbb{R}^N$ has at least four positive solutions for $\|h\|_{H^{-1}}$ sufficiently small. The second aim of our paper is also relaxing the condition $b(x) \geq b^\infty$ to show the existence of at least two positive solutions of (1.1) in Ω. Denote

$$b_{\sup} = \sup_{x\in\Omega} b(x) \tag{1.17}$$

and $S(\Omega) = [(2p/(p-2))\alpha_0^\infty(\Omega)]^{(2-p)/2p}$ is the best Sobolev constant of subcritical operator in $H_0^1(\Omega)$ (see Lin et al. [22] or Willem [3]). Then we have the following results.

Theorem 1.6. *Suppose that the domain Ω satisfies the conditions ($\Omega 1'$)-($\Omega 2'$) and $b(x)$ satisfies the condition (b1). If $h \geq 0$ and*

$$0 < \|h\|_{H^{-1}} < (p-2)\left(\frac{1}{p-1}\right)^{(p-1)/(p-2)} \left[b_{\sup} S^p(\Omega)\right]^{1/(2-p)}, \tag{1.18}$$

then (1.1) in Ω has at least two positive solutions.

Theorem 1.7. *Suppose that the domain* Ω *satisfies the condition* $(\Omega 3')$ *and* $b(x)$ *satisfies the condition* $(b2)$. *If* $h \geq 0$ *and*

$$0 < \|h\|_{H^{-1}} < (p-2)\left(\frac{1}{p-1}\right)^{(p-1)/(p-2)}\left[b_{\sup}S^{p}(\Omega)\right]^{1/(2-p)}, \tag{1.19}$$

then (1.1) in Ω *has at least two positive solutions.*

This paper is organized as follows. In Section 2, we describe various preliminaries. In Section 3, we use the shape of the domain Ω to prove that (1.5) in Ω has a ground-state solution. In Section 4, we modify the proof of Adachi and Tanaka [21], Tarantello [23], Cao and Zhu [18], and Zhu [17] to prove that (1.1) in Ω has at least two positive solutions.

2. Preliminary

We define the Palais-Smale (PS) sequences, (PS) values, and (PS) conditions in $H_0^1(\Omega)$ for J_h^b as follows.

Definition 2.1. (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h^b if $J_h^b(u_n) = \beta + o(1)$ and $(J_h^b)'(u_n) = o(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$;

(ii) $\beta \in \mathbb{R}$ is a (PS) value in $H_0^1(\Omega)$ for J_h^b if there is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h^b;

(iii) J_h^b satisfies the $(\mathrm{PS})_\beta$-condition in $H_0^1(\Omega)$ if every $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h^b contains a convergent subsequence;

(iv) J_h^b satisfies the (PS) condition in $H_0^1(\Omega)$ if for every $\beta \in \mathbb{R}$, J_h^b satisfies the $(\mathrm{PS})_\beta$-condition in $H_0^1(\Omega)$.

We need the following lemmas.

Lemma 2.2. *Let* $u_n \rightharpoonup u$ *weakly in* $H_0^1(\Omega)$. *Then there exists a subsequence* $\{u_n\}$ *such that*

(i) $\{u_n\}$ *is bounded in* $H_0^1(\Omega)$ *and* $\|u\|_{H^1} \leq \liminf_{n\to\infty}\|u_n\|_{H^1}$;

(ii) $u_n \rightharpoonup u$, $\nabla u_n \rightharpoonup \nabla u$ *weakly in* $L^2(\Omega)$, *and* $u_n \to u$ *a.e. in* Ω;

(iii) $\|u_n - u\|_{H^1}^2 = \|u_n\|_{H^1}^2 - \|u\|_{H^1}^2 + o(1)$.

The proof is clear by the routine arguments, and hence is omitted here.

Lemma 2.3 (Brézis-Lieb lemma). *Suppose that* $u_n \to u$ *a.e. in* Ω *and there exists* $c > 0$ *such that* $\|u_n\|_{L^p} \leq c$ *for* $n = 1, 2, \ldots$. *Then*

(i) $\|u_n - u\|_{L^p}^p = \|u_n\|_{L^p}^p - \|u\|_{L^p}^p + o(1)$;

(ii) $|u_n - u|^{p-2}(u_n - u) - |u_n|^{p-2}u_n + |u|^{p-2}u = o(1)$ *in* $L^{p/(p-1)}(\Omega)$.

For the proof, see Brézis and Lieb [24].

Lemma 2.4. *Let* $u_n \rightharpoonup u$ *weakly in* $H_0^1(\Omega)$ *and*

$$(J_h^b)'(u_n) = -\Delta u_n + u_n - b(x)|u_n|^{p-2}u_n + h(x) = o(1) \quad \text{in } H^{-1}(\Omega). \tag{2.1}$$

Then

(i) $|u_n - u|^{p-2}(u_n - u) - |u_n|^{p-2}u_n + |u|^{p-2}u = o(1)$ *in* $H^{-1}(\Omega)$*;*

(ii) $(J_0^\infty)'(w_n) = -\Delta w_n + w_n - b^\infty |w_n|^{p-2}w_n = o(1)$ *in* $H^{-1}(\Omega)$*, where* $w_n = u_n - u$*;*

(iii) *if* $\{u_n\}$ *is a* $(PS)_\beta$*-sequence in* $H_0^1(\Omega)$ *for* J_h^b *then* $\{w_n\}$ *is a* $(PS)_{(\beta - J_h^b(u))}$*-sequence in* $H_0^1(\Omega)$ *for* J_0^∞*.*

Proof. For (i), (ii), see Bahri and Lions [11]. (iii) Since $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence for J_h^b in $H_0^1(\Omega)$, by Lemmas 2.2, 2.3, and the Sobolev embedding theorem, there exists a subsequence $\{u_n\}$ such that $w_n \rightharpoonup 0$ in $H_0^1(\Omega)$,

$$\begin{aligned} ||w_n||_{H^1}^2 &= ||u_n||_{H^1}^2 - \|u\|_{H^1}^2 + o(1), \\ ||w_n||_{L^p}^p &= ||u_n||_{L^p}^p - \|u\|_{L^p}^p + o(1). \end{aligned} \tag{2.2}$$

Thus,

$$J_0^\infty(w_n) = J_h^b(w_n) + o(1) = J_h^b(u_n) - J_h^b(u) + o(1) = \beta - J_h^b(u) + o(1). \tag{2.3}$$

Therefore, by part (ii), $\{p_n\}$ is a $(\mathrm{PS})_{(\beta - J_h^b(u))}$-sequence in $H_0^1(\Omega)$ for J_0^∞. □

We need the following useful results.

Lemma 2.5. *Let* $\{u_n\}$ *be a sequence in* $H_0^1(\Omega)$*. Then* $\{u_n\}$ *is a* $(PS)_{\alpha_0^b(\Omega)}$*-sequence for* J_0^b *if and only if* $J_0^b(u_n) = \alpha_0^b(\Omega) + o(1)$ *and* $\int_\Omega |\nabla u_n|^2 + u_n^2 = \int_\Omega b(x)|u_n|^p + o(1)$*. In particular, every minimizing sequence* $\{u_n\}$ *in* $\mathbf{M}_0^b(\Omega)$ *of* $\alpha_0^b(\Omega)$ *is a* $(PS)_{\alpha_0^b(\Omega)}$*-sequence in* $H_0^1(\Omega)$ *for* J_0^b*.*

The proof is almost the same as that by Wang and Wu in [4, Lemma 7], and is omitted here.

We introduce the Nehari minimization problem for (1.1) as

$$\alpha_h^b(\Omega) = \inf_{u \in \mathbf{M}_h^b(\Omega)} J_h^b(u), \tag{2.4}$$

where $\mathbf{M}_h^b(\Omega) = \{u \in H_0^1(\Omega) \backslash \{0\} \mid \langle (J_h^b)'(u), u\rangle = 0\}$. Define

$$\psi(u) = \langle (J_h^b)'(u), u\rangle = \|u\|_{H^1}^2 - \int_\Omega b(x)|u|^p - \int_\Omega h(x)u. \tag{2.5}$$

Then we have the following result.

Lemma 2.6. *If* $\|h\|_{H^{-1}} < (p-2)(1/(p-1))^{(p-1)/(p-2)}[b_{\sup} S^p(\Omega)]^{1/(2-p)}$*, then for each* $u \in \mathbf{M}_h^b(\Omega)$*,*

$$\langle \psi'(u), u\rangle = \|u\|_{H^1}^2 - (p-1)\int_\Omega b(x)|u|^p \neq 0. \tag{2.6}$$

Proof. For $u \in \mathbf{M}_h^b(\Omega)$, we have

$$\|u\|_{H^1}^2 - \int_\Omega b(x)|u|^p - \int_\Omega h(x)u = 0. \tag{2.7}$$

Then

$$\begin{aligned}\langle \psi'(u),u\rangle &= 2\|u\|_{H^1}^2 - p\int_\Omega b(x)|u|^p - \int_\Omega h(x)u\\ &= \|u\|_{H^1}^2 - (p-1)\int_\Omega b(x)|u|^p.\end{aligned} \tag{2.8}$$

We claim that if $\|h\|_{H^{-1}}<(p-2)(1/(p-1))^{(p-1)/(p-2)}[b_{\sup}S^p(\Omega)]^{1/(2-p)}$, then $\langle\psi'(u),u\rangle\neq 0$ for all $u\in \mathbf{M}_h^b(\Omega)$. Let $I:\mathbf{M}_h^b(\Omega)\to\mathbb{R}$ be given by

$$I(u) = K(p)\left(\frac{\|u\|_{H^1}^{2p-2}}{\int_\Omega b(x)|u|^p}\right)^{1/(p-2)} - \int_\Omega h(x)u, \tag{2.9}$$

where $K(p)=(p-2)(1/(p-1))^{(p-1)/(p-2)}$. Then we have for $u\in\mathbf{M}_h^b(\Omega)$,

$$\begin{aligned}I(u) &= K(p)\left(\frac{\|u\|_{H^1}^{2p-2}}{\int_\Omega b(x)|u|^p}\right)^{1/(p-2)} - \int_\Omega h(x)u\\ &\geq K(p)\left(\frac{\|u\|_{H^1}^{2p-2}}{\int_\Omega b(x)|u|^p}\right)^{1/(p-2)} - \|h\|_{H^{-1}}\|u\|_{H^1}\\ &= \|u\|_{H^1}\left(K(p)\left(\frac{\|u\|_{H^1}^{p}}{\int_\Omega b(x)|u|^p}\right)^{1/(p-2)} - \|h\|_{H^{-1}}\right)\end{aligned} \tag{2.10}$$

since

$$\left(\frac{\|u\|_{H^1}^{p}}{\int_\Omega b(x)|u|^p}\right)^{1/(p-2)} \geq \left[b_{\sup}S^p(\Omega)\right]^{1/(2-p)} \quad \forall u\in H_0^1(\Omega)\setminus\{0\}. \tag{2.11}$$

Thus, for $\|h\|_{H^{-1}}<K(p)[b_{\sup}S^p(\Omega)]^{1/(2-p)}$, we have

$$I(u)>0 \quad \forall u\in\mathbf{M}_h^b(\Omega). \tag{2.12}$$

Assume that there is a $w\in\mathbf{M}_h^b(\Omega)$ such that $\langle\psi'(w),w\rangle=0$, then we have

$$\begin{aligned}\|w\|_{H^1}^2 &= (p-1)\int_\Omega b(x)|w|^p,\\ \int_\Omega h(x)w = \|w\|_{H^1}^2 - \int_\Omega b(x)|w|^p &= (p-2)\int_\Omega b(x)|w|^p.\end{aligned} \tag{2.13}$$

From (2.12) and (2.13),

$$\begin{aligned}0<I(w) &= K(p)\left(\frac{\|w\|_{H^1}^{2p-2}}{\int_\Omega b(x)|w|^p}\right)^{1/(p-2)} - \int_\Omega h(x)w\\ &= \left(\frac{1}{p-1}\right)^{(p-1)/(p-2)}(p-2)\left(\frac{(p-1)^{p-1}[\int_\Omega b(x)|w|^p]^{p-1}}{\int_\Omega b(x)|w|^p}\right)^{1/(p-2)} - (p-2)\int_\Omega h(x)w = 0,\end{aligned} \tag{2.14}$$

which is a contradiction. Thus, we can conclude that for

$$\|h\|_{H^{-1}} < (p-2)\left(\frac{1}{p-1}\right)^{(p-1)/(p-2)} \left[b_{\sup} S^p(\Omega)\right]^{1/(2-p)}, \tag{2.15}$$

we have $\langle \psi'(u), u \rangle \neq 0$ for all $u \in \mathbf{M}_h^b(\Omega)$. □

By Lemma 2.6, we write $\mathbf{M}_h^b(\Omega) = \mathbf{M}_h^{b+}(\Omega) \cup \mathbf{M}_h^{b-}(\Omega)$, where

$$\begin{aligned} \mathbf{M}_h^{b+}(\Omega) &= \left\{ u \in \mathbf{M}_h^b(\Omega) \mid \|u\|_{H^1}^2 - (p-1)\int_\Omega b(x)|u|^p > 0 \right\}, \\ \mathbf{M}_h^{b-}(\Omega) &= \left\{ u \in \mathbf{M}_h^b(\Omega) \mid \|u\|_{H^1}^2 - (p-1)\int_\Omega b(x)|u|^p < 0 \right\}, \end{aligned} \tag{2.16}$$

and define

$$\alpha_h^{b+}(\Omega) = \inf_{u \in \mathbf{M}_h^{b+}(\Omega)} J_h^b(u), \qquad \alpha_h^{b-}(\Omega) = \inf_{u \in \mathbf{M}_h^{b-}(\Omega)} J_h^b(u). \tag{2.17}$$

For each $u \in H_0^1(\Omega)\backslash\{0\}$, we write

$$t_{\max} = \left(\frac{\|u\|_{H^1}^2}{(p-1)\int_\Omega b(x)|u|^p}\right)^{1/(p-2)} > 0. \tag{2.18}$$

Similar as the proof of some results by Tarantello in [23], we have the following two lemmas.

Lemma 2.7. *For each $u \in H_0^1(\Omega)\backslash\{0\}$,*

(i) *there is a unique $t^- = t^-(u) > t_{\max} > 0$ such that $t^- u \in \mathbf{M}_h^{b-}(\Omega)$ and $J_h^b(t^- u) = \max_{t \geq t_{\max}} J_h^b(tu)$;*

(ii) *$t^-(u)$ is a continuous function for nonzero u;*

(iii) *$\mathbf{M}_h^{b-}(\Omega) = \{u \in H_0^1(\Omega)\backslash\{0\} \mid (1/\|u\|_{H^1})t^-(u/\|u\|_{H^1}) = 1\}$;*

(iv) *if $\int_\Omega hu > 0$, then there is a unique $0 < t^+ = t^+(u) < t_{\max}$ such that $t^+ u \in \mathbf{M}_h^{b+}(\Omega)$ and $J_h^b(t^+ u) = \min_{0 \leq t \leq t^-} J_h^b(tu)$.*

Lemma 2.8. (i) *For each $u \in \mathbf{M}_h^{b+}(\Omega)$, $\int_\Omega h(x)u > 0$ and $J_h^b(u) < 0$. In particular, $\alpha_h(\Omega) \leq \alpha_h^+(\Omega) < 0$;*

(ii) *J_h^b is coercive and bounded below on $\mathbf{M}_h^b(\Omega)$.*

Proof. (i) For each $u \in \mathbf{M}_h^{b+}(\Omega)$, $\|u\|_{H^1}^2 - (p-1)\int_\Omega b(x)|u|^p > 0$ and

$$\|u\|_{H^1}^2 = \int_\Omega b(x)|u|^p + \int_\Omega h(x)u. \tag{2.19}$$

Thus,

$$\int_\Omega h(x)u = \|u\|_{H^1}^2 - \int_\Omega b(x)|u|^p > (p-2)\int_\Omega b(x)|u|^p > 0, \tag{2.20}$$

and hence

$$\begin{aligned} J_h^b(u) &= \left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega b(x)|u|^p - \frac{1}{2}\int_\Omega h(x)u \\ &< \frac{p-2}{2p}\int_\Omega b(x)|u|^p - \frac{p-2}{2}\int_\Omega b(x)|u|^p \\ &= -\frac{(p-1)(p-2)}{2p}\int_\Omega b(x)|u|^p < 0. \end{aligned} \tag{2.21}$$

(ii) Is similar to the proof of Theorem 1 by Tarantello in [23]. □

3. Homogeneous problems

First, we present several (PS) conditions in $H_0^1(\Omega)$ for J_0^b which are used to prove our main results. As a consequence of Lemma 2.8(ii), for each $(\mathrm{PS})_\beta$-sequence $\{u_n\}$ in $H_0^1(\Omega)$ for J_0^b, there exist a subsequence $\{u_n\}$ and u_0 in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. Then u_0 is a solution of (1.5) in Ω. Moreover, we have the following lemma.

Let Ω be any unbounded domain and $\xi \in C^\infty([0,\infty))$ such that $0 \le \xi \le 1$ and

$$\xi(t) = \begin{cases} 0 & \text{for } t \in [0,1] \\ 1 & \text{for } t \in [2,\infty). \end{cases} \tag{3.1}$$

Let

$$\xi_n(z) = \xi\left(\frac{2|z|}{n}\right). \tag{3.2}$$

Then we have the following result.

Lemma 3.1. *Let $\{u_n\}$ be a $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J_0^b satisfying $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and let $v_n = \xi_n u_n$. Then there exists a subsequence $\{u_n\}$ such that*

(i) $\|u_n - v_n\|_{H^1} = o(1)$ *as* $n \to \infty$;
(ii) $\int_\Omega b(x)|u_n|^p = \int_\Omega b(x)|v_n|^p + o(1) = \int_\Omega b^\infty |v_n|^p + o(1)$;
(iii) $\int_\Omega |\nabla v_n|^2 + v_n^2 = \int_\Omega b^\infty |v_n|^p + o(1)$;
(iv) $\{v_n\}$ *is a* $(PS)_\beta$*-sequence in* $H_0^1(\Omega)$ *for* J_0^∞.

Proof. By the fact that

$$||u_n - v_n||_{H^1}^2 = ||u_n||_{H^1}^2 + ||v_n||_{H^1}^2 - 2\langle u_n, v_n\rangle_{H^1}, \tag{3.3}$$

thus it suffices to show that $\langle u_n, v_n\rangle_{H^1} = \|u_n\|_{H^1}^2 + o(1) = \|v_n\|_{H^1}^2 + o(1)$. Since

$$\langle u_n, v_n\rangle_{H^1} = \int_\Omega \nabla u_n \nabla v_n + u_n v_n = \int_\Omega \xi_n[|\nabla u_n|^2 + u_n^2] + \int_\Omega u_n \nabla u_n \nabla \xi_n, \tag{3.4}$$

$|\nabla \xi_n| \le c/n$ and $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_0^b, it follows that

$$\int_\Omega \xi_n^q u_n \nabla u_n \nabla \xi_n = o(1) \quad \text{for } q > 0. \tag{3.5}$$

Hence,

$$\langle u_n, v_n\rangle_{H^1} = \int_\Omega \xi_n[\,|\nabla u_n|^2 + u_n^2] + o(1). \tag{3.6}$$

Similarly, we have

$$\|v_n\|_{H^1}^2 = \int_\Omega \xi_n^2[\,|\nabla u_n|^2 + u_n^2] + o(1). \tag{3.7}$$

Given $r \geq 1$, since $\{\xi_n^r u_n\}$ is bounded in $H_0^1(\Omega)$, we have

$$\begin{aligned} o(1) &= \langle (J_0^b)'(u_n), \xi_n^r u_n\rangle \\ &= \int_\Omega (\xi_n^r |\nabla u_n|^2 + r\xi_n^{r-1} u_n \nabla\xi_n \nabla u_n + \xi_n^r u_n^2) - \int_\Omega b(x)\xi_n^r |u_n|^p. \end{aligned} \tag{3.8}$$

From (3.5), we can conclude that

$$\int_\Omega \xi_n^r(|\nabla u_n|^2 + u_n^2) = \int_\Omega b(x)\xi_n^r |u_n|^p + o(1). \tag{3.9}$$

Since $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and $b(x) \to b^\infty$ as $|x| \to \infty$, there exists a subsequence $\{u_n\}$ such that $u_n \to 0$ strongly in $L^p_{\text{loc}}(\Omega)$, or there exists a subsequence $\{u_n\}$ such that

$$\int_{Q(n)} b(x)|u_n|^p = o(1), \tag{3.10}$$

where $Q(n) = \Omega \cap B^N(0;n)$. Clearly,

$$\int_\Omega b(x)|u_n|^p = \int_\Omega b(x)\xi_n^r |u_n|^p + o(1) = \int_\Omega b^\infty \xi_n^r |u_n|^p + o(1). \tag{3.11}$$

By (3.6), (3.7), (3.9), and (3.11),

$$\begin{gathered} \langle u_n, v_n\rangle_{H^1} = \|u_n\|_{H^1}^2 + o(1) = \|v_n\|_{H^1}^2 + o(1), \\ \int_\Omega b(x)|u_n|^p = \int_\Omega b(x)|v_n|^p + o(1) = \int_\Omega b^\infty |v_n|^p + o(1). \end{gathered} \tag{3.12}$$

Therefore, $\|u_n - v_n\|_{H^1} = o(1)$ as $n \to \infty$. The results of (iii) and (iv), from (i), (ii) and Lemmas 2.4, 2.5. □

We need the following compactness results.

Proposition 3.2. *Suppose that the domain Ω satisfies the conditions $(\Omega 1')$-$(\Omega 2')$. If $\{u_n\}$ is a $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J_0^b with*

$$\alpha_0^b(\Omega) \leq \beta < \min\{\alpha_0^\infty(\Omega) + \alpha_0^b(\Omega), \alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\}, \tag{3.13}$$

then there exist a subsequence $\{u_n\}$ and $u_0 \neq 0$ such that $u_n \to u_0$ strongly in $H_0^1(\Omega)$ and $J_0^b(u_0) = \beta$.

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_0^b with

$$\alpha_0^b(\Omega) \le \beta < \min\{\alpha_0^\infty(\Omega) + \alpha_0^b(\Omega), \alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \dots, \alpha_0^\infty(\Omega_m)\}. \tag{3.14}$$

Since $\{u_n\}$ is bounded, there exist a subsequence $\{u_n\}$ and u_0 in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ and $u_n \to u_0$ a.e in Ω. Moreover, u_0 is a solution of (1.5) in Ω. If $u_0 \equiv 0$, by Lemma 3.1 there exists a subsequence $\{u_n\}$ such that $\{\xi_n u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_0^∞, where ξ_n is as in (3.2). Let $v_n = \xi_n u_n$, and we obtain

$$J_0^\infty(v_n) = \beta + o(1), \quad (J_0^\infty)'(v_n) = o(1) \quad \text{in } H^{-1}(\Omega). \tag{3.15}$$

Since $\Omega_i \cap \Omega_j$ is bounded for $i \neq j$ and Ω_l is also bounded for $m+1 \le l \le k$, there exists $n_0 \in \mathbb{N}$ such that $v_n = 0$ in $\overline{\Omega(n_0)}$ for $n > 2n_0$ and $\Omega_l \subset \Omega(n_0)$ for all $l \in \{m+1, m+2, \dots, k\}$, where $\Omega(n) = \Omega \cap B^N(0;n)$. Moreover, $v_n = v_n^1 + v_n^2 + \cdots + v_n^m$ and for $i = 1, 2, \dots, m$,

$$v_n^i(z) = \begin{cases} v_n(z) & \text{for } z \in \Omega_i, \\ 0, & \text{for } z \notin \Omega_i. \end{cases} \tag{3.16}$$

Then $v_n^i \in H_0^1(\Omega_i)$ and

$$\int_{\Omega_i} \left(|\nabla v_n^i|^2 + (v_n^i)^2 \right) = \int_{\Omega_i} b^\infty |v_n^i|^p + o(1). \tag{3.17}$$

By (3.15), we obtain

$$\begin{gathered} (J_0^\infty)'(v_n^i) = o(1) \quad \text{strongly in } H^{-1}(\Omega_i) \text{ for } i = 1, 2, \dots, m, \\ \beta = J_0^\infty(v_n) + o(1) = \sum_{i=1}^m J_0^\infty(v_n^i) + o(1). \end{gathered} \tag{3.18}$$

Assume that

$$J_0^\infty(v_n^i) = c_i + o(1) \quad \text{for } i = 1, 2, \dots, m, \tag{3.19}$$

then $c_1 + c_2 + \cdots + c_m = \beta$, since all of c_i are (PS)-values in $H_0^1(\Omega_i)$ for J_0^∞ and nonnegative. Thus, there exists $i_0 \in \{1, 2, \dots, m\}$ such that c_{i_0} are positive (PS)-values in $H_0^1(\Omega_i)$ for J_0^∞ and

$$\alpha_0^\infty(\Omega_{i_0}) \le c_{i_0} \le \beta, \tag{3.20}$$

which contradicts (3.14). Consequently, $u_0 \not\equiv 0$ and $\beta \ge J_0^b(u_0) \ge \alpha_0^b(\Omega)$. Let $p_n = u_n - u_0$. By Lemma 2.4, $\{p_n\}$ is a $(\mathrm{PS})_{(\beta - J_0^b(u_0))}$-sequence in $H_0^1(\Omega)$ for J_0^∞. Since $\beta < \alpha_0^\infty(\Omega) + \alpha_0^b(\Omega)$, $J_0^b(u_0) \ge \alpha_0^b(\Omega)$ and $\alpha_0^b(\Omega)$ is a smallest positive (PS)-value in $H_0^1(\Omega)$ for J_0^b. Thus, $\beta - J_0^b(u_0) = 0$. This implies that $u_n \to u_0$ strongly in $H_0^1(\Omega)$ and $J_0^b(u_0) = \beta$. $\square$

Proposition 3.3. *Suppose that the unbounded domain Ω satisfies the condition $(\Omega 3')$. If $\{u_n\}$ is a (PS)$_\beta$-sequence in $H_0^1(\Omega)$ for J_0^b with*

$$\alpha_0^b(\Omega) \le \beta < \min\left\{\alpha_0^\infty(\Omega) + \alpha_0^b(\Omega),\ \lim_{r\to\infty} \alpha_0^\infty(\Omega^c(r))\right\}, \tag{3.21}$$

then there exist a subsequence $\{u_n\}$ and $u_0 \neq 0$ such that $u_n \to u_0$ strongly in $H_0^1(\Omega)$ and $J_0^b(u_0) = \beta$.

Proof. Let $\{u_n\}$ be a (PS)$_\beta$-sequence in $H_0^1(\Omega)$ for J_0^b with

$$\alpha_0^b(\Omega) \le \beta < \min\left\{\alpha_0^\infty(\Omega) + \alpha_0^b(\Omega),\ \lim_{r\to\infty} \alpha_0^\infty(\Omega^c(r))\right\}. \tag{3.22}$$

Since $\{u_n\}$ is bounded, there exist a subsequence $\{u_n\}$ and u_0 in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ and $u_n \to u_0$ a.e in Ω. Moreover, u_0 is a solution of (1.5) in Ω. If $u_0 \equiv 0$, by Lemma 3.1 there exists a subsequence $\{u_n\}$ such that $\{\xi_n u_n\}$ is a (PS)$_\beta$-sequence in $H_0^1(\Omega)$ for J_0^∞, where ξ_n is as in (3.2). Let $v_n = \xi_n u_n$, we obtain $v_n \in H_0^1(\Omega^c(n))$ for each n,

$$J_0^\infty(v_n) = \beta + o(1), \quad (J_0^\infty)'(v_n) = o(1) \quad \text{in } H^{-1}(\Omega). \tag{3.23}$$

Moreover, there is an $s_n > 0$ such that $s_n v_n \in \mathbf{M}^\infty(\Omega^c(n))$ and $s_n = 1 + o(1)$. Then

$$J_0^\infty(s_n v_n) \ge \alpha_0^\infty(\Omega^c(n)). \tag{3.24}$$

By (3.23), (3.24), we obtain

$$\beta \ge \lim_{n\to\infty} \alpha_0^\infty(\Omega^c(n)), \tag{3.25}$$

which contradicts (3.22). Consequently, $u_0 \not\equiv 0$ and $\beta \ge J_0^b(u_0) \ge \alpha_0^b(\Omega)$. Let $p_n = u_n - u_0$. By Lemma 2.4, $\{p_n\}$ is a (PS)$_{(\beta - J_0^b(u_0))}$-sequence in $H_0^1(\Omega)$ for J_0^∞. Since $\beta < \alpha_0^\infty(\Omega) + \alpha_0^b(\Omega)$, $J_0^b(u_0) \ge \alpha_0^b(\Omega)$ and $\alpha_0^b(\Omega)$ is smallest positive (PS)-value in $H_0^1(\Omega)$ for J_0^b. Thus, $\beta - J_0^b(u_0) = 0$. This implies that $u_n \to u_0$ strongly in $H_0^1(\Omega)$ and $J_0^b(u_0) = \beta$. □

Now, we begin to show the proof of Theorem 1.1: since the domain Ω satisfies the conditions $(\Omega 1')$-$(\Omega 2')$, we have (1.11), and there exists a ground-state solution u_0 such that $J_0^\infty(u_0) = \alpha_0^\infty(\Omega)$. Let $s_0 > 0$ with $s_0 u_0 \in \mathbf{M}_0^b(\Omega)$. Then

$$s_0^2 \int_\Omega \left(|\nabla u_0|^2 + u_0^2\right) = s_0^p \int_\Omega b(x)|u_0|^p. \tag{3.26}$$

Since $b(x) \ge b^\infty(\alpha_0^\infty(\Omega)/\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\})^{(p-2)/2}$ and $b(x) \to b^\infty$ as $|x| \to \infty$, we apply (3.26) to obtain

$$s_0 < \left(\frac{\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \ldots, \alpha_0^\infty(\Omega_m)\}}{\alpha_0^\infty(\Omega)}\right)^{1/2}. \tag{3.27}$$

Thus,

$$\begin{aligned}\alpha_0^b(\Omega) \le J_0^b(s_0u_0) &= \left(\frac{1}{2}-\frac{1}{p}\right)s_0^2\int_\Omega\left(|\nabla u_0|^2+u_0^2\right)\\ &< \frac{\min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\dots,\alpha_0^\infty(\Omega_m)\}}{\alpha_0^\infty(\Omega)}\left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega\left(|\nabla u_0|^2+u_0^2\right)\\ &= \min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\dots,\alpha_0^\infty(\Omega_m)\}.\end{aligned} \tag{3.28}$$

By Proposition 3.2, (1.5) has a ground-state solution.

Now, we begin to show the proof of Theorem 1.3: since the domain Ω satisfies the condition $(\Omega 3')$, we have (1.11) in Ω, and there exists a ground-state solution u_0 such that $J_0^\infty(u_0)=\alpha_0^\infty(\Omega)$. Let $s_0>0$ with $s_0u_0\in\mathbf{M}_0^b(\Omega)$. Then

$$s_0^2\int_\Omega\left(|\nabla u_0|^2+u_0^2\right)=s_0^p\int_\Omega b(x)|u_0|^p. \tag{3.29}$$

Since $b(x)\ge b^\infty(\alpha_0^\infty(\Omega)/\lim_{r\to\infty}\alpha_0^\infty(\Omega^c(r)))^{(p-2)/2}$ and $b(x)\to b^\infty$ as $|x|\to\infty$, we apply (3.29) to obtain

$$s_0<\left(\frac{\lim_{r\to\infty}\alpha_0^\infty(\Omega^c(r))}{\alpha_0^\infty(\Omega)}\right)^{1/2}. \tag{3.30}$$

Thus,

$$\begin{aligned}\alpha_0^b(\Omega) \le J_0^b(s_0u_0) &= \left(\frac{1}{2}-\frac{1}{p}\right)s_0^2\int_\Omega\left(|\nabla u_0|^2+u_0^2\right)\\ &< \frac{\lim_{r\to\infty}\alpha_0^\infty(\Omega^c(r))}{\alpha_0^\infty(\Omega)}\left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega\left(|\nabla u_0|^2+u_0^2\right)\\ &= \lim_{r\to\infty}\alpha_0^\infty(\Omega^c(r)).\end{aligned} \tag{3.31}$$

By Proposition 3.3, (1.5) has a ground-state solution.

4. Nonhomogeneous problems

4.1. Existence of a local minimum. First, we establish the existence of a local minimum. Similar as the proof of Lemma 1.4 by Adachi and Tanaka in [21], we have the following lemma.

Lemma 4.1. *If* $\|h\|_{H^{-1}} < (p-2)(1/(p-1))^{(p-1)/(p-2)}[b_{\sup} S^p(\Omega)]^{1/(2-p)}$, *then*
(i) $\mathbf{M}_h^{b+}(\Omega) \subset B(0;r_0)$;
(ii) $J_h^b(u)$ *is strictly convex in* $B(0;r_0)$,
where $B(0;r_0) = \{u \in H^1(\Omega) \mid \|u\|_{H^1} < r_0\}$ *and* $r_0 = [(p-1)b_{\sup} S^p(\Omega)]^{1/(2-p)}$.

Furthermore, we have the following theorem.

Theorem 4.2. *If* r_0 *is as in Lemma 4.1, then the functional* J_h^b *has a unique critical point* $u_{\min}$ *in* $B(0;r_0)$ *and it satisfies*
(i) $u_{\min} \in \mathbf{M}_h^{b+}(\Omega)$ *and* $J_h^b(u_{\min}) = \alpha_h^{b+}(\Omega) = \alpha_h^b(\Omega)$;
(ii) $u_{\min}$ *is a positive solution of (1.1).*

Proof. Similar as the proof of Theorem 2.1 by Cao and Zhu in [18], there is a $u_{\min} \in \mathbf{M}_h^{b+}(\Omega)$ which is a critical point for J_h^b such that $J_h^b(u_{\min}) = \alpha_h^{b+} = \alpha_h^b$, since $\mathbf{M}_h^{b+}(\Omega) \subset B(0;r_0)$ and $J_h^b(u)$ is strictly convex in $B(0;r_0)$, so that $u_{\min}$ is a unique critical point of J_h^b in $B(0;r_0)$. Since $u_{\min}$ is a unique critical point of J_h^b in $B(0;r_0)$, we have that $u_{\min}$ is a nonnegative solution of (1.1). By the maximum principle, $u_{\min}$ is positive. □

4.2. Multiple positive solutions. Throughout this section, we let $u_{\min}$ be the local minimum for J_h^b in $H_0^1(\Omega)$ in Theorem 4.2 and

$$\|h\|_{H^{-1}} < (p-2)\left(\frac{1}{p-1}\right)^{(p-1)/(p-2)}\left[b_{\sup} S^p(\Omega)\right]^{1/(2-p)}. \tag{4.1}$$

Then we have the following restricted (PS) conditions.

Proposition 4.3. *Suppose that the domain* Ω *satisfies the conditions* $(\Omega1')$-$(\Omega2')$. *If* $\{u_n\}$ *is a* $(PS)_\beta$*-sequence in* $H_0^1(\Omega)$ *for* J_h^b *with*

$$\beta < \alpha_h^b(\Omega) + \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \dots, \alpha_0^\infty(\Omega_m)\}, \tag{4.2}$$

then there exist a subsequence $\{u_n\}$ *and* u *in* $H_0^1(\Omega)$ *such that* $u_n \to u$ *strongly in* $H_0^1(\Omega)$ *and* $J_h^b(u) = \beta$.

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_h^b. By Lemma 2.8(ii), $\{u_n\}$ is bounded. Then there exist a subsequence $\{u_n\}$ and a nonzero solution u of (1.1) such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Suppose that $u_n \not\to u$ strongly in $H_0^1(\Omega)$. Let $w_n = u_n - u$ for $n = 1, 2, \dots$. Then, by Lemma 2.4, $\{w_n\}$ is a $(\mathrm{PS})_{\beta - J_h^b(u)}$-sequence in $H_0^1(\Omega)$ for J_0^∞, since $w_n \rightharpoonup 0$ and $w_n \not\to 0$ strongly in $H_0^1(\Omega)$. Similar as the proof of Proposition 3.2,

$$\beta - J_h^b(u) \geq \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \dots, \alpha_0^\infty(\Omega_m)\}, \tag{4.3}$$

which is a contradiction. Thus $u_n \to u$ strongly in $H_0^1(\Omega)$. □

Proposition 4.4. *Suppose that the domain* Ω *satisfies the condition* $(\Omega3')$. *If* $\{u_n\}$ *is a* $(PS)_\beta$*-sequence in* $H_0^1(\Omega)$ *for* J_h^b *with*

$$\beta < \alpha_h(\Omega) + \lim_{r\to\infty} \alpha_0^\infty(\Omega^c(r)), \tag{4.4}$$

then there exist a subsequence $\{u_n\}$ *and* u *in* $H_0^1(\Omega)$ *such that* $u_n \to u$ *strongly in* $H_0^1(\Omega)$ *and* $J_h^b(u) = \beta$.

The proof is similar to the proof of Proposition 4.3.

LEMMA 4.5. *Suppose that the domain* Ω *satisfies the conditions* $(\Omega 1')$-$(\Omega 2')$ *and the coefficient* $b(x)$ *satisfies the condition* $(b1)$. *Let* $\overline{u}$ *be a positive solution of (1.11) in* Ω *such that* $J_0^\infty(\overline{u}) = \alpha_0^\infty(\Omega)$ *and let* $u_{\min}$ *be a local minimum in Theorem 4.2. Then*

$$\sup_{t\geq 0} J_h^b(u_{\min} + t\overline{u}) < J_h^b(u_{\min}) + \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \dots, \alpha_0^\infty(\Omega_m)\}. \tag{4.5}$$

Proof. Since $u_{\min}$ is a positive solution of (1.1). Let $f(s) = s^{p-1}$ for $s \geq 0$ and $F_b(u) = \int_\Omega b(x)\int_0^u f(s)ds\,dx = (1/p)\int_\Omega b(x)u^p$, then

$$\begin{aligned} J_h^b(u_{\min} + t\overline{u}) &= J_h^b(u_{\min}) + J_0^b(t\overline{u}) + t\left(\int_\Omega b(x)u_0^{p-1}\overline{u} + h(x)\overline{u}\right) - \int_\Omega h(x)t\overline{u} \\ &\quad + \frac{1}{p}\left[\int_\Omega b(x)u_0^p + \int_\Omega b(x)|t\overline{u}|^p - \int_\Omega b(x)\,|\,u_0 + t\overline{u}\,|^{\,p}\right] \\ &= J_h^b(u_{\min}) + J_0^b(t\overline{u}) - \int_\Omega b(x)\left\{\int_0^{t\overline{u}} [f(u_0+s) - f(s) - f(u_0)]ds\right\}. \end{aligned} \tag{4.6}$$

For $v > 0$ and $w > 0$, we have

$$\begin{aligned} f(v+w) &= (v+w)^{p-1} \\ &= (v+w)^{p-2}v + (v+w)^{p-2}w \\ &> v^{p-1} + w^{p-1} = f(v) + f(w). \end{aligned} \tag{4.7}$$

Thus, $J_h^b(u_{\min} + t\overline{u}) \leq J_h^b(u_{\min}) + J_0^b(t\overline{u})$. Since $J_0^b(t\overline{u}) \to -\infty$ as $t \to \infty$, there is a $t_0 > 0$ such that $J_h^b(u_{\min} + t\overline{u}) < J_h^b(u_0)$ for $t \geq t_0$. Hence,

$$\sup_{t\geq 0} J_h^b(u_{\min} + t\overline{u}) = \sup_{0\leq t\leq t_0} J_h^b(u_{\min} + t\overline{u}). \tag{4.8}$$

Let $g_1(t) = J_h^b(u_{\min} + t\overline{u})$ for $t \geq 0$. By the continuity of $g_1(t)$, given

$$\varepsilon = \frac{1}{2}\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \dots, \alpha_0^\infty(\Omega_m)\} > 0, \tag{4.9}$$

there exists $t_1 \in (0, t_0)$ such that

$$g_1(t) < g_1(0) + \frac{1}{2}\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \dots, \alpha_0^\infty(\Omega_m)\} \quad \text{for } t \in [0, t_1). \tag{4.10}$$

Then

$$\begin{aligned} \sup_{0\leq t\leq t_1} J_h^b(u_{\min} + t\overline{u}) &\leq J_h^b(u_{\min}) + \frac{1}{2}\min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \dots, \alpha_0^\infty(\Omega_m)\} \\ &< J_h^b(u_{\min}) + \min\{\alpha_0^\infty(\Omega_1), \alpha_0^\infty(\Omega_2), \dots, \alpha_0^\infty(\Omega_m)\}. \end{aligned} \tag{4.11}$$

Now, we only need to show that

$$\sup_{t_1\le t\le t_0} J_h^b(u_{\min}+t\overline{u}) < J_h^b(u_{\min}) + \min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\ldots,\alpha_0^\infty(\Omega_m)\}. \tag{4.12}$$

Let $g_2(t)=J_0^b(t\overline{u})$ for $t\ge 0$. Then

$$\begin{aligned} g_2'(t) &= t\int_\Omega (|\nabla\overline{u}|^2+\overline{u}^2) - t^{p-1}\int_\Omega b(x)\overline{u}^p,\\ g_2''(t) &= \int_\Omega (|\nabla\overline{u}|^2+\overline{u}^2) - (p-1)t^{p-2}\int_\Omega b(x)\overline{u}^p. \end{aligned} \tag{4.13}$$

There is a unique $\overline{t}=[\int_\Omega(|\nabla\overline{u}|^2+\overline{u}^2)/\int_\Omega b(x)\overline{u}^p]^{1/(p-2)}$ such that $g_2'(\overline{t})=0$ and $g_2''(\overline{t})<0$. Thus, $g_2(t)$ has an absolutely maximum at $\overline{t}$. Since

$$b(x)\ge b^\infty\left(\frac{\alpha_0^\infty(\Omega)}{\min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\ldots,\alpha_0^\infty(\Omega_m)\}}\right)^{(p-2)/2}, \tag{4.14}$$

we have

$$\overline{t}\le\left(\frac{\min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\ldots,\alpha_0^\infty(\Omega_m)\}}{\alpha_0^\infty(\Omega)}\right)^{1/2}. \tag{4.15}$$

Therefore,

$$\begin{aligned} \sup_{t\ge 0} J_0^b(t\overline{u}) = J_0^b(\overline{t}\overline{u}) &= \left(\frac{1}{2}-\frac{1}{p}\right)\overline{t}^2\int_\Omega(|\nabla\overline{u}|^2+\overline{u}^2)\\ &\le \min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\ldots,\alpha_0^\infty(\Omega_m)\}. \end{aligned} \tag{4.16}$$

By (4.6), (4.7), and (4.16), we obtain

$$\begin{aligned} &\sup_{t_1\le t\le t_0} J_h^b(u_{\min}+t\overline{u})\\ &\quad\le J_h^b(u_{\min})+\min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\ldots,\alpha_0^\infty(\Omega_m)\}\\ &\qquad-\inf_{t_1\le t\le t_0}\int_\Omega b(x)\left\{\int_0^{t\overline{u}}[f(u_{\min}+s)-f(s)-f(u_{\min})]\,ds\right\}\\ &\quad< J_h^b(u_{\min})+\min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\ldots,\alpha_0^\infty(\Omega_m)\}. \end{aligned} \tag{4.17}$$

Thus, $\sup_{t\ge 0} J_h^b(u_{\min}+t\overline{u}) < J_h^b(u_{\min})+\min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\ldots,\alpha_0^\infty(\Omega_m)\}$. □

LEMMA 4.6. *Suppose that the domain Ω satisfies the condition $(\Omega 3')$ and the coefficient $b(x)$ satisfies the condition $(b2)$. Let $\overline{u}$ be a positive solution of (1.11) in Ω such that $J_0^\infty(\overline{u})=\alpha_0^\infty(\Omega)$ and let $u_{\min}$ be the local minimum in Theorem 4.2. Then*

$$\sup_{t\ge 0} J_h^b(u_{\min}+t\overline{u}) < J_h^b(u_{\min})+\lim_{r\to\infty}\alpha_0^\infty(\Omega^c(r)). \tag{4.18}$$

The proof is similar to the proof of Lemma 4.5.

Now, we begin to show the proof of Theorem 1.6: for $u \in H_0^1(\Omega)$ with $\|u\|_{H^1} = 1$, by Lemma 2.7 there is a unique $t^-(u) > 0$ such that $t^-(u)$, $u \in \mathbf{M}_h^{b-}(\Omega)$ and

$$J_h^b(t^-(u)u) = \max_{t \geq t_{\max}} J_h^b(tu). \tag{4.19}$$

By Lemma 2.7(ii) and (iii), we have that $t^-(u)$ is a continuous function for nonzero u and

$$\mathbf{M}_h^{b-}(\Omega) = \left\{u \in H_0^1(\Omega)\backslash\{0\} \mid \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) = 1\right\}. \tag{4.20}$$

Let

$$\begin{aligned} A_1 &= \left\{u \in H_0^1(\Omega)\backslash\{0\} \mid \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) > 1\right\} \cup \{0\}, \\ A_2 &= \left\{u \in H_0^1(\Omega)\backslash\{0\} \mid \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) < 1\right\}. \end{aligned} \tag{4.21}$$

Then $\mathbf{M}_h^{b-}(\Omega)$ disconnects $H_0^1(\Omega)$ in two connected components A_1 and A_2 and $H_0^1(\Omega)\backslash \mathbf{M}_h^{b-}(\Omega) = A_1 \cup A_2$. For each $u \in \mathbf{M}_h^{b+}(\Omega)$, we have

$$1 < t_{\max}(u) < t^-(u). \tag{4.22}$$

Since $t^-(u) = (1/\|u\|_{H^1}) t^-(u/\|u\|_{H^1})$, then $\mathbf{M}_h^{b+}(\Omega) \subset A_1$. In particular, $u_{\min} \in A_1$. We claim that there exists $t_0 > 0$ such that $u_{\min} + t_0\overline{u} \in A_2$. First, we find a constant $c > 0$ such that $0 < t^-((u_{\min} + t\overline{u})/\|u_{\min} + t\overline{u}\|_{H^1}) < c$ for each $t \geq 0$. Otherwise, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and $t^-((u_{\min} + t_n\overline{u})/\|u_{\min} + t_n\overline{u}\|_{H^1}) \to \infty$ as $n \to \infty$. Let $v_n = (u_{\min} + t_n\overline{u})/\|u_{\min} + t_n\overline{u}\|_{H^1}$. Since $t^-(v_n)$, $v_n \in \mathbf{M}_h^{b-}(\Omega) \subset \mathbf{M}_h^b(\Omega)$, and by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \int_\Omega b(x) v_n^p &= \frac{1}{\|u_{\min} + t_n\overline{u}\|_{H^1}^p} \int_\Omega b(x)(u_{\min} + t_n\overline{u})^p \\ &= \frac{1}{\|u_{\min}/t_n + \overline{u}\|_{H^1}^p} \int_\Omega b(x)\left(\frac{u_{\min}}{t_n} + \overline{u}\right)^p \longrightarrow \frac{\int_\Omega b(x)\overline{u}^p}{\|\overline{u}\|_{H^1}^p} \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{4.23}$$

We have

$$\begin{aligned} J_h^b(t^-(v_n)v_n) &= \frac{1}{2}[t^-(v_n)]^2 - \frac{1}{p}[t^-(v_n)]^p \int_\Omega b(x) v_n^p \\ &\quad - t^-(v_n)\int_\Omega h v_n \longrightarrow -\infty \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{4.24}$$

But J_h^b is bounded below on $\mathbf{M}_h^b(\Omega)$, a contradiction. Let

$$t_0 = \frac{\left| c^2 - \|u_{\min}\|_{H^1}^2 \right|^{1/2}}{\|\overline{u}\|_{H^1}} + 1. \tag{4.25}$$

Then

$$\begin{aligned}||u_{\min}+t_0\overline{u}||_{H^1}^2 &= ||u_{\min}||_{H^1}^2+t_0^2||\overline{u}||_{H^1}^2+2t_0\langle u_{\min},\overline{u}\rangle \\ &> ||u_{\min}||_{H^1}^2+|c^2-||u_{\min}||_{H^1}^2|+2\int_\Omega b^\infty\overline{u}^{p-1}u_{\min} \\ &> c^2 > \left[t^-\left(\frac{u_{\min}+t_0\overline{u}}{||u_{\min}+t_0\overline{u}||_{H^1}}\right)\right]^2,\end{aligned} \tag{4.26}$$

that is, $u_{\min}+t_0\overline{u}\in A_2$. Define a path $\gamma(s)=u_{\min}+st_0\overline{u}$ for $s\in[0,1]$, then

$$\gamma(0)=u_{\min}\in A_1, \qquad \gamma(1)=u_{\min}+t_0\overline{u}\in A_2, \tag{4.27}$$

and there exists $s_0\in(0,1)$ such that $u_{\min}+s_0t_0\overline{u}\in\mathbf{M}_h^{b-}(\Omega)$. Thus, by Lemma 4.5,

$$\begin{aligned}\alpha_h^-(\Omega) &\le J_h^b(u_{\min}+s_0t_0\overline{u}) \le \max_{s\in[0,1]}J_h^b(\gamma(s)) \\ &< J_h^b(u_{\min})+\min\{\alpha_0^\infty(\Omega_1),\alpha_0^\infty(\Omega_2),\dots,\alpha_0^\infty(\Omega_m)\}.\end{aligned} \tag{4.28}$$

By the Ekeland variational principle [25], there exists a sequence $\{u_n\}$ in $\mathbf{M}_h^{b-}(\Omega)$ such that

$$\begin{aligned}J_h^b(u_n) &= \alpha_h^{b-}(\Omega)+o(1), \\ (J_h^b)'(u_n) &= o(1) \quad \text{strongly in } H^{-1}(\Omega).\end{aligned} \tag{4.29}$$

Then by Proposition 4.3, there exist a subsequence $\{u_n\}$ and $u^0\in\mathbf{M}_h^b(\Omega)$ such that $u_n\to u^0$ strongly in $H_0^1(\Omega)$, u^0 is a solution of (1.1), and $J_h^b(u^0)=\alpha_h^{b-}(\Omega)$. By the Sobolev imbedding theorem, we have $u_n\to u^0$ strongly in $L^p(\Omega)$. Thus,

$$||u^0||_{H^1}^2-(p-1)\int_\Omega b(x)|u^0|^p\le 0. \tag{4.30}$$

Then $u^0\in\mathbf{M}_h^{b-}(\Omega)$ and

$$J_h^b(u^0)=\alpha_h^{b-}(\Omega). \tag{4.31}$$

This implies that $u_{\min}$ and u^0 are distinct. Finally, since $h\ge 0$, by Lemma 2.7 there exists $t^-(|u^0|)>0$ such that

$$\begin{aligned}t^-(|u^0|)|u^0|\in\mathbf{M}_h^{b-}(\Omega), &\qquad t^-(|u^0|)>t_{\max}(|u^0|)=t_{\max}(u^0), \\ \alpha_h^{b-}(\Omega) &\le J_h^b(t^-(|u^0|)|u^0|)\le J_h^b(t^-(|u^0|)u^0) \\ &\le \max_{t\ge t_{\max}(u^0)}J_h^b(tu^0)=J_h^b(u^0)=\alpha_h^{b-}(\Omega).\end{aligned} \tag{4.32}$$

Thus,

$$J_h^b(t^-(|u^0|)|u^0|)=J_h^b(t^-(|u^0|)u^0)=\alpha_h^{b-}(\Omega). \tag{4.33}$$

We concluded that $\int_\Omega hu^0 = \int_\Omega h|u^0|$. Let

$$u_+^0 = \max\{u^0, 0\}, \qquad u_-^0 = \max\{-u^0, 0\}, \tag{4.34}$$

then $\int_\Omega hu_-^0 = 0$. Since $h \geq 0$ and $u_-^0 \geq 0$, we have $u_-^0 = 0$. Hence, u^0 is nonnegative. By the maximum principle, u^0 is positive. We complete the proof of Theorem 1.6.

Remark 4.7. The proof of Theorem 1.7 similar to Theorem 1.6.

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