NORMAL SOLVABILITY OF GENERAL LINEAR ELLIPTIC PROBLEMS

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The paper is devoted to general elliptic problems in the Douglis-Nirenberg sense. We obtain a necessary and sufficient condition of normal solvability in the case of unbounded domains. Along with the ellipticity condition, proper ellipticity and Lopatinsky condition that determine normal solvability of elliptic problems in bounded domains, one more condition formulated in terms of limiting problems should be imposed in the case of unbounded domains.

1. Introduction

In this work we study normal solvability of general elliptic problems in the Douglis-Nirenberg sense. If in the case of bounded domains with a sufficiently smooth boundary the normal solvability is completely determined by the conditions of ellipticity, proper ellipticity, and Lopatinsky condition (see [2, 3, 19, 20]), then in the case of unbounded domains one more condition related to behavior of solutions at infinity should be imposed.

If the coefficients of the operator have limits at infinity, and the domain is cylindrical or conical at infinity, then the additional condition is determined by the invertibility of limiting operators, that is of the operators with the limiting coefficients in the limiting domain. This situation is studied in a number of works for differential [4, 11, 21, 25, 26] and pseudodifferential operators [15, 17, 16].

If the coefficients do not have limits at infinity but the domain is the whole R^n, the notion of limiting operators was used in [12, 13]. Previously it was used in the one-dimensional case to study differential equations with quasi-periodic coefficients [6, 8, 9, 14] (see also [18]).

In the case of arbitrary domains, we need to introduce the notion of limiting domains and limiting problems. In the case of general elliptic problems and Hölder spaces it is done in [23]. In [22] we study scalar elliptic problems in Sobolev spaces (see below). We obtain conditions of normal solvability in terms of uniqueness of solutions of limiting problems. In this work we generalize these results to the case of systems.

We should note that the choice of function spaces plays important role. We introduce a generalization of Sobolev-Slobodetskii spaces that will be essentially used in the

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subsequent works where we consider nonhomogeneous adjoint problems, obtain a priori estimates for them and prove their normal solvability. These results are used to prove the Fredholm property of general elliptic problems. For the scalar equation the solvability conditions will be also formulated in terms of formally adjoint problems. We will use the approach developed in [10] for scalar elliptic problems in bounded domains.

1.1. Function spaces. Sobolev spaces $W^{s,p}$ proved to be very convenient in the study of elliptic problems in bounded domains. But more flexible spaces are needed for elliptic problems in unbounded domains. We need some generalization of the space $W^{s,p}$. More exactly, we mean such spaces which coincide with $W^{s,p}$ in bounded domains but have a prescribed behavior at infinity in unbounded domains. It turns out that such spaces can be constructed for arbitrary Banach spaces of distributions (not only Sobolev spaces) as follows.

Consider first functions defined on R^n. As usual we denote by D the space of infinitely differentiable functions with compact support and by D' its dual. Let $E \subset D'$ be a Banach space, the inclusion is understood both in algebraic and topological sense. Denote by E_{loc} the collection of all $u \in D'$ such that $fu \in E$ for all $f \in D$. Let $\omega(x) \in D$, $0 \le \omega(x) \le 1$, $\omega(x) = 1$ for $|x| \le 1/2$, $\omega(x) = 0$ for $|x| \ge 1$.

Definition 1.1. E_q $(1 \le q \le \infty)$ is the space of all $u \in E_{\mathrm{loc}}$ such that

$$\begin{gathered} \|u\|_{E_q} := \left(\int_{R^n} \|u(\cdot)\omega(\cdot - y)\|_E^q \, dy \right)^{1/q} < \infty, \quad 1 \le q < \infty, \\ \|u\|_{E_\infty} := \sup_{y \in R^n} \|u(\cdot)\omega(\cdot - y)\|_E < \infty. \end{gathered} \tag{1.1}$$

It is proved that E_q is a Banach space. If Ω is a domain in R^n, then by definition $E_q(\Omega)$ is the space of restrictions of E_q to Ω with the usual norm of restrictions. It is easy to see that if Ω is a bounded domain, then

$$E_q(\Omega) = E(\Omega), \quad 1 \le q \le \infty. \tag{1.2}$$

In particular, if $E = W^{s,p}$, then we denote $W_q^{s,p} = E_q$ $(1 \le q \le \infty)$. It is proved that

$$W_p^{s,p} = W^{s,p} \quad (s \ge 0,\ 1 < p < \infty). \tag{1.3}$$

Hence the spaces $W_q^{s,p}$ generalize the Sobolev spaces $(q < \infty)$ and the Stepanov spaces $(q = \infty)$ (see [8, 9, 14]).

If $E = L^p$, then $L_q^p = E_q$. It can be proved that if u belongs to L^p locally and $|u(x)| \le K|x|^{-\alpha}$ for $|x|$ sufficiently large, where K is a positive constant, and $\alpha q > n$, then $u \in L_q^p$. Unlike the spaces L^p for which there is no embedding $L^p(R^n)$ in $L^{p_1}(R^n)$ for any $1 < p$, $p_1 < \infty$, $p \ne p_1$, it is easy to prove that

$$L_q^p(R^n) \subset L_{q_1}^{p_1}(R^n) \quad (p \ge p_1,\ q \le q_1). \tag{1.4}$$

1.2. Elliptic problems. Consider the operators

$$\begin{aligned} A_i u &= \sum_{k=1}^{N} \sum_{|\alpha| \le \alpha_{ik}} a_{ik}^{\alpha}(x) D^{\alpha} u_k, \quad i = 1,\dots,N,\ x \in \Omega, \\ B_j u &= \sum_{k=1}^{N} \sum_{|\beta| \le \beta_{jk}} b_{jk}^{\beta}(x) D^{\beta} u_k, \quad i = 1,\dots,m,\ x \in \partial\Omega, \end{aligned} \tag{1.5}$$

where $u = (u_1,\dots,u_N)$, $\Omega \subset R^n$ is an unbounded domain that satisfy certain conditions given below. According to the definition of elliptic operators in the Douglis-Nirenberg sense [5] we suppose that

$$\alpha_{ik} \le s_i + t_k, \quad i,k = 1,\dots,N, \qquad \beta_{jk} \le \sigma_j + t_k, \quad j = 1,\dots,m,\ k = 1,\dots,N \tag{1.6}$$

for some integers s_i, t_k, σ_j such that $s_i \le 0$, $\max s_i = 0$, $t_k \ge 0$.

Denote by E the space of vector-valued functions $u = (u_1,\dots,u_N)$, where u_j belongs to the Sobolev space $W^{l+t_j,p}(\Omega)$, $j = 1,\dots,N$, $1 < p < \infty$, l is an integer, $l \ge \max(0, \sigma_j + 1)$, $E = \Pi_{j=1}^{N} W^{l+t_j,p}(\Omega)$. The norm in this space is defined as

$$\|u\|_E = \sum_{j=1}^{N} \|u_j\|_{W^{l+t_j,p}(\Omega)}. \tag{1.7}$$

The operator A_i acts from E to $W^{l-s_i,p}(\Omega)$, the operator B_j acts from E to $W^{l-\sigma_j-1/p,p}(\partial\Omega)$. Denote

$$\begin{gathered} L = (A_1,\dots,A_N,B_1,\dots,B_m), \\ F = \Pi_{i=1}^{N} W^{l-s_i,p}(\Omega) \times \Pi_{j=1}^{m} W^{l-\sigma_j-1/p,p}(\partial\Omega). \end{gathered} \tag{1.8}$$

We will consider the operator L as acting from E_∞ to F_∞.

Throughout the paper we assume that the operator L satisfies the condition of uniform ellipticity.

1.3. Limiting problems. We recall that the operator is normally solvable with a finite dimensional kernel if and only if it is proper, that is the inverse image of a compact set is compact in any closed bounded set. In this work we obtain necessary and sufficient conditions for a general elliptic operator to satisfy this property. Consider as example the following operator

$$Lu = a(x)u'' + b(x)u' + c(x)u \tag{1.9}$$

acting from $H^2(R)$ to $L^2(R)$. If we assume that there exist limits of the coefficients of the operator at infinity, then we can define the operators

$$L_\pm u = a_\pm u'' + b_\pm u' + c_\pm u, \tag{1.10}$$

where the subscripts $+$ and $-$ denote the limiting values at $+\infty$ and $-\infty$, respectively. As it is well known, the operator L satisfies the Fredholm property if and only if the equations $L_\pm u = 0$ do not have nonzero bounded solutions. We can easily write down this condition explicitly:

$$-a_\pm \xi^2 + b_\pm i\xi + c_\pm \neq 0 \tag{1.11}$$

if we look for solutions of this equations in the form $u = \exp(i\xi)$.

This simple approach is not applicable for general elliptic problems where limits of the coefficients may not exist and the domain can be arbitrary. In the next section we will define limiting problems in the general case. Construction of limiting domains can be briefly described as follows. Let $x_k \in \Omega$ be a sequence, which tends to infinity. Consider the shifted domains Ω_k corresponding to the shifted characteristic functions $\chi(x + x_k)$, where $\chi(x)$ is the characteristic function of the domain Ω. Consider a ball $B_r \subset R^n$ with the center at the origin and with the radius r. Suppose that for all k there are points of the boundaries $\partial\Omega_k$ inside B_r. If the boundaries are sufficiently smooth, we can expect that from the sequence $\Omega_k \cap B_r$ we can choose a subsequence that converges to some limiting domain Ω_*. After that we take a larger ball and choose a convergent subsequence of the previous subsequence. The usual diagonal process allows us to extend the limiting domain to the whole space.

To define limiting operators we consider shifted coefficients $a^\alpha(x + x_k)$, $b_j^\alpha(x + x_k)$ and choose subsequences that converge to some limiting functions $\hat{a}^\alpha(x)$, $\hat{b}_j^\alpha(x)$ uniformly in every bounded set. The limiting operator is the operator with the limiting coefficients. Limiting operators and limiting domains constitute limiting problems. It is clear that the same problem can have a family of limiting problems depending on the choice of the sequence x_k and on the choice of both converging subsequences of domains and coefficients.

We note that in the case where $\Omega = R^n$ the limiting domain is also R^n. In this case the limiting operators were introduced and used in [12, 13, 17, 18].

1.4. Normal solvability. The following condition determines normal solvability of elliptic problems.

Condition NS. Any limiting problem

$$\hat{L}u = 0, \quad x \in \Omega_*, \ u \in E_\infty(\Omega_*) \tag{1.12}$$

has only zero solution.

It is a necessary and sufficient condition for general elliptic operators considered in Hölder spaces to be normally solvable with a finite dimensional kernel [23]. For scalar elliptic problems in Sobolev spaces it was proved in [22]. In this work we generalize these results for elliptic systems. More precisely, we prove that the elliptic operator L is normally solvable and has a finite-dimensional kernel in the space $W_\infty^{l,p}$ $(1 < p < \infty)$ if and only if Condition NS is satisfied. Using this result it can be proved that the elliptic operator L is Fredholm (if the limiting operators are invertible) in the space $W_q^{l,p}$ for $1 < p < \infty$ and some q. This result will be published elsewhere.

It is easy to see how this condition is related to the condition formulated in terms of the Fourier transform. In fact, for operator (1.9) the nonzero solution of the limiting problem $L_{\pm}u = 0$ has the form $u_0(x) = e^{i\xi x}$, where ξ is the value for which the essential spectrum passes through 0. The function $u_0(x)$ belongs obviously to the Hölder spaces and also to the space $W^{2,p}_{\infty}(R)$. However it does not belong to the usual Sobolev space $W^{2,p}(R)$. So Condition NS cannot be obtained in terms of usual Sobolev spaces (see also [22] for counter-examples in R^n). This is one of the reasons why it is more convenient to work with $W^{s,p}_q$ spaces.

2. A priori estimates in the spaces $W^{s,p}_{\infty}$

In this section, we define the spaces $W^{s,p}_{\infty}$ and obtain a priori estimates of solutions, which are similar to those in usual Sobolev spaces.

Denote by $W^{k,p}_{\infty}(\Omega)$ the space of functions defined as the closure of smooth functions in the norm

$$\|u\|_{W^{k,p}_{\infty}(\Omega)} = \sup_{y\in\Omega} \|u\|_{W^{k,p}(\Omega\cap Q_y)}. \tag{2.1}$$

Here Ω is a domain in R^n, Q_y is a unit ball with the center at y, $\|\cdot\|_{W^{k,p}}$ is the Sobolev norm. We note that in bounded domains Ω the norms of the spaces $W^{k,p}(\Omega)$ and $W^{k,p}_{\infty}(\Omega)$ are equivalent. In the one-dimensional case with $k = 0$ similar spaces were used in [8, 9, 14]. This definition is equivalent to Definition 1.1.

We suppose that the boundary $\partial\Omega$ belongs to the Hölder space $C^{k+\theta}$, $0 < \theta < 1$, and that the Hölder norms of the corresponding functions in local coordinates are bounded independently of the point of the boundary. Then we can define the space $W^{k-1/p,p}_{\infty}(\partial\Omega)$ of traces on the boundary $\partial\Omega$ of the domain Ω,

$$\|\phi\|_{W^{k-1/p,p}_{\infty}(\partial\Omega)} = \inf \|v\|_{W^{k,p}_{\infty}(\Omega)}, \tag{2.2}$$

where the infimum is taken with respect to all functions $v \in W^{k,p}_{\infty}(\Omega)$ equal ϕ at the boundary, and $k > 1/p$.

The space $W^{k,p}_{\infty}(\Omega)$ with $k = 0$ will be denoted by $L^p_{\infty}(\Omega)$. We will use also the notations

$$\begin{gathered} E_{\infty} = \Pi_{j=1}^{N} W^{l+t_j,p}_{\infty}(\Omega), \\ F_{\infty} = \Pi_{i=1}^{N} W^{l-s_i,p}_{\infty}(\Omega) \times \Pi_{j=1}^{m} W^{l-\sigma_j-1/p,p}_{\infty}(\partial\Omega). \end{gathered} \tag{2.3}$$

We consider the operator L defined by (1.8) and denote $l_1 = \max(0, \sigma_j + 1)$. We suppose that the integer l in the definition of the spaces is such that $l \geq l_1$, and the boundary $\partial\Omega$ belongs to the class $C^{r+\theta}$ with r specified in Condition D below.

Theorem 2.1. *Let $u \in \Pi_{j=1}^{N} W^{l_1+t_j,p}_{\infty}(\Omega)$. Then for any $l \geq l_1$ we have $u \in E_{\infty}$ and*

$$\|u\|_{E_{\infty}} \leq c\Big(\|Lu\|_{F_{\infty}} + \|u\|_{L^p_{\infty}(\Omega)}\Big), \tag{2.4}$$

where the constant c does not depend on u.

Proof. Let $\omega(x)$ be an infinitely differentiable nonnegative function such that

$$\omega(x) = 1, \quad |x| \le \frac{1}{2}, \qquad \omega(x) = 0, \quad |x| \ge 1. \tag{2.5}$$

Denote $\omega_y(x) = \omega(x-y)$. Suppose $u(x)$ is a function satisfying the conditions of the theorem. Then $\omega_y u \in \Pi_{j=1}^N W_\infty^{l_1+t_j,p}(\Omega)$. Since the support of this function is bounded, we can use now a priori estimates of solutions [1]:

$$||\omega_y u||_E \le c\Big(||L(\omega_y u)||_F + ||\omega_y u||_{L^p(\Omega)}\Big), \quad \forall y \in R^n, \tag{2.6}$$

where the constant c does not depend on y. We now estimate the right-hand side of the last inequality. We have

$$A_i(\omega_y u) = \omega_y A_i u + T_i, \tag{2.7}$$

where

$$T_i = \sum_{k=1}^N \sum_{|\alpha|\le\alpha_{ik}} a_{ik}^\alpha \sum_{\beta+\gamma\le\alpha,\,|\beta|>0} c_{\beta\gamma} D^\beta \omega_y D^\gamma u_k, \tag{2.8}$$

and $c_{\beta\gamma}$ are some constants. If $|\tau| \le l - s_i$, then

$$||D^\tau(\omega_y A_i u)||_{L^p(\Omega)} \le M||A_i u||_{W_\infty^{l-s_i,p}(\Omega)}. \tag{2.9}$$

For any $\epsilon > 0$ we have the estimate

$$\begin{aligned} ||T_i||_{W^{l-s_i,p}(\Omega)} &\le \epsilon \sum_{k=1}^N ||u_k||_{W^{l+t_k,p}(\Omega\cap Q_y)} + C_\epsilon \sum_{k=1}^N ||u_k||_{L^p(\Omega\cap Q_y)} \\ &\le \epsilon \|u\|_{E_\infty} + C_\epsilon \|u\|_{L^p_\infty(\Omega)}, \end{aligned} \tag{2.10}$$

where Q_y is a unit ball with the center at y.

Thus

$$||A_i(\omega_y u)||_{W^{l-s_i,p}(\Omega)} \le M||A_i u||_{W_\infty^{l-s_i,p}(\Omega)} + \epsilon\|u\|_{E_\infty} + C_\epsilon\|u\|_{L^p_\infty(\Omega)}. \tag{2.11}$$

Consider next the boundary operators in the right-hand side of (2.6). We have

$$B_j(\omega_y u) = \omega_y \Phi_j + S_j, \tag{2.12}$$

where $\Phi_j = B_j u$,

$$S_j = \sum_{k=1}^N \sum_{|\beta|\le\beta_{jk}} b_{jk}^\beta \sum_{\alpha+\gamma\le\beta,\,|\alpha|>0} \lambda_{\alpha\gamma} D^\alpha \omega_y\, D^\gamma u_k, \tag{2.13}$$

and $\lambda_{\alpha\gamma}$ are some constants.

There exists a function $v \in W_\infty^{l-\sigma_j,p}(\Omega)$ such that $v = \Phi_i$ on $\partial\Omega$ and

$$\|v\|_{W_\infty^{l-\sigma_j,p}(\Omega)} \le 2\left\|\Phi_j\right\|_{W_\infty^{l-\sigma_j-1/p,p}(\partial\Omega)}. \tag{2.14}$$

Since $v \in W_\infty^{l-\sigma_j,p}(\Omega)$, then $\omega_y v \in W^{l-\sigma_j,p}(\Omega)$ and

$$\left\|\omega_y v\right\|_{W^{l-\sigma_j,p}(\Omega)} \le M\|v\|_{W_\infty^{l-\sigma_j,p}(\Omega)} \tag{2.15}$$

with a constant M independent of v. Since $\omega_y v = \omega_y \Phi_j$ on $\partial\Omega$, then

$$\left\|\omega_j \Phi_j\right\|_{W^{l-\sigma_j-1/p,p}(\partial\Omega)} \le M_1\left\|\Phi_j\right\|_{W_\infty^{l-\sigma_j-1/p,p}(\partial\Omega)}. \tag{2.16}$$

Further,

$$\begin{aligned}\left\|S_j\right\|_{W^{l-\sigma_j-1/p,p}(\partial\Omega)} \le \left\|S_j\right\|_{W^{l-\sigma_j,p}(\Omega)} &\le \epsilon\sum_{k=1}^{N}\left\|u_k\right\|_{W^{l+t_k,p}(\Omega\cap Q_y)} + C_\epsilon\sum_{k=1}^{N}\left\|u_k\right\|_{L^p(\Omega\cap Q_y)}\\ &\le \epsilon\|u\|_{E_\infty} + C_\epsilon\|u\|_{L_\infty^p(\Omega)}.\end{aligned} \tag{2.17}$$

Thus

$$\left\|B_j\left(\omega_y u\right)\right\|_{W^{l-\sigma_j-1/p,p}(\partial\Omega)} \le M\left\|\Phi_j\right\|_{W_\infty^{l-\sigma_j-1/p,p}(\partial\Omega)} + \epsilon\|u\|_{E_\infty} + C_\epsilon\|u\|_{L_\infty^p(\Omega)}. \tag{2.18}$$

From (2.6), (2.11), and (2.18) we obtain the estimate

$$\left\|\omega_y u\right\|_E \le c\Big(M_2\|Lu\|_{F_\infty} + \kappa\epsilon\|u\|_{E_\infty} + C_\epsilon\|u\|_{L_\infty^p(\Omega)}\Big) \tag{2.19}$$

with some constants M_2 and κ. Taking $\epsilon > 0$ sufficiently small, we obtain (2.4). The theorem is proved. □

3. Limiting problems

In this section, we define limiting domains and limiting operators. They determine limiting problems. We will restrict ourselves to the definitions and to the result, which we give without proofs, that will be used below. More detailed presentation including the proofs can be found in [22].

3.1. Limiting domains.

In this section, we define limiting domains for unbounded domains in R^n, show their existence and study some of their properties. We consider an unbounded domain $\Omega \subset R^n$, which satisfies the following condition.

Condition D. For each $x_0 \in \partial\Omega$ there exists a neighborhood $U(x_0)$ such that:

(1) $U(x_0)$ contains a sphere with the radius δ and the center x_0, where δ is independent of x_0,

(2) there exists a homeomorphism $\psi(x;x_0)$ of the neighborhood $U(x_0)$ on the unit sphere $B = \{y : |y| < 1\}$ in R^n such that the images of $\Omega \bigcap U(x_0)$ and $\partial\Omega \cap U(x_0)$ coincide with $B_+ = \{y : y_n > 0,\ |y| < 1\}$ and $B_0 = \{y : y_n = 0,\ |y| < 1\}$, respectively,

(3) the function $\psi(x;x_0)$ and its inverse belong to the Hölder space $C^{r+\theta}$, $0 < \theta < 1$. Their $\|\cdot\|_{r+\theta}$-norms are bounded uniformly in x_0.

For definiteness we suppose that $\delta < 1$. We assume also that

$$r \geq \max\left(l + t_i, l - s_i, l - \sigma_j + 1\right), \quad i = 1,\dots,N,\ j = 1,\dots,m. \tag{3.1}$$

The first expression in the maximum is used for a priori estimates of solutions, the second and the third will allow us to extend the coefficients of the operator (see Section 3.3).

To define convergence of domains we use the following Hausdorff metric space. Let M and N denote two nonempty closed sets in R^n. Denote

$$\varsigma(M,N) = \sup_{a\in M} \rho(a,N), \qquad \varsigma(N,M) = \sup_{b\in N} \rho(b,M), \tag{3.2}$$

where $\rho(a,N)$ denotes the distance from a point a to a set N, and let

$$\varrho(M,N) = \max\left(\varsigma(M,N), \varsigma(N,M)\right). \tag{3.3}$$

We denote by Ξ a metric space of bounded closed nonempty sets in R^n with the distance given by (3.3). We say that a sequence of domains Ω_m converges to a domain Ω in Ξ_{loc} if

$$\varrho\left(\bar{\Omega}_m \cap \bar{B}_R, \bar{\Omega} \cap \bar{B}_R\right) \longrightarrow 0, \quad m \longrightarrow \infty \tag{3.4}$$

for any $R > 0$ and $B_R = \{x : |x| < R\}$. Here the bar denotes the closure of domains.

Definition 3.1. Let $\Omega \subset R^n$ be an unbounded domain, $x_m \in \Omega$, $|x_m| \to \infty$ as $m \to \infty$; $\chi(x)$ be the characteristic function of Ω, and Ω_m be a shifted domain defined by the characteristic function $\chi_m(x) = \chi(x + x_m)$. We say that Ω_* is a *limiting domain* of the domain Ω if $\Omega_m \to \Omega_*$ in Ξ_{loc} as $m \to \infty$.

We denote by $\Lambda(\Omega)$ the set of all limiting domains of the domain Ω (for all sequences x_m). We will show below that if Condition D is satisfied, then the limiting domains exist and also satisfy this condition.

Theorem 3.2. *If a domain Ω satisfies Condition D, then there exists a function $f(x)$ defined in R^n such that:*

(1) $f(x) \in C^{k+\theta}(R^n)$, $k \geq r$,

(2) $f(x) > 0$ *if and only if* $x \in \Omega$,

(3) $|\nabla f(x)| \geq 1$ *for* $x \in \partial\Omega$,

(4) $\min(d(x),1) \leq |f(x)|$, *where* $d(x)$ *is the distance from* x *to* $\partial\Omega$.

Let Ω be an unbounded domain satisfying Condition D and $f(x)$ be a function satisfying conditions of Theorem 3.2. Consider a sequence $x_m \in \Omega$, $|x_m| \to \infty$. Denote

$$f_m(x) = f(x + x_m). \tag{3.5}$$

Theorem 3.3. *Let $f_m(x) \to f_*(x)$ in $C^k_{\text{loc}}(R^n)$, where k is not greater than that in Theorem 3.2. Denote*

$$\Omega_* = \{x : x \in R^n,\ f_*(x) > 0\}. \tag{3.6}$$

Then
(1) $f_*(x) \in C^{k+\theta}(R^n)$,
(2) Ω_* *is an nonempty open set.*
If $\Omega_* \neq R^n$, *then*
(3) $|\nabla f_*(x)|_{\partial\Omega_*} \geq 1$,
(4) $\min(d_*(x), 1) \leq |f_*(x)|$, *where* $d_*(x)$ *is the distance from* x *to* $\partial\Omega_*$.

Theorem 3.4. *If $f_m(x) \to f_*(x)$ in C^k_{loc} as $m \to \infty$, then $\partial\Omega_m \to \partial\Omega_*$ in Ξ_{loc}. Moreover, the limiting domain Ω_* either satisfies Condition D or $\Omega_* = R^n$.*

Theorem 3.5. *Let Ω be an unbounded domain satisfying Condition D, $x_m \in \Omega$, $|x_m| \to \infty$, and $f(x)$ be the function constructed in Theorem 3.2.*

Then there exists a subsequence x_{m_i} and a function $f_*(x)$ such that

$$f_{m_i}(x) \equiv f(x + x_{m_i}) \longrightarrow f_*(x) \tag{3.7}$$

in $C^k_{\text{loc}}(R^n)$, and the domain $\Omega_* = \{x : f_*(x) > 0\}$ either satisfies Condition D or $\Omega_* = R^n$. Moreover, $\bar{\Omega}_{m_i} \to \bar{\Omega}_*$ in Ξ_{loc}, where $\Omega_{m_i} = \{x : f_{m_i}(x) > 0\}$.

3.2. Convergence. In the previous section we have introduced limiting domains. Here we define the corresponding limiting problems.

Let Ω be a domain satisfying Condition D and $\chi(x)$ be its characteristic function. Consider a sequence $x_m \in \Omega$, $|x_m| \to \infty$ and the shifted domains Ω_m defined by the shifted characteristic functions $\chi_m(x) = \chi(x + x_m)$. We suppose that the sequence of domains Ω_m converge in Ξ_{loc} to some limiting domain Ω_*. In this section we suppose that $0 \leq k \leq r$.

Definition 3.6. Let $u_m \in W^{k,p}_\infty(\Omega_m)$, $m = 1, 2, \ldots$. We say that u_m converges to a limiting function $u_* \in W^{k,p}_\infty(\Omega_*)$ in $W^{k,p}_{\text{loc}}(\Omega_m \to \Omega_*)$ if there exists an extension $v_m(x) \in W^{k,p}_\infty(R^n)$ of $u_m(x)$, $m = 1, 2, \ldots$ and an extension $v_*(x) \in W^{k,p}_\infty(R^n)$ of $u_*(x)$ such that $v_m \to v_*$ in $W^{k,p}_{\text{loc}}(R^n)$.

Definition 3.7. Let $u_m \in W^{k-1/p,p}_\infty(\partial\Omega_m)$, $k > 1/p$, $m = 1, 2, \ldots$. We say that u_m converges to a limiting function $u_* \in W^{k-1/p,p}_\infty(\partial\Omega_*)$ in $W^{k-1/p,p}_{\text{loc}}(\partial\Omega_m \to \partial\Omega_*)$ if there exists an extension $v_m(x) \in W^{k,p}_\infty(R^n)$ of $u_m(x)$, $m = 1, 2, \ldots$ and an extension $v_*(x) \in W^{k,p}_\infty(R^n)$ of $u_*(x)$ such that $v_m \to v_*$ in $W^{k,p}_{\text{loc}}(R^n)$.

Definition 3.8. Let $u_m(x) \in C^k(\Omega_m)$, $m = 1, 2, \ldots$. We say that $u_m(x)$ converges to a limiting function $u_*(x) \in C^k(\Omega_*)$ in $C^k_{\text{loc}}(\Omega_m \to \Omega_*)$ if there exists an extension $v_m(x) \in C^k(R^n)$ of $u_m(x)$, $m = 1, 2, \ldots$ and an extension $v_*(x) \in C^k(R^n)$ of $u_*(x)$ such that

$$v_m \longrightarrow v_* \quad \text{in } C^k_{\text{loc}}(R^n). \tag{3.8}$$

Definition 3.9. Let $u_m(x) \in C^k(\partial\Omega_m)$, $m = 1,2,\ldots$. We say that $u_m(x)$ converges to a limiting function $u_*(x) \in C^k(\partial\Omega_*)$ in $C^k_{\text{loc}}(\partial\Omega_m \to \partial\Omega_*)$ if there exists an extension $v_m(x) \in C^k(R^n)$ of $u_m(x)$, $m = 1,2,\ldots$ and an extension $v_*(x) \in C^k(R^n)$ of $u_*(x)$ such that

$$v_m \longrightarrow v_* \quad \text{in } C^k_{\text{loc}}(R^n). \tag{3.9}$$

THEOREM 3.10. *The limiting function $u_*(x)$ in Definitions 3.6–3.9 does not depend on the choice of extensions $v_m(x)$ and $v_*(x)$.*

THEOREM 3.11. *Suppose that $0 < k \le r - 1$. Let*

$$u_m \in W^{k+1,p}_\infty(\Omega_m), \qquad ||u_m||_{W^{k+1,p}_\infty(\Omega_m)} \le M, \tag{3.10}$$

where the constant M does not depend on m. Then there exists a function $u_* \in W^{k+1,p}_\infty(\Omega_*)$ and a subsequence u_{m_i} such that $u_{m_i} \to u_*$ in $W^{k,p}_{\text{loc}}(\Omega_m \to \Omega_*)$.

THEOREM 3.12. *Suppose that $0 < k \le r - 1$. Let $u_m \in W^{k+1-1/p,p}_\infty(\partial\Omega_m)$,*

$$||u_m||_{W^{k+1-1/p,p}_\infty(\partial\Omega_m)} \le M, \tag{3.11}$$

where the constant M does not depend on m. Then there exists a function

$$u_* \in W^{k+1-1/p,p}_\infty(\partial\Omega_*) \tag{3.12}$$

and a subsequence u_{m_i} such that

$$u_{m_i} \longrightarrow u_* \quad \text{in } W^{k+1-\epsilon-1/p,p}_{\text{loc}}(\partial\Omega_m \longrightarrow \partial\Omega_*), \tag{3.13}$$

where $0 < \epsilon < k + 1 - 1/p$.

THEOREM 3.13. *Let $u_m \in C^{k+\theta}(\Omega_m)$, $\|u_m\|_{C^{k+\theta}} \le M$, where the constant M is independent of m. Then there exists a function $u_* \in C^{k+\theta}(\Omega_*)$ and a subsequence u_{m_k} such that $u_{m_k} \to u_*$ in $C^k_{\text{loc}}(\Omega_{m_k} \to \Omega_*)$.*

Let $u_m \in C^{k+\theta}(\partial\Omega_m)$, $\|u_m\|_{C^{k+\theta}} \le M$. Then there exists a function $u_* \in C^{k+\theta}(\partial\Omega_*)$ and a subsequence u_{m_k} such that $u_{m_k} \to u_*$ in $C^k_{\text{loc}}(\partial\Omega_{m_k} \to \partial\Omega_*)$.

3.3. Limiting operators. Suppose that we are given a sequence $\{x_\nu\}$, $\nu = 1,2,\ldots$, $x_\nu \in \Omega$, $|x_\nu| \to \infty$. Consider the shifted domains Ω_ν with the characteristic functions $\chi_\nu(x) = \chi(x + x_\nu)$ where $\chi(x)$ is the characteristic function of Ω, and the shifted coefficients of the operators A_i and B_j:

$$a^\alpha_{ik,\nu}(x) = a^\alpha_{ik}(x + x_\nu), \qquad b^\beta_{jk,\nu}(x) = b^\beta_{jk}(x + x_\nu). \tag{3.14}$$

We suppose that

$$a^\alpha_{ik}(x) \in C^{l-s_i+\theta}(\bar{\Omega}), \qquad b^\beta_{jk}(x) \in C^{l-\sigma_j+\theta}(\partial\Omega), \tag{3.15}$$

where $0 < \theta < 1$, and that these coefficients can be extended to R^n:

$$a_{ik}^{\alpha}(x) \in C^{l-s_i+\theta}(R^n), \qquad b_{jk}^{\beta}(x) \in C^{l-\sigma_j+\theta}(R^n). \tag{3.16}$$

Therefore

$$\|a_{ik,\nu}^{\alpha}(x)\|_{C^{l-s_i+\theta}(R^n)} \le M, \qquad \left\|b_{jk,\nu}^{\beta}(x)\right\|_{C^{l-\sigma_j+\theta}(R^n)} \le M \tag{3.17}$$

with some constant M independent of ν. It follows from Theorem 3.5 that there exists a subsequence of the sequence Ω_ν, for which we keep the same notation, such that it converges to a limiting domain Ω_*. From (3.17) it follows that this subsequence can be chosen such that

$$a_{ik,\nu}^{\alpha} \longrightarrow \hat{a}_{ik}^{\alpha} \quad \text{in } C^{l-s_i}(R^n) \text{ locally}, \qquad b_{jk,\nu}^{\beta} \longrightarrow \hat{b}_{jk}^{\beta} \quad \text{in } C^{l-\sigma_j}(R^n) \text{ locally}, \tag{3.18}$$

where $\hat{a}_{ik}^{\alpha}$ and $\hat{b}_{jk}^{\beta}$ are limiting coefficients,

$$\hat{a}_{ik}^{\alpha} \in C^{l-s_i+\theta}(R^n), \qquad \hat{b}_{jk}^{\beta} \in C^{l-\sigma_j+\theta}(R^n). \tag{3.19}$$

We have constructed limiting operators:

$$\begin{gathered}
\hat{A}_i u = \sum_{k=1}^{N} \sum_{|\alpha| \le \alpha_{ik}} \hat{a}_{ik}^{\alpha}(x) D^{\alpha} u_k, \quad i = 1,\dots,N,\ x \in \Omega_*, \\
\hat{B}_j u = \sum_{k=1}^{N} \sum_{|\beta| \le \beta_{jk}} \hat{b}_{jk}^{\beta}(x) D^{\beta} u_k, \quad i = 1,\dots,m,\ x \in \partial\Omega_*, \\
\hat{L} = (\hat{A}_1,\dots,\hat{A}_N,\hat{B}_1,\dots,\hat{B}_m).
\end{gathered} \tag{3.20}$$

We consider them as acting from $E_\infty(\Omega_*)$ to $F_\infty(\Omega_*)$.

4. A priori estimates with condition ns

In Section 5, we will prove that Condition NS (Section 1.4) is necessary and sufficient in order for the operator L to be normal solvable with a finite dimensional kernel. In this section we will use it to obtain a priori estimates of solutions stronger than those given by Theorem 2.1. Estimates of this type are first obtained in [12, 13] for elliptic operators in the whole R^n.

Theorem 4.1. *Let Condition NS be satisfied. Then there exist numbers M_0 and R_0 such that the following estimate holds:*

$$\|u\|_{E_\infty} \le M_0 \Big(\|Lu\|_{F_\infty} + \|u\|_{L^p(\Omega_{R_0})} \Big), \quad \forall u \in E_\infty. \tag{4.1}$$

Here $\Omega_{R_0} = \Omega \cap \{|x| \le R_0\}$.

Proof. Suppose that the assertion of the theorem is not right. Let $M_k \to \infty$ and $R_k \to \infty$ be given sequences. Then there exists $u_k \in E_\infty$ such that

$$||u_k||_{E_\infty} > M_k\Big(||Lu_k||_{F_\infty} + ||u_k||_{L^p(\Omega_{R_k})}\Big). \tag{4.2}$$

We can suppose that

$$||u_k||_{E_\infty} = 1. \tag{4.3}$$

Then

$$||Lu_k||_{F_\infty} + ||u_k||_{L^p(\Omega_{R_k})} < \frac{1}{M_k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.4}$$

From Theorem 2.1 we obtain

$$||Lu_k||_{F_\infty} + ||u_k||_{L^p_\infty(\Omega)} \geq \frac{1}{c}. \tag{4.5}$$

It follows from (4.4) that $\|Lu_k\|_{F_\infty} \to 0$. Hence

$$||u_k||_{L^p_\infty(\Omega)} > \frac{1}{2c} \quad \text{for } k \geq k_0 \tag{4.6}$$

with some k_0. Since

$$||u_k||_{L^p_\infty(\Omega)} = \sup_{y\in\Omega} ||u_k||_{L^p(Q_y\cap\Omega)}, \tag{4.7}$$

then it follows from (4.6) that there exists $y_k \in \Omega$ such that

$$||u_k||_{L^p(Q_{y_k}\cap\Omega)} > \frac{1}{2c}. \tag{4.8}$$

From (4.4)

$$||u_k||_{L^p(\Omega_{R_k})} \longrightarrow 0. \tag{4.9}$$

This convergence and (4.8) imply that $|y_k| \to \infty$.

Denote

$$Lu_k = f_k. \tag{4.10}$$

From (4.4) we get

$$||f_k||_{F_\infty} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.11}$$

Denote next $x = y + y_k$,

$$w_k(y) = u_k(y + y_k). \tag{4.12}$$

We rewrite (4.10) in the detailed form

$$\begin{aligned}\sum_{h=1}^{N}\sum_{|\alpha|\le\alpha_{ih}} a_{ih}^{\alpha}(x)D^{\alpha}u_{hk} &= f_{ik}, \quad i=1,\dots,N,\ x\in\Omega,\\ \sum_{h=1}^{N}\sum_{|\beta|\le\beta_{jh}} b_{jh}^{\beta}(x)D^{\beta}u_{hk} &= f_{j\ k}^{b}, \quad i=1,\dots,m,\ x\in\partial\Omega,\end{aligned} \tag{4.13}$$

where

$$f_k = \left(f_{1k},\dots,f_{Nk},f_{1\ k}^{b},\dots,f_{mk}^{b}\right), \qquad u_k = \left(u_{1k},\dots,u_{Nk}\right). \tag{4.14}$$

Denoting

$$a_{ih}{}_{k}^{\alpha}(y) = a_{ih}^{\alpha}\left(y+y_k\right), \qquad b_{jh}{}_{k}^{\beta}(y) = b_{jh}^{\beta}\left(y+y_k\right), \tag{4.15}$$

we obtain from (4.13)

$$\sum_{h=1}^{N}\sum_{|\alpha|\le\alpha_{ih}} a_{ih}{}_{k}^{\alpha}(y)D^{\alpha}w_{hk}(y) = f_{ik}\left(y+y_k\right), \quad i=1,\dots,N,\ x\in\Omega_k, \tag{4.16}$$

$$\sum_{h=1}^{N}\sum_{|\beta|\le\beta_{jh}} b_{jh}{}_{k}^{\beta}(y)D^{\beta}w_{hk}(y) = f_{j\ k}^{b}\left(y+y_k\right), \quad i=1,\dots,m,\ x\in\partial\Omega_k, \tag{4.17}$$

Ω_k is the shifted domain. From (4.3) we have

$$\left|\left|w_k\right|\right|_{E_\infty(\Omega_k)} = 1. \tag{4.18}$$

We have $w_k = (w_{1k},\dots,w_{Nk})$, and (4.18) can be written in the form

$$\sum_{i=1}^{N}\left|\left|w_{ik}\right|\right|_{W_\infty^{l+t_i,p}(\Omega_k)} = 1. \tag{4.19}$$

We suppose that w_{ik} are extended to R^n such that their $W_\infty^{l+t_i,p}(R^n)$-norms are uniformly bounded. Passing to a subsequence and retaining the same notation, we can suppose that

$$w_{ik} \longrightarrow w_{i0} \quad \text{in } W^{l+t_i-\epsilon,p}\left(R^n\right) \text{ locally}, \quad (\epsilon>0), \tag{4.20}$$

$$w_{ik} \longrightarrow w_{i0} \quad \text{in } W^{l+t_i,p}\left(R^n\right) \text{ locally weakly} \tag{4.21}$$

for some w_{i0} as $k\to\infty$, and

$$w_{i0} \in W_\infty^{l+t_i,p}\left(R^n\right), \quad i=1,\dots,N. \tag{4.22}$$

Denote $w_0 = (w_{10},\dots,w_{N0})$. We prove that

$$\hat{L}w_0 = 0 \tag{4.23}$$

for a limiting operator $\hat{L}$. To do this we pass to the limit in (4.16), (4.17) by a subsequence of k. We choose this subsequence such that Ω_k converges to a limiting domain, $\Omega_k \to \Omega_*$, and keep for it the same notation.

We begin with (4.16). For any $x_0 \in \Omega_*$ we take a neighborhood U in such a way that $U \subset \Omega_k$ for k sufficiently large. For any $\phi \in D$ with the support in U we get from (4.16):

$$\int_U \sum_{h=1}^{N} \sum_{|\alpha|\le\alpha_{ih}} a_{ih_k}^{\alpha}(y)D^{\alpha}w_{hk}(y)\phi(y)dy = \int_U f_{ik}(y+y_k)\phi(y)dy. \tag{4.24}$$

We can suppose, passing to a subsequence, that

$$a_{ih,k}^{\alpha}(y) \longrightarrow \hat{a}_{ih}^{\alpha}(y) \quad \text{in } C^{l-s_i}(R^n) \text{ locally} \tag{4.25}$$

(see (3.17)), where $\hat{a}_{ih}^{\alpha}(y)$ are the coefficients of the limiting operator. It follows from (4.21) that $D^{\alpha}w_{hk}$ ($|\alpha| \le \alpha_{ih}$) converges locally weakly in $W^{l-s_i,p}$ to $D^{\alpha}w_{h0}$ as $k \to \infty$. Hence we can pass to the limit in (4.24).

From (4.11) it follows that

$$||f_{ik}(\,\cdot+y_k)||_{W_{\infty}^{l-s_i,p}(\Omega_k)} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.26}$$

Hence the right-hand side in (4.24) tends to zero. Passing to the limit in this equation, we obtain

$$\sum_{h=1}^{N} \sum_{|\alpha|\le\alpha_{ih}} \hat{a}_{ih}^{\alpha}(y)D^{\alpha}w_{h0}(y) = 0, \quad y \in \Omega_*. \tag{4.27}$$

Consider now (4.17). From (4.20) it follows that $D^{\beta}w_{hk}$ ($|\beta| \le \beta_{ih}$) tends to $D^{\beta}w_{h0}$ in $W^{l-\sigma_j-\epsilon,p}(R^n)$ locally. Hence (3.17) implies that

$$\sum_{h=1}^{N} \sum_{|\beta|\le\beta_{jh}} b_{jh_k}^{\beta}(y)D^{\beta}w_{hk}(y) \longrightarrow \sum_{h=1}^{N} \sum_{|\beta|\le\beta_{jh}} \hat{b}_{jh}^{\beta}(y)D^{\beta}w_{h0}(y) \tag{4.28}$$

in $W_{\text{loc}}^{l-\sigma_j-\epsilon,p}(R^n)$. Therefore this convergence takes place also in $W_{\text{loc}}^{l-\sigma_j-\epsilon,p}(\Omega_*)$ and, consequently, in $W_{\text{loc}}^{l-\sigma_j-\epsilon-1/p,p}(\partial\Omega_*)$. In other words, we have proved that the convergence (4.28) is in $W_{\text{loc}}^{l-\sigma_j-\epsilon-1/p,p}(\partial\Omega_k \to \partial\Omega_*)$ (see Definition 3.7).

Consider next the right-hand side in (4.17). According to (4.11) we have

$$\left|\left|f_{jk}^{b}(\,\cdot\,+y_k)\right|\right|_{W_\infty^{l-\sigma_j-1/p,p}(\partial\Omega_k)} \longrightarrow 0 \quad \text{as } k\to\infty. \tag{4.29}$$

We can extend $f_{jk}^{b}(y+y_k)$ to the whole R^n in such a way that

$$f_{jk}^{b}(\,\cdot\,+y_k) \longrightarrow 0 \quad \text{in } W_\infty^{l-\sigma_j,p}(R^n). \tag{4.30}$$

Therefore

$$f_{jk}^{b}(\,\cdot\,+y_k) \longrightarrow 0 \quad \text{in } W_{\text{loc}}^{l-\sigma_j-1/p,p}(\partial\Omega_k \longrightarrow \partial\Omega_*). \tag{4.31}$$

From this and convergence (4.28) it follows

$$\sum_{h=1}^{N}\sum_{|\beta|\le\beta_{jh}} \hat{b}_{jh}^{\beta}(y)D^{\beta}w_{h0}(y)=0, \quad y\in\partial\Omega_*. \tag{4.32}$$

From (4.22) it follows that the left-hand side of this equality belongs to $W_\infty^{l-\sigma_j-1/p,p}(\partial\Omega_*)$. Hence it can be regarded as an equality in $W_\infty^{l-\sigma_j-1/p,p}(\partial\Omega_*)$.

From (4.27) and (4.32) we conclude that w_0 is a solution of the limiting problem (4.23). We prove now that $w_0\neq 0$. From (4.8) and (4.12) we have

$$||w_k||_{L^p(\Omega_k\cap Q_0)} > \frac{1}{2c}, \tag{4.33}$$

where Q_0 is the unit ball with the center at the origin. We prove that

$$||w_0||_{L^p(\Omega_*\cap Q_0)} \ge \frac{1}{2c}. \tag{4.34}$$

Indeed, from (4.20),

$$w_k \longrightarrow w_0 \quad \text{in } L_{\text{loc}}^{p}(R^n). \tag{4.35}$$

Denote $S_k=\Omega_k\cap Q_0$, $S_*=\Omega_*\cap Q_0$. Then

$$\begin{aligned}
&\left|\,||w_k||_{L^p(S_k)}-||w_0||_{L^p(S_*)}\right| \\
&\quad\le \left|\,||w_k||_{L^p(S_k)}-||w_0||_{L^p(S_k)}\right| + \left|\,||w_0||_{L^p(S_k)}-||w_0||_{L^p(S_*)}\right| \\
&\quad\equiv A_k+B_k.
\end{aligned} \tag{4.36}$$

Further

$$\begin{aligned} A_k &\leq \|w_k - w_0\|_{L^p(S_k)} = \sum_{i=1}^{N} \|w_{ik} - w_{i0}\|_{L^p(S_k)} \\ &= \sum_{i=1}^{N} \left(\int_{S_k} |w_{ik} - w_{i0}|^p dx \right)^{1/p} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \\ B_k &\leq \sum_{i=1}^{N} \left| \|w_{i0}\|_{L^p(S_k)} - \|w_{i0}\|_{L^p(S_*)} \right| \\ &\leq M \sum_{i=1}^{N} \left(\int_{S_k \triangle S_*} |w_{i0}|^p dx \right)^{1/p} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty \end{aligned} \tag{4.37}$$

since the measure of the symmetric difference $S_k \triangle S_*$ converges to 0.

We have proved that

$$\|w_k\|_{L^p(\Omega_k \cap Q_0)} \longrightarrow \|w_0\|_{L^p(\Omega_* \cap Q_0)} \tag{4.38}$$

and (4.34) follows from (4.33).

Thus there exists a limiting problem with a nonzero solution. This contradicts Condition NS. The theorem is proved. □

Denote

$$\omega_\mu = e^{\mu\sqrt{1+|x|^2}}, \tag{4.39}$$

where μ is a real number.

THEOREM 4.2. *Let Condition NS be satisfied. Then there exist numbers $M_0 > 0$, $R_0 > 0$ and $\mu_0 > 0$ such that for all μ, $0 < \mu < \mu_0$ the following estimate holds:*

$$\|\omega_\mu u\|_{E_\infty} \leq M_0 \left(\|\omega_\mu L u\|_{F_\infty} + \|\omega_\mu u\|_{L^p(\Omega_{R_0})} \right), \quad \forall u \in E_\infty. \tag{4.40}$$

More complete proof of this theorem is given in [24].

Proof. According to (4.1) we have

$$\|\omega_\mu u\|_{E_\infty} \leq M(\|L(\omega_\mu u)\|_{F_\infty} + \|\omega_\mu u\|_{L^p(\Omega_{R_0})}). \tag{4.41}$$

By (1.8), $L = (A_1, \ldots, A_N, B_1, \ldots, B_m)$. Consider first the operator

$$A_i(\omega_\mu u) = \sum_{k=1}^{N} \sum_{|\alpha| \leq \alpha_{ik}} a_{ik}^{\alpha}(x) D^{\alpha}(\omega_\mu u_k), \quad i = 1, \ldots, N. \tag{4.42}$$

We have

$$A_i(\omega_\mu u) = \omega_\mu A_i(u) + \Phi_i, \tag{4.43}$$

where

$$\Phi_i = \sum_{k=1}^{N} \sum_{|\alpha| \le \alpha_{ik}} \sum_{\beta+\gamma=\alpha, |\beta|>0} a_{ik}^{\alpha}(x) c_{\beta\gamma} D^{\beta} \omega_{\mu} D^{\gamma} u_k, \tag{4.44}$$

and $c_{\beta\gamma}$ are some constants. Direct calculations give the following estimate

$$||\Phi_i||_{W_\infty^{l-s_i,p}} \le M_1 \mu ||\omega_\mu u||_{E_\infty(\Omega)}. \tag{4.45}$$

For the boundary operators we have

$$B_j(\omega_\mu u) = \sum_{k=1}^{N} \sum_{|\beta| \le \beta_{jk}} b_{jk}^{\beta}(x) D^{\beta}(\omega_\mu u_k). \tag{4.46}$$

As above we get

$$\begin{gathered} B_j(\omega_\mu u) = \omega_\mu B_j(u) + \Psi_j, \\ ||\Psi_j||_{W_\infty^{l-\sigma_j-1/p,p}} \le M_2 \mu ||\omega_\mu u||_{E_\infty(\Omega)}. \end{gathered} \tag{4.47}$$

From (4.43), (4.45), and (4.47) we obtain

$$||L(\omega_\mu u)||_{F_\infty} \le ||\omega_\mu L u||_{F_\infty} + M\mu ||\omega_\mu u||_{E_\infty}. \tag{4.48}$$

The assertion of the theorem follows from this estimate and (4.41). The theorem is proved. □

Corollary 4.3. *If* $0 < \mu < \mu_0$, $u \in E_\infty$, *and* $\omega_\mu L u \in F_\infty$, *then* $\omega_\mu u \in E_\infty$. *In particular, if* $u \in E_\infty$ *and* $Lu = 0$, *then* $\omega_\mu u \in E_\infty$.

5. Normal solvability

We recall that an operator L acting in Banach spaces is normally solvable if its range is closed. It is called n-normally solvable if it is normally solvable and has a finite dimensional kernel (see, e.g., [7])

Theorem 5.1. *Let Condition NS be satisfied. Then the elliptic operator* $L : E_\infty(\Omega) \to F_\infty(\Omega)$ *is normally solvable and has a finite dimensional kernel.*

Proof. It is known that a linear operator has a finite dimensional kernel and a closed range if and only if its restriction to any bounded closed set is proper.

Let $Lu_n = f_n$, $u_n \in E_\infty(\Omega)$, $f_n \in F_\infty(\Omega)$. Suppose that $\|u_n\|_{E_\infty} \le M$ and f_n is convergent. It is sufficient to prove that the sequence u_n is compact. This follows from Theorem 4.1. The theorem is proved. □

In the next theorem we prove that Condition NS is necessary for the operator L to be normally solvable with a finite dimensional kernel. To simplify the construction we impose a stronger regularity condition on the boundary of the domain, $\partial\Omega \in C^{r+1+\theta}$. We will use the following lemma.

Lemma 5.2. *Let Ω_k and Ω_* be a shifted and a limiting domains, respectively. Then for any N there exists k_0 such that for $k > k_0$ there exists a diffeomorphism*

$$h_k(x) : \bar{\Omega}_k \bigcap B_N \longrightarrow \bar{\Omega}_* \bigcap B_N \tag{5.1}$$

satisfying the condition

$$\left|\left|h_k(x) - x\right|\right|_{C^{l+\theta_0}(\bar{\Omega}_k \bigcap B_N)} \longrightarrow 0 \tag{5.2}$$

as $k \to \infty$. Here $0 < \theta_0 < \theta$.

The proof is given in [22]

Theorem 5.3. *Suppose that a limiting problem for the operator L has a nonzero solution. Then the operator L is not n-normally solvable.*

Explanation. To prove the theorem we construct a sequence u_n such that it is not compact in $E_\infty(\Omega)$ but Lu_n converges to zero in $F_\infty(\Omega)$. The idea of the construction is rather simple but its technical realization is rather long. This is why we preface the proof by a short description of the construction.

Let us consider a ball $B_R(x_k)$ of a fixed radius R with the center at x_k. From the definition of limiting problems it follows that we can choose the sequence x_k in such a way that inside $B_R(x_k)$ the domain Ω is close to the limiting domain, and the coefficients of the operator are close to the coefficients of the limiting operator. Moreover, the domain and the coefficients converge to their limits as $k \to \infty$. Thus we move the ball $B_R(x_k)$ to infinity and superpose it on the domain Ω in the places where the operator and the domain are close to their limits and converge to them.

If u_0 is a nonzero solution of the limiting problem, then we shift it to the ball $B_R(x_k)$. Denote the shifted function by u_k. Then inside $B_R(x_k)$, Lu_k tends to zero as $k \to \infty$. The sequence u_k is not compact.

If u_0 had a bounded support, the construction would be finished. Since it is not necessarily the case, we multiply u_0 by an infinitely differentiable function ϕ with a bounded support. Of course, this product is not an exact solution of the limiting problem any more. However, all terms of the difference $\hat{L}(\phi u_0) - \phi \hat{L} u_0$ contain derivatives of ϕ. If the support of ϕ is sufficiently large, then the derivatives of ϕ can be done sufficiently small. Hence when we move the ball $B_R(x_k)$ to infinity, we should also increase its radius and also increase supports of functions ϕ_k.

Proof. Suppose that there exists a limiting operator $\hat{L}$ such that

$$\hat{L}u_0 = 0, \quad u_0 \in E_\infty(\Omega),\ u_0 \neq 0. \tag{5.3}$$

Consider an infinitely differentiable function $\varphi(x)$, $x \in R^n$ such that $0 \leq \varphi(x) \leq 1$, $\varphi(x) = 1$ for $|x| < 1$, $\varphi(x) = 0$ for $|x| > 2$. If $\{x_k\}$ is the sequence for which the limiting operator $\hat{L}$ is defined, denote

$$\varphi_k(x) = \varphi\left(\frac{x}{r_k}\right), \tag{5.4}$$

where $r_k \to \infty$ and $r_k \le |x_k|/3$. Some other conditions on the sequence r_k will be formulated below.

Let $V_j = \{y : y \in R^n, |y| < j\}$, $j = 1,2,\dots$. Denote by n_j a number such that for $k \ge n_j$ the diffeomorphism h_k defined in Lemma 5.2 can be constructed in $\Omega_k \bigcap V_{j+1}$ and

$$||h_k(y) - y||_{C^{l+\theta_0}(\Omega_k \bigcap V_{j+1})} < \delta, \tag{5.5}$$

where $\delta > 0$ is taken so small that $|h'_k - I| < 1/2$, h'_k is the Jacobian matrix and I is the identity matrix.

For arbitrary $k_j \ge n_j$ we take $r_{k_j} = \min(j/2, |x_{k_j}|/3)$. Let

$$\begin{gathered} v_{k_j}(y) = \varphi_{k_j}(y)\, u_0(h_{k_j}(y)) \quad \text{for } y \in \Omega_{k_j} \bigcap V_{j+1}, \\ v_{k_j}(y) = 0 \quad \text{for } y \in \Omega_{k_j},\ |y| \ge j+1. \end{gathered} \tag{5.6}$$

Denote

$$u_{k_j}(x) = v_{k_j}(x - x_{k_j}), \quad x \in \Omega. \tag{5.7}$$

It is easy to see that $u_{k_j} \in E_\infty(\Omega)$ and

$$||u_{k_j}||_{E_\infty(\Omega)} \le M, \tag{5.8}$$

where M does not depend on k_j. Indeed, obviously

$$\varphi_{k_j}(y) = 0 \tag{5.9}$$

for y outside V_j. Therefore to prove (5.8) it is sufficient to show that

$$||v_{k_j}||_{E_\infty(\Omega_{k_j} \bigcap V_{j+1})} \le M_1, \tag{5.10}$$

or

$$||u_0(h_{k_j}(y))||_{E_\infty(\Omega_{k_j} \bigcap V_{j+1})} \le M_2, \tag{5.11}$$

where M_1 and M_2 do not depend on k_j. This follows from (5.5) and the fact that $u_0 \in E_\infty(\Omega_*)$.

We prove now that the choice of k_j in (5.7) can be specified in such a way that

(i) $Lu_{k_j} \to 0$ in $F_\infty(\Omega)$ as $k_j \to \infty$,

(ii) the sequence $\{u_{k_j}\}$ is not compact in $E_\infty(\Omega)$.

The assertion of the theorem will follow from this.

(i) We consider first the operators A_i, $i = 1,\dots,N$, and then the operator B_j, $j = 1,\dots,m$. For any $k = k_j \ge n_j$ we have

$$A_i u_k = A_i^1 u_k + A_i^2 u_k, \tag{5.12}$$

where

$$A_i^1 u_k(x) = \varphi_k(x - x_k) \sum_{r=1}^{N} \sum_{|\alpha| \le \alpha_{ir}} a_{ir}^\alpha(x) D^\alpha u_{0r}(h_k(x - x_k)), \quad x \in \Omega, \tag{5.13}$$

and A_i^2 contains derivatives of φ_k. Obviously

$$||A_i^2 u_k||_{W_\infty^{l-s_i,p}(\Omega)} \longrightarrow 0 \tag{5.14}$$

as $k \to \infty$.

Denote $y = x - x_k$. From (5.13) we obtain

$$A_i^1 u_k(y+x_k) = \varphi_k(y) T_{ik}(y), \quad y \in \Omega_k, \tag{5.15}$$

where

$$T_{ik}(y) = \sum_{r=1}^{N} \sum_{|\alpha| \le \alpha_{ir}} a_{ir}{}_k^\alpha(y) D^\alpha u_{0r}(h_k(y)), \quad y \in \Omega_k, \tag{5.16}$$

$a_{ir}{}_k^\alpha(y) = a_{ir}^\alpha(y+x_k)$. We prove that for any fixed j

$$||T_{ik}||_{W_\infty^{l-s_i,p}(\Omega_k \bigcap V_{j+1})} \longrightarrow 0 \tag{5.17}$$

as $k \to \infty$. Indeed, by the definition of u_0 the following equality holds:

$$\sum_{r=1}^{N} \sum_{|\alpha| \le \alpha_{ir}} \hat{a}_{ir}^\alpha(x) D^\alpha u_{0r}(x) = 0, \quad x \in \Omega_*. \tag{5.18}$$

Here $\hat{a}_{ir}^\alpha(x)$ are the limiting coefficients. Hence

$$T_{ik}(y) = \sum_{r=1}^{N} \sum_{|\alpha| \le \alpha_{ir}} \Big[S_{ir}{}_k^\alpha(y) + P_{ir}{}_k^\alpha(y) \Big], \tag{5.19}$$

where

$$S_{ir}{}_k^\alpha(y) = a_{ir}{}_k^\alpha(y) \Big[D_y^\alpha u_{0r}(h_k(y)) - D_x^\alpha u_{0r}(h_k(y)) \Big], \tag{5.20}$$

$$P_{ir}{}_k^\alpha(y) = \Big[a_{ir}{}_k^\alpha(y) - \hat{a}_{ir}^\alpha(h_k(y)) \Big] D_x^\alpha u_{0r}(h_k(y)). \tag{5.21}$$

The first factor in the right-hand side of (5.20) is bounded in the norm $C^{l-s_i}(\Omega_k)$ since

$$||a_{ir}{}_k^\alpha||_{C^{l-s_i}(\Omega_k)} = ||a_{ir}^\alpha||_{C^{l-s_i}(\Omega)}. \tag{5.22}$$

From Lemma 5.2 it follows that the second factor tends to 0 in the norm $W_\infty^{l-s_i,p}(\Omega_k \bigcap V_{j+1})$ as $k \to \infty$. Consequently,

$$||S_{ir}{}_k^\alpha||_{W_\infty^{l-s_i,p}(\Omega_k \bigcap V_{j+1})} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{5.23}$$

Consider (5.21). Using (5.5) we easily prove that

$$\|D_x^\alpha u_0(h_k(y))\|_{W_\infty^{l-s_i,p}(\Omega_k\cap V_{j+1})} \le M_3 \tag{5.24}$$

with M_3 independent of k.

To prove (5.17) it remains to show that

$$\|a_{ir}{}_k^\alpha(\cdot) - \hat{a}_{ir}^\alpha(h_k(\cdot))\|_{C^{l-s_i}(\Omega_k\cap V_{j+1})} \longrightarrow 0 \quad \text{as } k\to\infty. \tag{5.25}$$

We recall that it is supposed that $a_{ir}{}_k^\alpha(y)$ and $\hat{a}_{ir}^\alpha(y)$ are defined for $y\in R^n$,

$$\|a_{ir}{}_k^\alpha\|_{C^{l-s_i+\theta}(R^n)} \le M \tag{5.26}$$

with M independent of k, $\hat{a}_{ir}^\alpha(y)\in C^{l-s_i+\theta}(R^n)$ and

$$a_{ir}{}_k^\alpha(y) \longrightarrow \hat{a}_{ir}^\alpha(y) \tag{5.27}$$

in $C_{\text{loc}}^{l-s_i}(R^n)$ as $k\to\infty$. We have

$$\begin{aligned}&\|a_{ir}{}_k^\alpha(y) - \hat{a}_{ir}^\alpha(h_k(y))\|_{C^{l-s_i}(\Omega_k\cap V_{j+1})}\\ &\quad\le \|a_{ir}{}_k^\alpha(y) - \hat{a}_{ir}^\alpha(y)\|_{C^{l-s_i}(\Omega_k\cap V_{j+1})} + \|\hat{a}_{ir}^\alpha(y) - \hat{a}_{ir}^\alpha(h_k(y))\|_{C^{l-s_i}(\Omega_k\cap V_{j+1})}.\end{aligned} \tag{5.28}$$

The first term on the right tends to zero as $k\to\infty$ according to (5.27). The second term tends to zero by the properties of the function $\hat{a}_{ir}^\alpha$ mentioned above, by Lemma 5.2 and by inequality (5.5). Thus (5.17) is proved.

Now we specify the choice of k_j in (5.7). According to (5.17) for any j we can take p_j in such a way that

$$\|T_{ik}\|_{W_\infty^{l-s_i,p}(\Omega_k\cap V_{j+1})} < \frac{1}{j} \tag{5.29}$$

for $k\ge p_j$. We put $k_j=\max(n_j,p_j)$. Then obviously

$$\|\varphi_{k_j}T_{ik_j}\|_{W_\infty^{l-s_i,p}(\Omega_{k_j})} \longrightarrow 0 \quad \text{as } k\longrightarrow\infty. \tag{5.30}$$

Consider now the boundary operators B_i. According to our assumptions, the coefficients $b_{ih}^\beta(x)$ of the operators B_i $(i=1,\dots,m)$ are defined in the domain $\overline{\Omega}$ and belong to the space $C^{l-\sigma_i+\theta}(\overline{\Omega})$. By the same arguments, which we used for the operator A_i, we prove that

$$\|B_iu_{k_j}\|_{W_\infty^{l-\sigma_i-1/p,p}(\partial\Omega)} \longrightarrow 0 \quad \text{as } k_j\longrightarrow\infty. \tag{5.31}$$

We repeat the same construction as above and obtain the following operator:

$$T_{ik}(y) = \sum_{h=1}^{N}\sum_{|\beta|\le\beta_{ih}} b_{ih}{}_k^\beta(y)D^\beta u_{0h}(h_k(y)), \quad y\in\Omega_k, \tag{5.32}$$

where $b_{ih_k}^{\beta}(y) = b_{ih}^{\beta}(y + x_k)$. We prove that

$$||T_{ik}||_{W_\infty^{l-\sigma_i-1/p,p}(\partial\Omega_k \cap V_{j+1})} \longrightarrow 0. \tag{5.33}$$

Indeed, denote

$$g_i(x) = \sum_{h=1}^{N} \sum_{|\beta| \le \beta_{ih}} \hat{b}_{ih}\beta(y) D^\beta u_{0h}(x), \quad x \in \Omega_*. \tag{5.34}$$

This expression equals 0 only at the boundary $\partial\Omega_*$. Therefore instead of what is written above for the operator A_i, we have now

$$T_{ik}(y) = Q_{ik}(y) + g_i(h_k(y)), \tag{5.35}$$

where

$$Q_{ik}(y) = \sum_{h=1}^{N} \sum_{|\beta| \le \beta_{ih}} \left[S_{ih_k}^{\beta}(y) + P_{ih_k}^{\beta}(y) \right]. \tag{5.36}$$

Here S and P are the same as for the operator A but the coefficients a are replaced by b. Exactly as we have done for the operator A we prove that

$$||Q_k||_{W_\infty^{l-\sigma_i,p}(\Omega_k \cap V_{j+1})} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{5.37}$$

It follows that

$$||Q_k||_{W_\infty^{l-\sigma_i-1/p,p}(\partial\Omega_k \cap V_{j+1})} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{5.38}$$

Since for $y \in \partial\Omega_k$ we have $h_k(y) \in \partial\Omega_*$, we have $g_i(h_k(y)) = 0$ for $y \in \partial\Omega_k$. From this, (5.35) and (5.38) we get (5.33). Thus the assertion (i) is proved.

(ii) We prove now that sequence (5.7) does not have a convergent subsequence. Obviously $u_{k_j}(x) = 0$ for $|x| < r_{k_j}$ and, consequently,

$$\int_\Omega u_{k_j}(x)\omega(x)dx \longrightarrow 0 \tag{5.39}$$

as $k_j \to \infty$ for any continuous $\omega(x)$ with a compact support.

For any subsequence s_i of k_j there exists N such that

$$\int_\Omega |u_{s_i}(x)|^p dx \ge \rho \tag{5.40}$$

for $s_i > N$ and some $\rho > 0$. Indeed, let $y = x - x_{s_i}$. Then

$$\begin{aligned} T_i \equiv \int_{\Omega} |u_{s_i}(x)|^p dx &= \int_{\Omega_{s_i}} |v_{s_i}(y)|^p dy \\ &= \int_{\Omega_{s_i} \bigcap V_{j+1}} |\varphi_{s_i}(y) u_0(h_{s_i}(y))|^p dy \\ &\geq \int_{\Omega_{s_i} \bigcap V_{r s_i}} |u_0(h_{s_i}(y))|^p dy. \end{aligned} \tag{5.41}$$

We do the change of variables $y = h_{s_i}^{-1}(x)$ in the last integral. Then

$$T_i \geq \int_{\Omega_* \bigcap W_{s_i}} |u_0(x)|^p \left| \frac{dh_{s_i}^{-1}(x)}{dx} \right| dx, \tag{5.42}$$

where $W_{s_i} = h_{s_i}(V_{r_{s_i}})$.

Since $\|u_0\|_{L^p(\Omega_*)} \neq 0$, there exists a ball $B_l = \{x : |x| < l\}$ and a number $\rho_0 > 0$ such that

$$\int_{\Omega_* \bigcap B_l} |u_0(x)|^p dx \geq \rho_0. \tag{5.43}$$

Increasing N, if necessary, we can suppose that $B_l \subset W_{s_i}$ and $|dh_{s_i}^{-1}(x)/dx| \geq \varepsilon$ for $x \in B_l$ and some $\varepsilon > 0$. The last inequality follows from the fact that according to (5.5) the derivatives of $h_{s_i}(y)$ are uniformly bounded. By (5.43) we get $T_i \geq \varepsilon \rho_0$ and (5.40) is proved.

If (5.7) has a convergent subsequence: $u_{s_i} \to u_*$ in $E(\Omega)$, then this convergence is also in $L^p(\Omega)$. From (5.39) it follows that $u_* = 0$ which contradicts (5.40). Thus the sequence (5.7) is not compact in $E(\Omega)$. The theorem is proved. □

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