A NOTE ON PROPERTIES THAT IMPLY THE FIXED POINT PROPERTY

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We give relationships between some Banach-space geometric properties that guarantee the weak fixed point property. The results extend some known results of Dalby and Xu.

1. Introduction

A Banach space X is said to satisfy the weak fixed point property (fpp) if every nonempty weakly compact convex subset C, and every nonexpansive mapping $T : C \to C$ (i.e., $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$) has a fixed point, that is, there exists $x \in C$ such that $T(x) = x$. Many properties have been shown to imply fpp. The most recent one is the uniform nonsquareness which is proved by Mazcuñán [20] solving a long stand open problem. Other well known properties include Opial property (Opial [21]), weak normal structure (Kirk [17]), property (M) (García-Falset and Sims [12]), $R(X) < 2$ (García-Falset [10]), and UCED (Garkavi [13]). Connection between these properties were investigated in Dalby [3] and Xu et al. [27]. We aim to continue the study in this direction. In contrast to [3], we do not assume that all Banach spaces are separable.

2. Preliminaries

Let X be a Banach space. For a sequence (x_n) in X, $x_n \xrightarrow{w} x$ denotes the weak convergence of (x_n) to $x \in X$. When $x_n \xrightarrow{w} 0$, we say that (x_n) is a weakly null sequence. $B(X)$ and $S(X)$ stand for the unit ball and the unit sphere of X, respectively. It becomes a common ingredient that when working with a weak null sequence (x_n), we consider the type function $\limsup_{n\to\infty} \|x_n - x\|$ for all $x \in X$. As for a starting point, we recall Opial property.

Opial property [21] states that

$$\text{if } x_n \xrightarrow{w} 0, \quad \text{then } \limsup_{n\to\infty} \|x_n\| < \limsup_{n\to\infty} \|x_n - x\| \ \forall x \in X,\ x \neq 0. \tag{2.1}$$

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If the strict inequality becomes $\leq$, this condition becomes a nonstrict Opial property. On the other hand, if for every $\epsilon > 0$, for each $x_n \overset{w}{\to} 0$ with $\|x_n\| \to 1$, there is an $r > 0$ such that

$$1 + r \leq \limsup_{n\to\infty} \|x_n + x\| \tag{2.2}$$

for each $x \in X$ with $\|x\| \geq \epsilon$, then we have the locally uniformly Opial property (see [27]).

The coefficient $R(X)$, introduced in García-Falset [9], is defined as

$$R(X) := \sup\Big\{\liminf_{n\to\infty} \|x_n - x\| : x_n \overset{w}{\longrightarrow} 0,\ \|x_n\| \leq 1\ \forall n,\ \|x\| \leq 1\Big\}. \tag{2.3}$$

So $1 \leq R(X) \leq 2$ and it is not hard to see that in the definition of $R(X)$, "lim inf" can be replaced by "lim sup." Some values of $R(X)$ are $R(c_0) = 1$ and $R(l_p) = 2^{1/p}, 1 < p < \infty$.

A Banach space X has property (M) if whenever $x_n \overset{w}{\to} 0$, then $\limsup_{n\to\infty} \|x_n - x\|$ is a function of $\|x\|$ only. Property (M) which is introduced by Kalton [15] is equivalent to:

$$\text{if } x_n \overset{w}{\longrightarrow} 0, \quad \|u\| \leq \|v\|, \text{ then } \limsup_{n\to\infty} \|x_n + u\| \leq \limsup_{n\to\infty} \|x_n + v\|. \tag{2.4}$$

Sims [23] introduced a property called weak orthogonality (WORTH) for Banach spaces. A Banach space X is said to have property WORTH if,

$$\text{for every } x_n \overset{w}{\longrightarrow} 0,\ x \in X, \quad \limsup_{n\to\infty} \|x_n + x\| = \limsup_{n\to\infty} \|x_n - x\|. \tag{2.5}$$

It remains unknown if property WORTH implies fpp. In many situations, the fixed point property can be easily obtained when we assume, in addition, that the spaces being considered have the property WORTH. For examples, WORTH and ε_0-inquadrate for some $\epsilon_0 < 2$ ([24]), WORTH and 2-UNC ([11]) imply fpp.

The following results will be used in Section 3.

Proposition 2.1 [12, Proposition 2.1]. *For the following conditions on a Banach space X, we have (i)⇒(ii)⇒(iii)⇒(iv).*

(i) *X has property (M).*

(ii) *X has property WORTH.*

(iii) *If $x_n \overset{w}{\to} 0$, then for each $x \in X$ we have* $\limsup_{n\to\infty} \|x_n - tx\|$ *is an increasing function of t on* $[0, \infty)$.

(iv) *X satisfies the nonstrict Opial property.*

Property (M) implies the nonstrict Opial property but not weak normal structure. c_0 has property (M) but does not have weak normal structure. In [3, 25] it had been shown that $R(X) = 1$ implies X has property (M).

A generalization of uniform convexity of Banach spaces which is due to Sullivan [26] is now recalled. Let $k \geq 1$ be an integer. Then a Banach space X is said to be k-UR (k-uniformly rotund) if given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $\{x_1, \ldots, x_{k+1}\} \subset B(X)$

satisfying $V(x_1,\dots,x_{k+1}) \geq \varepsilon$, then

$$\left\| \frac{\sum_{i=1}^{k+1} x_i}{k+1} \right\| \leq \delta(\varepsilon). \tag{2.6}$$

Here, $V(x_1,\dots,x_{k+1})$ is the volume enclosed by the set $\{x_1,\dots,x_{k+1}\}$, that is,

$$V(x_1,\dots,x_{k+1}) = \sup \left\{ \left| \begin{matrix} 1 & \cdots & 1 \\ f_1(x_1) & \cdots & f_1(x_{k+1}) \\ \vdots & \ddots & \vdots \\ f_k(x_1) & \cdots & f_k(x_{k+1}) \end{matrix} \right| \right\}, \tag{2.7}$$

where the supremum is taken over all $f_1,\dots,f_k \in B(X^*)$.

Let K be a weakly compact convex subset of a Banach space X and (x_n) a bounded sequence in X. Define a function f on X by

$$f(x) = \limsup_{n\to\infty} ||x_n - x||, \quad x \in X. \tag{2.8}$$

Let

$$\begin{aligned} r \equiv r_K((x_n)) &:= \inf\{f(x) : x \in K\}, \\ A \equiv A_K((x_n)) &:= \{x \in K : f(x) = r\}. \end{aligned} \tag{2.9}$$

Recall that r and A are, respectively, called the asymptotic radius and center of (x_n) relative to K. As K is weakly compact convex, we see that A is nonempty, weakly compact and convex (see [14]). In [18], Kirk proved that the asymptotic center of a bounded sequence w.r.t a bounded closed convex subset of a k-uniformly convex spaces X is compact. This fact will be used in proving Theorem 3.8.

Being k-UR and Opial property are related in the following way.

Theorem 2.2 [19, Theorem 3.5]. *If X is k-UR and satisfies the Opial property, then X satisfies locally uniform Opial property.*

One last concept we need to mention is ultrapowers of Banach spaces. Ultrapowers of a Banach space are proved to be useful in many branches of mathematics. Many results can be seen more easily when treated in this setting. We recall some basic facts about the ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i\in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I, i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the fact that

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and

(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I,X_i)$ denote the subspace of the product space $\Pi_{i\in I} X_i$ equipped with the norm $\|(x_i)\| := \sup_{i\in I} \|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \left\{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\right\}. \tag{2.10}$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I,X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_i)_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from (ii) above and the definition of the quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|. \tag{2.11}$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an *ultrapower* of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see [1] or [22]).

3. Main results

Recall that a Banach space X is said to have Schur's property if

$$\text{for every sequence } (x_n), \quad x_n \xrightarrow{w} 0 \text{ implies } x_n \longrightarrow 0. \tag{3.1}$$

An element $x \in X$ is said to be an H-point if

$$x_n \xrightarrow{w} x, \quad \|x_n\| \longrightarrow \|x\| \text{ imply } x_n \longrightarrow x. \tag{3.2}$$

X has property (H) if every element of X is an H-point. These concepts are related, in conjunction with the condition $R(X) = 1$, as follow.

Theorem 3.1. *A Banach space X has Schur's property if and only if $R(X) = 1$ and X has at least one H-point.*

Proof. "⇒" It is well known that Schur's property implies property (H). From the definition of $R(X)$ and Schur's property, we have

$$\begin{aligned} R(X) &= \sup\left\{\liminf_{n\to\infty} \|x_n - x\| : x_n \xrightarrow{w} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\right\} \\ &= \sup\{\|x\| : \|x\| \le 1\} = 1. \end{aligned} \tag{3.3}$$

"⇐" Suppose that there exists a sequence (x_n) converges weakly to 0 but $\|x_n\| \nrightarrow 0$. By passing through a subsequence if necessary, we can assume that $\|x_n\| \to a \neq 0$. Put $y_n = x_n/a$. Clearly $y_n \xrightarrow{w} 0$ and $\|y_n\| \to 1$. Let x_0 be an H-point. If $x_0 = 0$, we are done. We assume now that $x_0 \neq 0$ and in fact we assume that $x_0 \in S(X)$. Thus, as $R(X) = 1$ and the weak lower semicontinuity of the norm,

$$(x_0 - y_n) \xrightarrow{w} x_0, \quad \liminf_{n\to\infty} \|x_0 - y_n\| = 1. \tag{3.4}$$

Choose a subsequence $(y_{n'})$ of (y_n) such that

$$\lim_{n'\to\infty} \|x_0 - y_{n'}\| = 1. \tag{3.5}$$

We see that $(x_0 - y_n') \to x_0$ and $y_n' \to 0$. Thus $\|y_n'\| \to 0$ and $0 = a$, a contradiction. □

A Banach space X has property m_p (resp., m_∞) (cf. [27]) if for all $x \in X$, whenever $x_n \xrightarrow{w} 0$,

$$\limsup_{n\to\infty} \|x + x_n\|^p = \|x\|^p + \limsup_{n\to\infty} \|x_n\|^p$$
$$\left(\text{resp., } \limsup_{n\to\infty} \|x + x_n\| = \max\left\{\|x\|, \limsup_{n\to\infty} \|x_n\|\right\}\right). \tag{3.6}$$

Clearly the above properties imply property (M) and property m_1 implies Opial property.

Property m_1 implies property (H). For, if $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$ for some sequence (x_n) and $x \in X$, we have, by m_1,

$$\|x\| = \limsup_{n\to\infty} \|x_n\| = \limsup_{n\to\infty} \|(x_n - x) + x\| = \|x\| + \limsup_{n\to\infty} \|x_n - x\|. \tag{3.7}$$

This implies that $\limsup_{n\to\infty} \|x_n - x\| = 0$ and thus $x_n \to x$.

It also turns out that property m_∞ and the condition $R(X) = 1$ coincide as the following result shows.

Theorem 3.2. *A Banach space X has property m_∞ if and only if $R(X) = 1$.*

Proof. "⇒" Suppose that X has property m_∞. Thus,

$$R(X) = \sup\left\{\limsup_{n\to\infty} \|x_n - x\| : x_n \xrightarrow{w} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\right\}$$
$$= \sup\left\{\max\left\{\|x\|, \limsup_{n\to\infty} \|x_n\|\right\} : x_n \xrightarrow{w} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\right\} = 1. \tag{3.8}$$

"⇐" To show that X has property m_∞. Given $x_n \xrightarrow{w} 0$ and $x \in X - \{0\}$. Put $a = \max\{\|x\|, \limsup_{n\to\infty} \|x_n\|\}$. Clearly, $\limsup_{n\to\infty}(\|x_n\|/a) \le 1$ and $\|x\|/a \in B(X)$. We note here that $R(X) = 1$ implies property (M) and it in turn implies the nonstrict Opial property. By the weak lower semicontinuity of $\|\cdot\|$ and the nonstrict Opial property, we see that $\|x\| \le \limsup_{n\to\infty} \|x_n - x\|$ and $\limsup_{n\to\infty} \|x_n\| \le \limsup_{n\to\infty} \|x_n - x\|$. Thus $a \le \limsup_{n\to\infty} \|x_n - x\|$. On the other hand, as $R(X) = 1$, we can show that $\limsup_{n\to\infty} \|x_n/a - x/a\| \le 1$. So we can conclude that,

$$\limsup_{n\to\infty} \left\|\frac{x_n}{a} - \frac{x}{a}\right\| = 1, \tag{3.9}$$

and thus $\limsup_{n\to\infty} \|x_n - x\| = a = \max\{\|x\|, \limsup_{n\to\infty} \|x_n\|\}$ and the proof is complete. □

For $p < \infty$, we have the following proposition.

Proposition 3.3. *If X has property $m_p (1 \le p < \infty)$, then $R(X) \le 2^{1/p}$. Moreover, if in addition X does not have Schur's property, then $R(X) = 2^{1/p}$.*

Proof. Define

$$R_p(X) := \sup\left\{\limsup_{n\to\infty} \|x_n - x\|^p : x_n \xrightarrow{w} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\right\}. \tag{3.10}$$

By property m_p, we have

$$R_p(X) = \sup\left\{\|x\|^p + \limsup_{n\to\infty} \|x_n\|^p : x_n \xrightarrow{w} 0,\ \|x_n\| \le 1\ \forall n, \|x\| \le 1\right\}. \tag{3.11}$$

Thus, $R_p(X) \le 2$ which implies $R(X) \le 2^{1/p}$. On the other hand, if, in addition, X does not have Schur's property, then there exists a weakly null sequence (x_n) such that $x_n \nrightarrow 0$. From this we can construct a weakly null sequence (y_n) in the unit sphere. We can now see that $R_p(X) \ge 2$ and hence $R(X) \ge 2^{1/p}$. Therefore $R(X) = 2^{1/p}$. □

Example 3.4. In $l_p\,(1 < p < \infty)$, we have $e_n \in S(X)$ and $e_n \xrightarrow{w} 0$, where (e_n) is the standard basis. Clearly

$$\|e_n - e_1\| \xrightarrow{n\to\infty} 2^{1/p}, \tag{3.12}$$

thus $R(l_p) = 2^{1/p}$. Note that l_p has property m_p (cf. [27]).

Some properties are equivalent in a space X with $R(X) = 1$.

Theorem 3.5. *Let X be a Banach space with $R(X) = 1$. The following conditions are equivalent:*

(i) *X has property m_1;*
(ii) *X satisfies Opial property;*
(iii) *X has Schur's property.*

Proof. (i)⇒(ii) and (iii)⇒(i) are clear. It needs to prove (ii) ⇒ (iii).

Let $x_n \xrightarrow{w} 0$. To show $x_n \to 0$, let $0 \ne x \in X$. By Opial property together with property m_∞, we have

$$\limsup_{n\to\infty} \|x_n\| < \limsup_{n\to\infty} \|x_n + x\| = \max\left\{\|x\|, \limsup_{n\to\infty} \|x_n\|\right\}. \tag{3.13}$$

Thus

$$\limsup_{n\to\infty} \|x_n\| < \|x\|, \tag{3.14}$$

for all $x \in X - \{0\}$. This means that $\limsup_{n\to\infty} \|x_n\| = 0$ and thus $\lim_{n\to\infty} \|x_n\| = 0$. Consequently, $x_n \to 0$, and therefore X has Shur's property. □

The Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ of X is defined by

$$\begin{aligned} C_{\mathrm{NJ}}(X) &= \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x,y\in X \text{ not both zero}\right\} \quad ([2]) \\ &= \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x\in S(X),\ y\in B(X)\right\} \quad ([16]). \end{aligned} \tag{3.15}$$

Another important constant which is closely related to $C_{\mathrm{NJ}}(X)$ is the James constant $J(X)$ defined by Gao and Lau [7] as:

$$\begin{aligned} J(X) &= \sup\{\|x+y\|\wedge\|x-y\| : x,y\in S(X)\} \\ &= \sup\{\|x+y\|\wedge\|x-y\| : x,y\in B(X)\}. \end{aligned} \tag{3.16}$$

In general we have

$$\frac{1}{2}J(X)^2 \le C_{\mathrm{NJ}}(X) \le \frac{J(X)^2}{(J(X)-1)^2+1} \quad ([16]). \tag{3.17}$$

With or without having WORTH, Mazcuñán [20] showed that $R(1,X) < 2$ whenever $C_{\mathrm{NJ}}(X) < 2$. In gerneral, $R(1,X) \le R(X)$. The constant $R(a,X)$ is introduced by Dominguez [6] as: for a given real number a

$$R(a,X) := \sup\left\{\liminf_{n\to\infty}||x+x_n||\right\}, \tag{3.18}$$

where the supremum is taken over all $x\in X$ with $\|x\|\le a$ and all weakly null sequences (x_n) in the unit ball of X such that

$$\limsup_{n\to\infty}\left(\limsup_{n\to\infty}||x_n-x_m||\right) \le 1. \tag{3.19}$$

Replacing $R(1,X)$ in [20] by $R(X)$ we obtain the following theorem.

Theorem 3.6. *If X has property WORTH and $C_{\mathrm{NJ}}(X)<2$, then $R(X)<2$.*

Proof. Suppose on the contrary that $R(X)=2$. Thus there exist sequences $(x_n^m),(x^m)\in B(X)$ such that for each m, $x_n^m \xrightarrow{w} 0$ as $n\to\infty$ and

$$\liminf_{n\to\infty}||x_n^m - x^m|| > 2-\frac{1}{m} \tag{3.20}$$

for all $m\in N$. Now, by WORTH, we have, for each m,

$$\frac{||x_n^m+x^m||^2+||x_n^m-x^m||^2}{2(||x_n^m||^2+||x^m||^2)} > \frac{2(2-1/m)^2}{4} = 2-\frac{2}{m}+\frac{1}{2m^2} \tag{3.21}$$

for all large n. This implies $C_{\mathrm{NJ}}(X)=2$, a contradiction, and therefore $R(X)<2$ as desired. □

Remark 3.7. Theorem 3.6 says that every Banach space X with property WORTH has fpp or $C_{NJ}(X) = 2 = R(X)$.

Theorem 3.8. *If X is k-UR and satisfies property (M), then X satisfies Opial property.*

Proof. Suppose that there exist $x_n \xrightarrow{w} 0$ and $0 \neq x_0 \in X$ such that

$$\limsup_{n\to\infty} ||x_n|| \geq \limsup_{n\to\infty} ||x_n - x_0||. \tag{3.22}$$

Observe that X is therefore not finite dimensional. By the nonstrict Opial property (see Proposition 2.1) we have

$$\limsup_{n\to\infty} ||x_n|| = \limsup_{n\to\infty} ||x_n - x_0|| = \alpha \neq 0. \tag{3.23}$$

We may assume that $\|x_0\| = 1$. Define the type function by

$$f(u) = \limsup_{n\to\infty} ||x_n - u||. \tag{3.24}$$

Then f is a function of $\|u\|$ and is also nondecreasing in $\|u\|$. Now since $f(0) = f(x_0) = \alpha$ and since $\|x_0\| = 1$, it follows that $f(u) \equiv \alpha$ for all $u \in B(X)$. This implies that $A_{B(X)}(x_n) = B(X)$. Since X is k-UR, Kirk [18] implies that $A_{B(X)}(x_n)$ and so $B(X)$ is compact, that is, X is finite dimensional, a contradiction. □

Corollary 3.9. *If X is k-UR and has property (M), then X has the locally uniform Opial property. In particular, properties UR and (M) imply the locally uniform Opial property.*

Proof. This follows from Theorem 2.2 and Theorem 3.8. □

Definition 3.10. Let X be a Banach space.

(i) We say that X has property strict (M) [27, Definition 2.2] if, for each weakly null sequence (x_n), for $u, v \in X$ such that $\|u\| < \|v\|$, $\limsup_{n\to\infty} \|x_n - u\| < \limsup_{n\to\infty} \|x_n - v\|$.

(ii) We say that X has property strict (W) if, for each weakly null sequence (x_n), for $x \in X$ we have $\limsup_{n\to\infty} \|x_n - tx\|$ is an strictly increasing function of t on $[0, \infty)$.

It is easy to see that

$$\text{property strict } (M) \Longrightarrow \text{property strict } (W) \Longrightarrow \text{Opial property}. \tag{3.25}$$

Proposition 3.11. *Let X be a Banach space, then X has property strict (M) if and only if it has both properties (M) and strict (W).*

Proof. "⇒" Clear.

"⇐" Suppose X has properies (M) and strict (W). Let (x_n) be a weakly null sequence, $u, v \in X$ with $\|u\| < \|v\|$. By property strict (W) we have

$$\limsup_{n\to\infty} ||x_n - u|| < \limsup_{n\to\infty} \left|\left| x_n - \frac{\|v\|}{\|u\|} u \right|\right|. \tag{3.26}$$

Since $\|(\|v\|/\|u\|)u\| = \|v\|$, so by property (M) we have $\limsup_{n\to\infty}\|x_n - (\|v\|/\|u\|)u\| = \limsup_{n\to\infty}\|x_n - v\|$. Hence

$$\limsup_{n\to\infty}\|x_n - u\| < \limsup_{n\to\infty}\|x_n - v\|. \tag{3.27}$$

This shows that X has property strict (M). □

Proposition 3.12. *Let X be a Banach space which satisfies Opial property and has property (M). Then X satisfies the locally uniform Opial property.*

Proof. Let (x_n) be a weakly null sequence in X satisfying $\|x_n\| \to 1$ and $c > 0$. Set $r = \limsup_{n\to\infty}\|x_n - (c/\|x\|)x\| - 1$, where $x \in X - \{0\}$. Since X satisfies Opial property, we have $r > 0$. Hence, for $u \in X$ such that $\|u\| \geq c$, we have

$$\limsup_{n\to\infty}\|x_n - u\| \geq \limsup_{n\to\infty}\left\|x_n - \frac{c}{\|u\|}u\right\| = \limsup_{n\to\infty}\left\|x_n - \frac{c}{\|x\|}x\right\| = 1 + r. \tag{3.28}$$

Thus, X satisfies the locally uniform Opial property. □

Corollary 3.13 [27, Theorem 2.1]. *Let X be a Banach space which has property strict (M). Then X satisfies the locally uniform Opial property.*

Recall that a Banach space X is uniformly convex in every direction (UCED) Day et al. [4] if, for each $z \in X$ such that $\|z\| = 1$ and $\epsilon > 0$, we have

$$\delta_z(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1,\ \|y\| \leq 1,\ x - y = tz,\ |t| \geq \epsilon\right\} > 0. \tag{3.29}$$

Theorem 3.14. *Suppose that a Banach space X has property WORTH and is also UCED. Then X has the property strict (W).*

Proof. Suppose X fails to have the property strict (W), then there exist a weakly null sequence (x_n), $x \in S(X)$, $t_1, t_2 \in [0,\infty)$, where $t_1 < t_2$, with

$$\limsup_{n\to\infty}\|x_n + t_1 x\| \geq \limsup_{n\to\infty}\|x_n + t_2 x\|. \tag{3.30}$$

By property WORTH we must have equality. Put $a = \limsup_{n\to\infty}\|x_n + t_1 x\|$, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\left\|x_n + \frac{t_1+t_2}{2}x\right\| &= \limsup_{n\to\infty}\left\|\frac{x_n + t_1 x + x_n + t_2 x}{2}\right\| \\ &\leq a\left[1 - \delta_x\left(\frac{t_2 - t_1}{a}\right)\right] < a = \limsup_{n\to\infty}\|x_n + t_1 x\|\end{aligned} \tag{3.31}$$

contradicting to having WORTH of X. □

From Proposition 3.11 and Theorem 3.14 we have the following corollary.

Corollary 3.15. *Suppose that a Banach space X has property (M) and is also UCED. Then X has property strict (M).*

Finally, we improve the latest upper bound of the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ at $(3+\sqrt{5})/4$ for X to have uniform normal structure which is proved in [5].

Theorem 3.16. *If $C_{\mathrm{NJ}}(X) < (1+\sqrt{3})/2$, then X has uniform normal structure.*

Proof. Since $C_{\mathrm{NJ}}(X) < 2$, X is uniformly nonsquare, and consequently, X is reflexive. Thus, normal structure and weak normal structure coincide. By [8, Theorem 5.2], it suffices to prove that X has weak normal structure.

Suppose on the contrary that X does not have weak normal structure. Thus, there exists a weak null sequence (x_n) in $S(X)$ such that for $C := \bar{\mathrm{co}}\{x_n : n \geq 1\}$,

$$\lim_{n\to\infty} ||x_n - x|| = \operatorname{diam} C = 1 \quad \forall x \tag{3.32}$$

(cf. [24]). Let $\alpha = \sqrt{1+\sqrt{3}}$. We choose first an $x \in C$ with $\|x\| = 1$. We will consider, without loss of generality

$$\begin{aligned}\lim_{n\to\infty} ||x_n + x|| &\leq R(1,X) \leq J(X) \quad ([20]) \\ &\leq \sqrt{2C_{\mathrm{NJ}}(X)} \quad ([16]) < \alpha.\end{aligned} \tag{3.33}$$

By Hanh-Banach theorem there exist $f_n, g \in S(X^*)$ satisfying $f_n(x_n - (1/2)x) = \|x_n - (1/2)x\|$, $\forall n \in \mathbb{N}$ and $g(x) = 1$. Set $\tilde{f} = \widetilde{(f_n)}$. Then $\tilde{f}, \dot{g} \in S(\tilde{X}^*)$ and satisfy

$$\tilde{f}\left(\widetilde{(x_n)}\right) = 1, \qquad \tilde{f}(\dot{x}) = 0, \qquad \dot{g}\left(\widetilde{(x_n)}\right) = 0, \qquad \dot{g}(\dot{x}) = 1. \tag{3.34}$$

Now consider

$$\begin{aligned}||\tilde{f} - \dot{g}|| &\geq (\tilde{f} - \dot{g})\left(\widetilde{(x_n)} - \dot{x}\right) \\ &= \tilde{f}\left(\widetilde{(x_n)}\right) - \tilde{f}(\dot{x}) - \dot{g}\left(\widetilde{(x_n)}\right) + \dot{g}(\dot{x}) \\ &= 1 + 0 - 0 + 1 \geq 2.\end{aligned} \tag{3.35}$$

On the other hand,

$$\begin{aligned}||\tilde{f} + \dot{g}|| &\geq (\tilde{f} + \dot{g})\left(\frac{1}{\alpha}\left(\widetilde{(x_n)} + \dot{x}\right)\right) \\ &= \tilde{f}\left(\frac{1}{\alpha}\widetilde{(x_n)}\right) + \tilde{f}\left(\frac{1}{\alpha}\dot{x}\right) - \dot{g}\left(\frac{1}{\alpha}\widetilde{(x_n)}\right) + \dot{g}\left(\frac{1}{\alpha}\dot{x}\right) \\ &= \frac{1}{\alpha} + 0 - 0 + \frac{1}{\alpha} = \frac{2}{\alpha}.\end{aligned} \tag{3.36}$$

Thus we have

$$C_{\mathrm{NJ}}(\tilde{X}^*) \geq \frac{||\tilde{f} + \dot{g}||^2 + ||\tilde{f} - \dot{g}||^2}{2(\|\tilde{f}\|^2 + \|\dot{g}\|^2)} \geq \frac{4 + 4/\alpha^2}{4} = 1 + \frac{1}{\alpha^2}. \tag{3.37}$$

Since the Jordan-von Neumann constants of X^*, X, $\widetilde{X}$, and $\widetilde{X}^*$ are all equal, we must have $C_{\mathrm{NJ}}(X) \geq 1 + 1/\alpha^2$, that is,

$$C_{\mathrm{NJ}}(X) \geq \frac{1+\sqrt{3}}{2}, \tag{3.38}$$

a contradiction. □

The following corollary is a consequence of the proof of Theorem 3.16.

Corollary 3.17. *If $C_{\mathrm{NJ}}(X) < 1 + 1/J(X)^2$, then X has uniform normal structure.*

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