EXACT CONTROLLABILITY FOR A SEMILINEAR WAVE EQUATION WITH BOTH INTERIOR AND BOUNDARY CONTROLS

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The exact controllability of a semilinear wave equation in a bounded open domain of R^n, with controls on a part of the boundary and in the interior, is shown. Feedback laws are established.

1. Introduction

The purpose of this paper is to prove the existence of the exact controllability of a semilinear wave equation with both interior and boundary controls.

Let Ω be a bounded open subset of R^n with a smooth boundary, let $f(y)$ be an accretive mapping of $L^2(0,T;H^{-1}(\Omega))$ into $L^2(0,T;H_0^1(\Omega))$ with respect to a duality mapping $J, D(f) = L^2(0,T;L^2(\Omega))$ and having at most a linear growth in y. Consider the initial boundary value problem

$$\begin{gathered} y'' - \Delta y + f(y) = u\chi_\omega \quad \text{in } \Omega \times (0,T), \\ y(x,t) = 0 \quad \text{on } \Gamma_0 \times (0,T), \qquad y(x,t) = v(u) \quad \text{on } \Gamma_1 \times (0,T), \\ y(x,0) = \alpha_0, \qquad y'(x,0) = \alpha_1 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

with

$$\Gamma_0 \bigcup \Gamma_1 = \partial\Omega, \qquad \Gamma_0 \bigcap \Gamma_1 = \varnothing, \qquad \Gamma_1 \neq \varnothing. \tag{1.2}$$

The characteristic function of the subset ω of Ω is χ_ω and the control function u is in a closed, bounded, convex subset $\mathcal{U}$ of $L^2(0,T;L^2(\Omega))$. Given $T > T_0$ and

$$\{\alpha_0, \alpha_1\};\ \{\beta_0, \beta_1\} \quad \text{in } L^2(\Omega) \times H^{-1}(\Omega), \tag{1.3}$$

the aim of the paper is to prove the existence of an optimal $\{\tilde{u}, v(\tilde{u})\} \in \mathcal{U} \times L^2(0,T;L^2(\Gamma_1))$ such that the solution $\tilde{y}$ of (1.1) satisfies

$$y(x,T) = \beta_0, \quad y'(x,T) = \beta_1 \quad \text{in } \Omega. \tag{1.4}$$

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The exact boundary controllability of the wave equation, using the Hilbert uniqueness method of Lions [3, 4], has been extensively investigated, both theoretically and numerically. For the semilinear wave equation, the local controllability was established by Russell [5] and others, using the implicit function theorem. More recently, Zuazua in [8, 9] introduced a variant of the Hilbert uniqueness method and treated the exact boundary controllability of the semilinear wave equation

(i) in the space $L^2(\Omega)\times H^{-1}(\Omega)$ for asymptotically linear mappings f in $W^{1,\infty}_{\text{loc}}(R)$,
(ii) in the space $\bigcup_{\gamma>0} H_0^{\gamma}(\Omega)\times H^{\gamma-1}(\Omega)$ for mappings f with f' in $L^\infty(R)$. The pair $\{\Gamma_0, T\}$ is assumed to have the unique continuation property for the wave equation with zero potential.

In order to handle the nonlinear term, some compactness is needed and thus the introduction in [9] of a smaller space for the exact controllability, where delicate estimates based on interpolation are used. A different approach is taken in this paper, it is based on the theory of accretive operators of Browder [1], Kato [2], and others. By assuming that f is accretive in the appropriate spaces, the passage to the limit can be obtained and the target space is still the largest one, namely, $L^2(\Omega)\times H^{-1}(\Omega)$. The accretiveness hypothesis will replace the condition f' in $L^\infty(R)$.

Exact controllability for the linear wave equation, with both controls in the interior and on the boundary, has been studied by the author in [6] and feedback laws were given. Dirichlet boundary exact controllability of the wave equation has been treated by Triggiani in [7].

Notations, the basic assumptions of the paper, and some preliminary results are given in Section 2. The exact controllability of (1.1)–(1.4) is established in Section 3. Optimal controls are shown in Section 4 and feedback laws are established in Section 5.

2. Notations, assumptions, preliminary results

Throughout the paper, we will denote by $(\cdot,\cdot)$ the $L^2(\Omega)$ inner product as well as the pairing between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$. Let J be the duality mapping of the Hilbert space $L^2(0,T;H^{-1}(\Omega))$ into $(L^2(0,T;H^{-1}(\Omega)))^* = L^2(0,T;H_0^1(\Omega))$ with gauge function $\Phi(r)=r$. We have

$$\begin{gathered}\|Jy\|_{L^2(0,T;H_0^1(\Omega))} = \Phi\big(\|y\|_{L^2(0,T;H^{-1}(\Omega))}\big) = \|y\|_{L^2(0,T;H^{-1}(\Omega))},\\ \int_0^T (Jy,y)\,dt = \|y\|^2_{L^2(0,T;H^{-1}(\Omega))}, \quad \forall y\in L^2(0,T;H^{-1}(\Omega)).\end{gathered} \tag{2.1}$$

Definition 2.1. Let g be a mapping in $L^2(0,T;H^{-1}(\Omega))$, with $D(g)=L^2(0,T;L^2(\Omega))$ and values in $L^2(0,T;H^{-1}(\Omega))$, said to be accretive with respect to J if

$$\int_0^T (g(y)-g(z),J(y-z))\,dt \geq 0 \quad \forall y,z\in L^2(0,T;H^{-1}(\Omega)). \tag{2.2}$$

We will consider mappings f of $L^2(0,T;L^2(\Omega))$ into $L^2(0,T;L^2(\Omega))$ satisfying the following assumption.

Assumption 2.2. Let f be a Lipschitz continuous mapping of $L^2(0,T;L^2(\Omega))$ into $L^2(0,T;L^2(\Omega))$. Suppose that

(i) $\|f(y)\|_{L^2(0,T;L^2(\Omega))} \le C\{1+\|y\|_{L^2(0,T;L^2(\Omega))}\}$ for all $y \in L^2(0,T;L^2(\Omega))$;
(ii) $\lambda I + f$ is accretive in the sense of Definition 2.1 for some $\lambda > \lambda_0 > 0$.

Lemma 2.3. *Let f be as in Assumption 2.2 and suppose that*

$$\{y_n, f(y_n)\} \longrightarrow \{y, \psi\} \tag{2.3}$$

in $\{L^2(0,T;H^{-1}(\Omega)) \cap (L^2(0,T;L^2(\Omega)))_{\text{weak}}\} \times (L^2(0,T;L^2(\Omega)))_{\text{weak}}$. *Then* $\psi = f(y)$.

Proof. (1) From the definition of accretiveness, we get

$$\int_0^T (\lambda(y_n - z) + f(y_n) - f(z), J(y_n - z))\,dt \ge 0, \quad \forall z \in L^2(0,T;L^2(\Omega)). \tag{2.4}$$

It is well known that the duality mapping J is monotone and continuous from the strong topology of $L^2(0,T;H^{-1}(\Omega))$ to the weak topology of $L^2(0,T;H_0^1(\Omega))$.Thus,

$$J(y_n - z) \longrightarrow J(y - z) \quad \text{in } (L^2(0,T;H_0^1(\Omega)))_{\text{weak}}. \tag{2.5}$$

On the other hand,

$$\begin{aligned} &||J(y_n - z)||_{L^2(0,T;H_0^1(\Omega))} \\ &\quad = ||y_n - z||_{L^2(0,T;H^{-1}(\Omega))} \longrightarrow \|y - z\|_{L^2(0,T;H^{-1}(\Omega))} \\ &\quad = \|J(y - z)\|_{L^2(0,T;H_0^1(\Omega))}. \end{aligned} \tag{2.6}$$

But $L^2(0,T;H_0^1(\Omega))$ is a Hilbert space, and thus

$$J(y_n - z) \longrightarrow J(y - z) \quad \text{in } L^2(0,T;H_0^1(\Omega)),\ \forall z \in L^2(0,T;H^{-1}(\Omega)). \tag{2.7}$$

(2) Since

$$||y_n - z||^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T (y_n - z, J(y_n - z))\,dt, \tag{2.8}$$

we obtain

$$\begin{aligned} &\int_0^T ([\lambda + \mu](y_n - z) + f(y_n) - f(z)), J(y_n - z)\,dt \\ &\quad \ge \mu||y_n - z||^2_{L^2(0,T;H^{-1}(\Omega))}, \quad \mu > 0,\ \forall z \in L^2(0,T;L^2(\Omega)). \end{aligned} \tag{2.9}$$

Let $n \to \infty$, and we have

$$\begin{aligned} &\int_0^T ([\lambda + \mu](y - z) + \psi - f(z), J(y - z))\,dt \\ &\quad \ge \mu\|y - z\|^2_{L^2(0,T;H^{-1}(\Omega))}, \quad \forall z \in L^2(0,T;L^2(\Omega)). \end{aligned} \tag{2.10}$$

Since f is Lipschitz continuous, a simple argument using the method of successive approximations shows that $R([\lambda+\mu]I+f)=L^2(0,T;L^2(\Omega))$ for large $\lambda>0$, and $([\lambda+\mu]I+f)$ is 1-1. Therefore, $([\lambda+\mu]I+f)^{-1}$ exists and maps $L^2(0,T;L^2(\Omega))$ into $L^2(0,T;L^2(\Omega))$. Thus for a given $\alpha\in L^2(0,T;L^2(\Omega))$, there exists a unique z_ε such that

$$z_\varepsilon=([\lambda+\mu]I+f)^{-1}\{[\lambda+\mu]y+\psi-\varepsilon\alpha\}. \tag{2.11}$$

Then (2.9), with $z=z_\varepsilon$, becomes

$$\int_0^T(\varepsilon\alpha,J(y-z_\varepsilon))\,dt\geq 0,\quad \forall\alpha\in L^2(0,T;L^2(\Omega)). \tag{2.12}$$

We have

$$([\lambda+\mu]I+f)^{-1}(y+\psi-\varepsilon\alpha)=z_\varepsilon\longrightarrow([\lambda+\mu]I+f)^{-1}(y+\psi) \tag{2.13}$$

in $L^2(0,T;H^{-1}(\Omega))\cap(L^2(0,T;L^2(\Omega)))_{\text{weak}}$ as

$$\begin{aligned}\mu\|z_\varepsilon-z_\nu\|^2_{L^2(0,T;H^{-1}(\Omega))}&\leq(\varepsilon+\nu)\|\alpha\|_{L^2(0,T;L^2(\Omega))}\|J(z_\varepsilon-z_\nu)\|_{L^2(0,T;H_0^1(\Omega))}\\&\leq(\varepsilon+\nu)\|\alpha\|_{L^2(0,T;L^2(\Omega))}\|z_\varepsilon-z_\nu\|_{L^2(0,T;H^{-1}(\Omega))}.\end{aligned} \tag{2.14}$$

We get

$$\begin{aligned}\lim_{\varepsilon\to0}\int_0^T(\alpha,J(y-z_\varepsilon))\,dt&=\lim_{\varepsilon\longrightarrow0}\int_0^T\left(\alpha,J\left(y-([\lambda+\mu]I+f)^{-1}([\lambda+\mu]y+\psi-\varepsilon\alpha)\right)\right)dt\\&=\int_0^T\left(\alpha,J\left(y-([\lambda+\mu]I+f)^{-1}([\lambda+\mu]y+\psi)\right)\right)dt\\&\geq0,\quad\forall\alpha\in L^2(0,T;L^2(\Omega)).\end{aligned} \tag{2.15}$$

Therefore,

$$y=([\lambda+\mu]I+f)^{-1}([\lambda+\mu]y+\psi),\quad\text{i.e.,}\quad[\lambda+\mu]y+f(y)=[\lambda+\mu]y+\psi;f(y)=\psi. \tag{2.16}$$

The lemma is proved. □

Remark 2.4. Suppose that f is a continuous mapping of $L^2(0,T;L^2(\Omega))$ into itself and that f' is in $L^\infty(R)$ with

$$\sup_R|f'|\leq c. \tag{2.17}$$

Then $(\lambda I + f)$ is accretive in $L^2(0,T;H^{-1}(\Omega))$, with respect to the duality mapping J, for $\lambda > c$. Indeed, we have

$$\begin{aligned}
&\int_0^T (\lambda(y-z) + f(y) - f(z), J(y-z))dt \\
&\qquad \ge \int_0^T (\lambda(y-z), J(y-z))dt \\
&\qquad\quad - c\|y-z\|_{L^2(0,T;H^{-1}(\Omega))}\|J(y-z)\|_{L^2(0,T;H_0^1(\Omega))} \\
&\qquad \ge (\lambda - c)\int_0^T (y-z, J(y-z))dt \\
&\qquad = (\lambda - c)\|y-z\|^2_{L^2(0,T;H^{-1}(\Omega))} \ge 0
\end{aligned} \tag{2.18}$$

for all y,z in $L^2(0,T;L^2(\Omega))$.

3. Existence theorem

The main result of the section is the following theorem.

Theorem 3.1. *Let f be as in Assumption 2.2, let*

$$\alpha = \{\alpha_0,\alpha_1\}, \quad \beta = \{\beta_0,\beta_1\} \quad \text{be in } L^2(\Omega)\times H^{-1}(\Omega);\ u \text{ in } \mathcal{U}. \tag{3.1}$$

Then for $T \ge T_0$, there exists a solution y of (1.1)–(1.4). Moreover,

$$\|y\|_{C(0,T;L^2(\Omega))} + \|y'\|_{C(0,T;H^{-1}(\Omega))} \le \mathscr{E}(u,\alpha,\beta) \tag{3.2}$$

with

$$\mathscr{E}(u;\alpha,\beta) = \Big\{\|u\|_{L^2(0,T;L^2(\Omega))} + ||\alpha_0||_{L^2(\Omega)} + ||\alpha_1||_{H^{-1}(\Omega)} + ||\beta_0||_{L^2(\Omega)} + ||\beta_1||_{H^{-1}(\Omega)}\Big\}. \tag{3.3}$$

The constant C is independent of u, α, β.

Consider the exact controllability of the linear wave equation

$$\begin{gathered}
y_1'' - \Delta y_1 = u\chi_\omega \quad \text{in } \Omega\times(0,T), \\
y_1(x,t) = 0 \quad \text{on } \Gamma_0\times(0,T), \qquad y_1(x,t) = v_1(u) \quad \text{on } \Gamma_1\times(0,T), \\
y_1(x,0) = \alpha_0, \qquad y_1'(x,0) = \alpha_1 \quad \text{in } \Omega, \\
y_1(x,T) = \beta_0, \qquad y_1(x,T) = \beta_1 \quad \text{in } \Omega.
\end{gathered} \tag{3.4}$$

The following result has been proved by the author in [6].

Lemma 3.2. *Let $u \in \mathcal{U}$ and let $\{\alpha,\beta\}$ be in $L^2(\Omega)\times H^{-1}(\Omega)$,then for $T \ge T_0$, there exist $v_1(u) \in L^2(0,T;L^2(\Gamma_1))$ and a unique solution y_1 of (3.4). Moreover,*

$$||y_1||_{C(0,T;L^2(\Omega))} + ||y_1'||_{C(0,T;H^{-1}(\Omega))} + ||v_1||_{L^2(0,T;L^2(\Gamma_1))} \le C\mathscr{E}(u;\alpha,\beta). \tag{3.5}$$

The constant C is independent of u, α, β.

Consider the initial boundary value problem

$$\begin{gathered} y_2'' - \Delta y_2 = 0 \quad \text{in } \Omega \times (0,T), \\ y_2(x,t) = 0 \quad \text{on } \Gamma_0 \times (0,T), \qquad y_2(x,t) = v_2 \quad \text{on } \Gamma_1 \times (0,T), \\ y_2(x,0) = 0, \qquad y_2'(x,0) = 0 \quad \text{in } \Omega, \end{gathered} \tag{3.6}$$

where $v_2 = n \cdot \nabla \varphi$ with φ being the unique solution of the initial boundary value problem

$$\begin{gathered} \varphi'' - \Delta \varphi = 0 \quad \text{in } \Omega \times (0,T), \\ \varphi = 0 \quad \text{on } \partial\Omega \times (0,T), \\ \varphi(x,T) = g_0, \quad \varphi'(x,T) = g_1 \quad \text{in } \Omega. \end{gathered} \tag{3.7}$$

We have the following known result.

Lemma 3.3. *Let $\{g_0, g_1\}$ be in $H_0^1(\Omega) \times L^2(\Omega)$, then there exists a unique solution y_2 of (3.6). Moreover,*

$$\|y_2\|_{C(0,T;L^2(\Omega))} + \|y_2'\|_{C(0,T;H^{-1}(\Omega))} + \|v_2\|_{L^2(0,T;L^2(\Gamma_1))} \leq C\Big\{\|g_0\|_{H_0^1(\Omega)} + \|g_1\|_{L^2(\Omega)}\Big\}. \tag{3.8}$$

The constant C is independent of g_0, g_1.

Let Λ be the mapping of $H_0^1(\Omega) \times L^2(\Omega)$ into its dual $H^{-1}(\Omega) \times L^2(\Omega)$, defined by

$$\Lambda(\mathbf{g}) = \{y_2'(x,T), -y_2(x,T)\}. \tag{3.9}$$

It is well known in the Hilbert uniqueness method that Λ is an isomorphism of $H_0^1(\Omega) \times L^2(\Omega)$ onto $H^{-1}(\Omega) \times L^2(\Omega)$.

We now consider the nonlinear initial boundary value problem

$$\begin{gathered} y_3'' - \Delta y_3 = -f(y_1 + y_2 + y_3) \quad \text{in } \Omega \times (0,T), \\ y_3(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ y_3(x,0) = 0, \quad y_3'(x,0) = 0 \quad \text{in } \Omega. \end{gathered} \tag{3.10}$$

Lemma 3.4. *Let f be as in Assumption 2.2 and let $\{y_1, y_2\}$ be as in Lemmas 3.2 and 3.3. Then there exists a solution y_3 of (3.10). Moroever,*

$$\|y_3\|_{C(0,T;H_0^1(\Omega))} + \|y_3'\|_{C(0,T;L^2(\Omega))} \leq C\Big\{\mathscr{E}(u;\alpha,\beta) + \|g_0\|_{H_0^1(\Omega)} + \|g_1\|_{L^2(\Omega)}\Big\}. \tag{3.11}$$

The constant C is independent of u, α, β_0, $\mathbf{g}$.

Proof. (1) Consider the system

$$
\begin{gathered}
y'' - \Delta y = -f(y_1 + y_2 + z) \quad \text{in } \Omega \times (0,T), \\
y = 0 \quad \text{on } \partial\Omega \times (0,T), \\
y(x,0) = 0, \quad y'(x,0) = 0 \quad \text{in } \Omega.
\end{gathered}
\tag{3.12}
$$

Let z be an element of the set

$$
\begin{aligned}
\mathcal{B}_C = \Big\{ z : \|z\|_{L^2(0,t;H^1(\Omega))} + \|z'\|_{L^2(0,t;L^2(\Omega))} \\
\leq C\Big\{ \mathcal{E}(u;\alpha,\beta) + \|g_0\|_{H_0^1(\Omega)} + \|g_1\|_{L^2(\Omega)} \Big\} \exp(Ct);\ t \in [0,T] \Big\}.
\end{aligned}
\tag{3.13}
$$

Clearly, there exists a unique solution y of the above initial boundary value problem with

$$
\begin{aligned}
\|y(\cdot,t)\|_{H_0^1(\Omega)} + \|y'(\cdot,t)\|_{L^2(\Omega)} \\
\leq C\Big\{ \|y_1\|_{L^2(0,t;L^2(\Omega))} + \|y_2\|_{L^2(0,t;L^2(\Omega))} + \|z\|_{L^2(0,t;L^2(\Omega))} \Big\}.
\end{aligned}
\tag{3.14}
$$

Taking into account the estimates of Lemmas 3.2 and 3.3, we obtain

$$
\begin{aligned}
\|y(\cdot,t)\|_{H_0^1(\Omega)} + \|y'(\cdot,t)\|_{L^2(\Omega)} \\
\leq C\Big\{ \mathcal{E}(u;\alpha,\beta) + \|g_1\|_{L^2(\Omega)} + \int_0^t \|z(\cdot,s)\|_{L^2(\Omega)}\, ds \Big\}.
\end{aligned}
\tag{3.15}
$$

Since z is in $\mathcal{B}_C$, it follows that

$$
\begin{aligned}
\|y(\cdot,t)\|_{H_0^1(\Omega)} + \|y'(\cdot,t)\|_{L^2(\Omega)} \\
\leq C\Big\{ \mathcal{E}(u;\alpha,\beta) + \|g_0\|_{H_0^1(\Omega)} + \|g_1\|_{L^2(\Omega)} \Big\} \exp(Ct)
\end{aligned}
\tag{3.16}
$$

for all $t \in [0,T]$, and thus $y \in \mathcal{B}_C$.

(2) Let $\mathcal{A}$ be the nonlinear mapping of $\mathcal{B}_C$, considered as a closed convex subset of $L^2(0,T;L^2(\Omega))$ into $L^2(0,T;L^2(\Omega))$ defined by

$$
\mathcal{A}(z) = y. \tag{3.17}
$$

We will show that $\mathcal{A}$ satisfies the hypotheses of the Schauder fixed point theorem.

Let $\{z_n\}$ be in $\mathcal{B}_C$ and let $y_n = \mathcal{A}(z_n)$. From Aubin's theorem we get subsequences, denoted again by $\{y_n, z_n\}$ such that $\{y_n, z_n\} \to \{y, z\}$ in

$$
\Big\{ L^2(0,T;L^2(\Omega)) \cap \big(L^\infty(0,T;H^1(\Omega))\big)_{\text{weak}^*} \Big\}^2. \tag{3.18}
$$

Since f is a continuous mapping of $L^2(0,T;L^2(\Omega))$ into $L^2(0,T;L^2(\Omega))$, we get $\mathcal{A}(z) = y$. It follows from the Schauder fixed point theorem that there exists y_3 in $\mathcal{B}_C$, solution of (3.10). With f being Lipschitz continuous, the solution is unique and the lemma is proved. □

Let $\mathcal{L}$ be the nonlinear mapping of $H_0^1(\Omega)\times L^2(\Omega)$ into $H^{-1}(\Omega)\times L^2(\Omega)$ defined by

$$\mathcal{L}(\{g_0,g_1\}) = \mathcal{L}(\mathbf{g}) = \{-y_3'(x,T), y_3(x,T)\}. \tag{3.19}$$

Since Λ is an isomorphism of $H_0^1(\Omega)\times L^2(\Omega)$ onto $H^{-1}(\Omega)\times L^2(\Omega)$, its inverse Λ^{-1} is well defined. We consider the nonlinear mapping

$$\mathcal{K} = \Lambda^{-1}\mathcal{L}. \tag{3.20}$$

It is clear that $\mathcal{K}$ is a nonlinear mapping of $H_0^1(\Omega)\times L^2(\Omega)$ into $H_0^1(\Omega)\times L^2(\Omega)$. We will now show that $\mathcal{K}$ has a fixed point

$$\mathcal{K}(\mathbf{g}) = \mathbf{g}, \quad \text{i.e.,} \quad \mathcal{L}(\mathbf{g}) = \Lambda(\mathbf{g}), \tag{3.21}$$

and thus

$$\{-y_3'(x,T), y_3(x,T)\} = \{y_2'(x,T), -y_2(x,T)\}. \tag{3.22}$$

Let $\mathcal{B}_{\hat{C}}$ be the set

$$\mathcal{B}_{\hat{C}} = \Big\{\mathbf{g} : \mathbf{g} = \{g_0,g_1\}; \|g_0\|_{H_0^1(\Omega)} + \|g_1\|_{L^2(\Omega)} \le \mathcal{E}(u;\alpha,\beta)\Big\}. \tag{3.23}$$

It follows from the Sobolev embedding theorem that $\mathcal{B}_{\hat{C}}$ is a compact convex subset of $L^2(\Omega)\times H^{-1}(\Omega)$.

Since Λ is an isomorphism of $H_0^1(\Omega)\times L^2(\Omega)$ onto $H^{-1}(\Omega)\times L^2(\Omega)$, we have

$$c\|\mathbf{h}\|_{H_0^1(\Omega)\times L^2(\Omega)} \le \|\Lambda(\mathbf{h})\|_{H^{-1}(\Omega)\times L^2(\Omega)} \le C\|\mathbf{h}\|_{H_0^1(\Omega)\times L^2(\Omega)} \tag{3.24}$$

for all $\mathbf{h}\in H_0^1(\Omega)\times L^2(\Omega)$.

Lemma 3.5. *Let $\mathcal{K}$ be as in (3.20), then it maps $\mathcal{B}_{\hat{C}}$ into $\mathcal{B}_{\hat{C}}$ with*

$$\hat{C} = \sup\{c^{-1}C\mathcal{E}(u;\alpha,\beta), C\mathcal{E}(u;\alpha,\beta)\}. \tag{3.25}$$

Proof. (1) Let $\mathbf{g}$ be in $\mathcal{B}_{\hat{C}}$, then

$$\mathcal{L}(\mathbf{g}) = \{-y_3'(x,T), y_3(x,T)\}, \tag{3.26}$$

and we obtain from the estimates of Lemma 3.4

$$\|\mathcal{L}(\mathbf{g})\|_{H^{-1}(\Omega)\times L^2(\Omega)} \le C\mathcal{E}(u;\alpha,\beta). \tag{3.27}$$

Thus,

$$\begin{aligned} \|\Lambda^{-1}\mathcal{K}(\mathbf{g})\|_{H_0^1(\Omega)\times L^2(\Omega)} &\le c^{-1}\|\mathcal{K}(\mathbf{g})\|_{L^2(\Omega)\times H^{-1}(\Omega)} \\ &\le c^{-1}C\mathcal{E}(u;\alpha,\beta) \le \hat{C}. \end{aligned} \tag{3.28}$$

□

LEMMA 3.6. *Let $\mathcal{K}$ be given by (3.20), then it has a fixed point in $\mathcal{B}_{\hat{C}}$.*

Proof. In view of Lemma 3.5, it suffices to show that $\mathcal{K}$ is a continuous mapping of $L^2(\Omega)\times H^{-1}(\Omega)$ into $L^2(\Omega)\times H^{-1}(\Omega)$ as the set $\mathcal{B}_{\hat{C}}$ is a compact convex subset of $L^2(\Omega)\times H^{-1}(\Omega)$.

Let $\mathbf{g}^n \in \mathcal{B}_{\hat{C}}$, then there exists a subsequence such that

$$\mathbf{g}^n \longrightarrow \tilde{\mathbf{g}} \quad \text{in } L^2(\Omega)\cap \big(H_0^1(\Omega)\big)_{\text{weak}} \times H^{-1}(\Omega)\cap \big(L^2(\Omega)\big)_{\text{weak}}. \tag{3.29}$$

Set

$$\mathcal{L}\mathbf{g}^n = \{-y'_{3,n}(\cdot,T), y_{3,n}(\cdot,T)\}, \tag{3.30}$$

where $y_{3,n}$ is the solution of (3.10), $y_{2,n}$ is the solution of (3.6) with $\mathbf{g}=\mathbf{g}^n$.

It follows from the estimates of Lemmas 3.3 and 3.4 that

$$\{y_{2,n}, y'_{2,n}, v_{2,n}\} \longrightarrow \{y_2, y'_2, v_2\} \tag{3.31}$$

in

$$\begin{aligned} C(0,T;H^{-1}(\Omega)) \cap \big(L^\infty(0,T;L^2(\Omega))\big)_{\text{weak}^*} &\times \big(L^\infty(0,T;H^{-1}(\Omega))\big)_{\text{weak}^*} \\ &\times \big(L^2(0,T;L^2(\Gamma_1))\big)_{\text{weak}} \end{aligned} \tag{3.32}$$

and $\{y_{3,n}, y'_{3,n}\} \to \{y_3, y'_3\}$ in

$$\begin{aligned} C(0,T;L^2(\Omega)) \cap \big(L^\infty(0,T;H_0^1(\Omega))\big)_{\text{weak}^*} &\times \big(L^\infty(0,T;L^2(\Omega))\big)_{\text{weak}^*} \\ &\cap L^2(0,T;H^{-1}(\Omega)). \end{aligned} \tag{3.33}$$

It is trivial to check that y_2 is the solution of (3.6). We now use Assumption 2.2 to show that y_3 is the solution of (3.10). Indeed,

$$y_{2,n} + y_{3,n} \longrightarrow y_2 + y_3 \quad \text{in } L^2(0,T;H^{-1}(\Omega)) \cap \big(L^2(0,T;L^2(\Omega))\big)_{\text{weak}} \tag{3.34}$$

and $f(y_1 + y_{2,n} + y_{3,n}) \to \psi$ weakly in $L^2(0,T;L^2(\Omega))$. Since f is accretive in $L^2(0,T;H^{-1}(\Omega))$, it follows from Lemma 2.3 that $\psi = f(y_1+y_2+y_3)$, and hence

$$\mathcal{L}\mathbf{g}^n \longrightarrow \mathcal{L}\mathbf{g} \quad \text{in } L^2(\Omega)\cap \big(H_0^1(\Omega)\big)_{\text{weak}} \times H^{-1}(\Omega)\cap \big(L^2(\Omega)\big)_{\text{weak}}. \tag{3.35}$$

Let

$$\mathcal{K}\mathbf{g}^n = \Lambda^{-1}\mathcal{L}(\mathbf{g}^n) = \mathbf{h}^n, \tag{3.36}$$

then

$$\Lambda(\mathbf{h}^n) = \mathcal{L}(\mathbf{g}^n) = \{-y'_{3,n}(\cdot,T), y_{3,n}(\cdot,T)\} = \{\hat{y}'_{2,n}(\cdot,T), -\hat{y}_{2,n}(\cdot,T)\} \tag{3.37}$$

and $\hat{y}_{2,n}$ is the unique solution of (3.6) with $\mathbf{g}=\mathbf{h}^n$.

With $\mathbf{h}^n \in \mathcal{B}_{\hat{C}}$ and with the estimates of Lemma 3.3, we get

$$\{\mathbf{h}^n, \hat{y}_{2,n}, \hat{y}'_{2,n}\} \longrightarrow \{\mathbf{h}, \hat{y}_2, \hat{y}'_2\} \tag{3.38}$$

in

$$\begin{aligned}(H_0^1(\Omega))_{\text{weak}} \cap L^2(\Omega) \times (L^2(\Omega))_{\text{weak}} \cap H^{-1}(\Omega) \times (L^\infty(0,T;L^2(\Omega)))_{\text{weak}^*} \\ \times (L^\infty(0,T;H^{-1}(\Omega)))_{\text{weak}^*}.\end{aligned} \tag{3.39}$$

Furthermore,

$$\{\hat{y}_{2,n}(\cdot,T), \hat{y}'_{2,n}(\cdot,T)\} \longrightarrow \{\hat{y}_2(\cdot,T), \hat{y}'_2(\cdot,T)\} \quad \text{in } H^{-1}(\Omega) \times H^{-2}(\Omega). \tag{3.40}$$

Moreover, $\Lambda(\mathbf{h}) = \{\hat{y}'_2(\cdot,T), -\hat{y}_2(\cdot,T)\}$. It follows that

$$\Lambda(\mathbf{h}) = \{\hat{y}'_2(\cdot,T), -\hat{y}_2(\cdot,T)\} = \{-y'_3(\cdot,T), y_3(\cdot,T)\} = \mathcal{L}(\mathbf{g}). \tag{3.41}$$

Hence,

$$\mathbf{h}^n = \Lambda^{-1}\mathcal{L}\mathbf{g}^n = \mathcal{K}\mathbf{g}^n \longrightarrow \mathbf{h} = \Lambda^{-1}\mathcal{L}(\mathbf{g}) = \mathcal{K}\mathbf{g} \tag{3.42}$$

in $(L^2(\Omega)) \times (H^{-1}(\Omega))$. The nonlinear mapping $\mathcal{K}$ satisfies the hypotheses of the Schauder fixed point theorem, and thus there exists $\tilde{\mathbf{g}} \in \mathcal{B}_{\hat{C}}$ such that

$$\mathcal{K}\tilde{\mathbf{g}} = \Lambda^{-1}\mathcal{L}\tilde{\mathbf{g}} = \tilde{\mathbf{g}}. \tag{3.43}$$

□

Proof of Theorem 3.1. In view of Lemma 3.6, there exists $\tilde{\mathbf{g}} \in \mathcal{B}_{\hat{C}}$ such that

$$\{-y'_3(\cdot,T), y_3(\cdot,T)\} = \{y'_2(\cdot,T), -y_2(\cdot,T)\} \tag{3.44}$$

with y_2, y_3 being the unique solution of (3.6), (3.10), respectively, and with $\mathbf{g} = \tilde{\mathbf{g}}$.

Let $u \in \mathcal{U}$, then it is clear that

$$\tilde{y} = y_1 + y_2 + y_3, \qquad \tilde{v}(u) = v_1 + v_2 \tag{3.45}$$

are a solution of (1.1)–(1.4). The estimate of the theorem is an immediate consequence of those of Lemmas 3.2–3.6. □

4. Optimal control

We associate with (1.1)–(1.4) the cost function

$$\mathcal{J}(y;u;\alpha;\beta) = \int_0^T \int_\Omega |y(x,t)|\,dx\,dt, \tag{4.1}$$

where y is a solution of (1.1)–(1.4) given by Theorem 3.1. The main result of the section is the following theorem.

Theorem 4.1. *Let f be as in Assumption 2.2, let*

$$\{\alpha_0,\alpha_1\},\{\beta_0,\beta_1\} \quad \text{be in } L^2(\Omega)\times H^{-1}(\Omega), \tag{4.2}$$

then for $T > T_0$, there exists $\tilde{u}\in\mathcal{U}$, and

$$\{\tilde{y},\tilde{y}',\tilde{v}(\tilde{u})\}\in C(0,T;L^2(\Omega))\times C(0,T;H^{-1}(\Omega))\times L^2(0,T;L^2(\Gamma_1)) \tag{4.3}$$

such that

$$V(\alpha,\beta)=\mathcal{J}(\tilde{y};\tilde{u};\alpha,\beta)=\inf\{\mathcal{J}(y;u;\alpha,\beta):\forall u\in\mathcal{U}\}. \tag{4.4}$$

Proof. (1) Let $\{u_n,v_n,y_n\}$ be a minimizing sequence of the optimization problem (4.4) with

$$\mathcal{J}(y_n;u_n;\alpha,\beta)\le V(\alpha,\beta)+1/n. \tag{4.5}$$

From the estimates of Theorem 3.1, we have

$$\begin{aligned}||y_n||_{C(0,T;L^2(\Omega))}+||y_n'||_{C(0,T;H^{-1}(\Omega))}+||v_n||_{L^2(0,T;L^2(\Gamma_1))}&\le C\mathcal{E}(u_n;\alpha,\beta)\\&\le C\{1+\mathcal{E}(u;\alpha,\beta)\},\end{aligned} \tag{4.6}$$

as $\mathcal{U}$ is a bounded subset of $L^2(0,T;L^2(\Omega))$. Thus there exists a subsequence such that

$$\{y_n,y_n',u_n,v_n\}\longrightarrow\{\tilde{y},\tilde{y}',\tilde{u},\tilde{v}\} \tag{4.7}$$

in

$$\begin{gathered}C(0,T;H^{-1}(\Omega))\cap(L^\infty(0,T;L^2(\Omega)))_{\text{weak}^*}\times(L^\infty(0,T;H^{-1}(\Omega)))_{\text{weak}^*},\\C(0,T;H^{-2}(\Omega))\times(L^2(0,T;L^2(\Omega)))_{\text{weak}}\times(L^2(0,T;L^2(\Gamma_1)))_{\text{weak}}.\end{gathered} \tag{4.8}$$

We now show that $\{\tilde{y},\tilde{u},\tilde{v}\}$ is a solution of (1.1)–(1.4) and it is clear that it suffices to prove that

$$f(y_n)\longrightarrow f(\tilde{y}) \quad \text{in } (L^2(0,T;L^2(\Omega)))_{\text{weak}}. \tag{4.9}$$

Since $f(y_n)\to\psi$ weakly in $L^2(0,T;L^2(\Omega))$ and since f is accretive in $L^2(0,T;H^{-1}(\Omega))$, it follows from Lemma 2.3 that $\psi=f(\tilde{y})$. The theorem is now an immediate consequence of (3.7). □

Lemma 4.2. *Let V be as the value function associated with (1.1)–(1.4) and the cost function (4.1). Then,*

$$|V(\alpha;\beta)-V(\gamma;\beta)|\le C\Big\{||\alpha_0-\gamma_0||_{L^2(\Omega)}+||\alpha_1-\gamma_1||_{H^{-1}(\Omega)}\Big\} \tag{4.10}$$

for all α, β, γ in $L^2(\Omega)\times H^{-1}(\Omega)$. The constant C is independent of α, β, γ.

Proof. Let α, β be in $L^2(\Omega) \times H^{-1}(\Omega)$, then it follows from Theorem 4.1 that

$$V(\alpha,\beta) = \mathcal{J}(\tilde{y};\tilde{u};\tilde{v}(\tilde{u});\alpha,\beta). \tag{4.11}$$

Then,

$$\begin{aligned} V(\gamma,\beta) - V(\alpha,\beta) &\le \mathcal{J}(z;\tilde{u},\hat{v}(\tilde{u}),\gamma,\beta) - \mathcal{J}(\tilde{y};\tilde{u},\tilde{v}(\tilde{u}),\gamma,\beta) \\ &\le \int_0^T \int_\Omega \{|z| - |\tilde{y}|\} dx\, dt \\ &\le \int_0^T \int_\Omega |z - \tilde{y}| dx\, dt \\ &\le C\|z - \tilde{y}\|_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{4.12}$$

On the other hand, we have

$$\begin{gathered} (z-\tilde{y})'' - \Delta(z-\tilde{y}) = f(\tilde{y}) - f(z) \quad \text{in } \Omega \times (0,T), \\ z - \tilde{y} = 0 \quad \text{on } \Gamma_0 \times (0,T), \qquad z - \tilde{y} = \hat{v} - \tilde{v} \quad \text{on } \Gamma_1 \times (0,T), \\ z - \tilde{y}|_{t=0} = \gamma_0 - \alpha_0, \qquad (z-\tilde{y})'|_{t=0} = \gamma_1 - \alpha_1 \quad \text{in } \Omega, \\ z - \tilde{y}|_{t=0} = 0 = (z-\tilde{y})'|_{t=0} \quad \text{in } \Omega. \end{gathered} \tag{4.13}$$

Applying Theorem 3.1 with

$$y = z - \tilde{y}, \qquad v_* = \hat{v} - \tilde{v}, \tag{4.14}$$

we have

$$V(\gamma,\beta) - V(\alpha,\beta) \le C\Big\{\|\alpha_0 - \gamma_0\|_{L^2(\Omega)} + \|\alpha_1 - \gamma_1\|_{H^{-1}(\Omega)}\Big\}. \tag{4.15}$$

Thus,

$$V(\gamma,\beta) - V(\alpha,\beta) \le C\Big\{\|\alpha_0 - \gamma_0\|_{L^2(\Omega)} + \|\alpha_1 - \gamma_1\|_{H^{-1}(\Omega)}\Big\}. \tag{4.16}$$

Reversing the role played by α, γ, we get the stated result. □

Consider the following initial boundary value poblem for the heat equation

$$\begin{gathered} \varphi' - \Delta\varphi = h \quad \text{in } \Omega \times (0,T), \\ \varphi(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad \varphi(x,0) = 0 \quad \text{in } \Omega. \end{gathered} \tag{4.17}$$

Let S be the linear mapping of $L^2(0,T;H^{-1}(\Omega))$ into $L^2(0,T;H_0^1(\Omega))$ given by

$$Sh = \varphi, \tag{4.18}$$

where φ is the unique solution of (4.17). Then S is a compact linear mapping of $L^2(0,T;H^{-1}(\Omega))$ into $L^2(0,T;L^2(\Omega))$ and

$$\|\varphi\|_{C(0,T;L^2(\Omega))} = \|Sh\|_{C(0,T;L^2(\Omega))} \le C\|h\|_{L^2(0,T;H^{-1}(\Omega))}. \tag{4.19}$$

Set

$$\mathcal{V}(\alpha,\beta) = V(S\alpha_0, S\alpha_1;\beta). \tag{4.20}$$

The following lemma will be needed in the establishment of the feedback laws.

Lemma 4.3. *Let $\alpha(\tau)$, β be in $L^2(\Omega)\times H^{-1}(\Omega)$ and let $\mathcal{V}(\cdot;\tau)$ be as in (4.20). Then $\partial_0\mathcal{V}(\alpha_0(\tau),\alpha_1(\tau);\beta;\tau)$, the subgradient of $\mathcal{V}$ with respect to $S\alpha_0$, exists and is a set-valued mapping of $L^2(\Omega)$ into the closed convex subsets of $L^2(\Omega)$. Moreover,*

$$||p(\alpha(\tau);\beta;\cdot)||_{L^2(\Omega)} \le C, \quad \forall p \in \partial_0\mathcal{V}(\alpha_0(\tau),\alpha_1(\tau);\beta;\tau). \tag{4.21}$$

Proof. From Lemma 4.2, we have

$$\begin{aligned} |V(S\alpha_0(\tau),S\alpha_1(\tau);\beta;\tau) - V(S\gamma_0,S\alpha_1;\beta;\tau)| &\le C||S\alpha_0(\cdot,\tau) - S\gamma_0(\cdot,\tau)||_{L^2(\Omega)} \\ &\le C||\alpha_0(\tau)-\gamma_0(\tau)||_{L^2(\Omega)}. \end{aligned} \tag{4.22}$$

The constant C is independent of τ, α, β. Hence, the generalized Clarke subgradient $\partial_0\mathcal{V}(\alpha_0,\alpha_1;\beta;\tau)$ with respect to $S\alpha_0$ exists and is a set-valued mapping of $L^2(\Omega)$ into the closed convex subsets of $L^2(\Omega)$. Furthermore,

$$||p(\alpha(\tau),\beta;\cdot)||_{L^2(\Omega)} \le C, \quad \forall p \in \partial_0\mathcal{V}(\alpha_0(\tau),\alpha_1(\tau);\beta;\tau). \tag{4.23}$$

The lemma is proved. □

With $y \in C(0,T;L^2(\Omega))\times C(0,T;H^{-1}(\Omega))$, we have $\partial_0\mathcal{V}(\gamma_0;\gamma_1;\beta;\tau)$ in $L^2(0,T;L^2(\Omega))$.

5. Feedback laws

In order to establish the feedback laws, we will first consider a nonlinear semilinear wave equation. Let f be as in Assumption 2.2 and let $\mathcal{V}$ be as in Lemma 4.3. Consider the problem

$$\begin{gathered} y'' - \Delta y + f(y) = \mathcal{P}p(y;\alpha,\beta;t)\chi_\omega \quad \text{in } \Omega\times(0,T), \\ y(x,t) = 0 \quad \text{on } \Gamma_0\times(0,T), \qquad y(x,t) = v(y) \quad \text{on } \Gamma_1\times(0,T), \\ y(x,0) = \alpha_0, \qquad y'(x,0) = \alpha_1 \quad \text{in } \Omega, \\ y(x,T) = \beta_0, \qquad y'(x,T) = \beta_1 \quad \text{in } \Omega, \end{gathered} \tag{5.1}$$

where $p(y;\alpha,\beta;t)$ is the element of minimum $L^2(0,T;L^2(\Omega))$-norm of the closed convex set $\partial_0\mathcal{V}(y(\cdot,t),y'(\cdot,t);\beta;t)$ and $v(y)\in L^2(0,T;L^2(\Gamma_1))$. The projection of $L^2(0,T;L^2(\Omega))$ onto the closed bounded convex set $\mathcal{U}$ is denoted by $\mathcal{P}$.

Theorem 5.1. *Let f be as in Assumption 2.2 and let $\{\alpha,\beta\}$ be in $L^2(\Omega)\times H^{-1}(\Omega)$. Then for $T > T_0$, there exists*

$$\{\tilde{y},\tilde{y}',v(y)\} \in C(0,T;L^2(\Omega))\times C(0,T;H^{-1}(\Omega))\times L^2(0,T;L^2(\Gamma_1)), \tag{5.2}$$

solution of (5.1).

Let $c = \inf_{\mathcal{U}}\{\|u\|_{L^2(0,T;L^2(\Omega))}\}$ and set

$$\mathcal{B}_{(\tilde{C})} = \Big\{y : \|y\|_{C(0,T;L^2(\Omega))} + \|y'\|_{C(0,T;H^{-1}(\Omega))} \\ \leq C\Big\{c + \|\alpha_0\|_{L^2(\Omega)} + \|\alpha_1\|_{H^{-1}(\Omega)} + \|\beta_0\|_{L^2(\Omega)} + \|\beta_1\|_{H^{-1}(\Omega)}\Big\}\Big\}. \tag{5.3}$$

Let $z \in \mathcal{B}_{\tilde{C}}$ and consider the exact controllability problem

$$\begin{gathered} y'' - \Delta y + f(y) = \mathcal{P}p(z;\alpha,\beta;t)\chi_\omega \quad \text{in } \Omega\times(0,T), \\ y(x,t) = 0 \quad \text{on } \Gamma_0\times(0,T), \qquad y(x,t) = v(z) \quad \text{on } \Gamma_1\times(0,T), \\ y(x,0) = \alpha_0, \qquad y'(x,0) = \alpha_1 \quad \text{in } \Omega, \\ y(x,T) = \beta_0, \qquad y'(x,T) = \beta_1 \quad \text{in } \Omega. \end{gathered} \tag{5.4}$$

Lemma 5.2. *Suppose the hypotheses of Theorem 5.1 are satisfied and let z be in $\mathcal{B}_{\tilde{C}}$, then for $T > T_0$, there exist $v(z) \in L^2(0,T;L^2(\Gamma_1))$ and a unique y, solution of (5.4). Moreover,*

$$\|y\|_{C(0,T;L^2(\Omega))} + \|y'\|_{C(0,T;H^{-1}(\Omega))} \leq \tilde{C} \tag{5.5}$$

with

$$\tilde{C} = C\Big\{c + \|\alpha_0\|_{L^2(\Omega)} + \|\alpha_1\|_{H^{-1}(\Omega)} + \|\beta_0\|_{L^2(\Omega)} + \|\beta_1\|_{H^{-1}(\Omega)}\Big\}. \tag{5.6}$$

Furthermore,

$$\|v(z)\|_{L^2(0,T;L^2(\Gamma_1))} \leq C(\alpha,\beta). \tag{5.7}$$

The constants C are independent of z, α, β.

Proof. With $u = \mathcal{P}p(z;\alpha,\beta)$, the lemma follows from Theorem 3.1. □

Let $\mathcal{A}$ be the nonlinear mapping of $\mathcal{B}_{\tilde{C}}$, considered as a subset of $L^2(0,T;H^{-1}(\Omega))$ into $L^2(0,T;H^{-1}(\Omega))$ defined by

$$\mathcal{A}z = y, \tag{5.8}$$

where y is the unique solution of (5.4). It is clear that $\mathcal{B}_{\tilde{C}}$ is a compact, convex subset of $L^2(0,T;H^{-1}(\Omega))$. We now show that $\mathcal{A}$ has a fixed point.

Lemma 5.3. *The nonlinear mapping $\mathcal{A}$ given by (5.8) maps $\mathcal{B}_{\tilde{C}}$ into $\mathcal{B}_{\tilde{C}}$.*

Proof. The lemma is obvious. □

Lemma 5.4. *Let $\mathcal{A}$ be as in (5.8), then it is continuous from $L^2(0,T;H^{-1}(\Omega))$ into $L^2(0,T;H^{-1}(\Omega))$.*

Proof. (1) Let $z_n \in \mathcal{B}_{\tilde{C}}$ and let $y_n = \mathcal{A} z_n$. Then we have

$$\begin{aligned} \|y_n\|_{C(0,T;L^2(\Omega))} + \|y_n'\|_{C(0,T;H^{-1}(\Omega))} + \|\nu(z_k)\|_{L^2(0,T;L^2(\Gamma_1))} \\ + \|p(z_n;\alpha,\beta)\|_{L^2(0,T;L^2(\Omega))} \le M. \end{aligned} \tag{5.9}$$

The constant M is independent of n. We obtain, by taking subsequences,

$$\{y_n, z_n, y_n', z_n', p(z_n;\alpha,\beta)\} \longrightarrow \{y, z, y', z'\Psi\} \tag{5.10}$$

in

$$\begin{aligned} \left\{L^2(0,T;H^{-1}(\Omega)) \cap (L^\infty(0,T;L^2(\Omega)))_{\text{weak}^*}\right\}^2 \times \left\{(L^\infty(0,T;H^{-1}(\Omega)))_{\text{weak}^*}\right\}^2 \\ \times (L^2(0,T;L^2(\Omega)))_{\text{weak}}. \end{aligned} \tag{5.11}$$

Furthermore,

$$\nu(z_n) \longrightarrow \tilde{\nu} \quad \text{in } (L^2(0,T;L^2(\Gamma_1)))_{\text{weak}}, \tag{5.12}$$

$$\{Sz_n, Sz_n'\} \longrightarrow \{Sz, Sz'\} \quad \text{in } (L^2(0,T;L^2(\Omega)))^2. \tag{5.13}$$

(2) Since f is accretive in $L^2(0,T;H^{-1}(\Omega))$ and

$$y_n \longrightarrow y \quad \text{in } L^2(0,T;H^{-1}(\Omega)) \cap (L^\infty(0,T;L^2(\Omega)))_{\text{weak}^*}, \tag{5.14}$$

it follows from Lemma 2.3 that

$$f(y_n) \longrightarrow f(y) \quad \text{in } (L^2(0,T;L^2(\Omega)))_{\text{weak}}. \tag{5.15}$$

(3) We now show that $\Psi = p(z;\alpha,\beta)$ with $p(z;\alpha,\beta)$ being the unique element of minimum $L^2(0,T;L^2(\Omega))$-norm of the closed convex set $\partial_0 \mathcal{V}(z_0,z_1;\beta)$. From the definition of a subgradient, we obtain

$$\int_0^T \{V(S\gamma, Sz_n';\beta;t) - V(Sz_n, Sz_n';\beta,t)\}\,dt \ge \int_0^T (p(z_n,z_n';\beta), S\gamma - Sz_n)\,dt \tag{5.16}$$

for all $\gamma \in L^2(0,T;L^2(\Omega))$. We have

$$\int_0^T V(S\gamma, Sz_n';\beta,t)\,dt \longrightarrow \int_0^T V(S\gamma, Sz';\beta,t)\,dt \tag{5.17}$$

since

$$\int_0^T |V(S\gamma, Sz_n';\beta,t) - V(S\gamma, Sz';\beta,t)|\,dt \le C\|Sz_n' - Sz'\|_{L^2(0,T;L^2(\Omega))}. \tag{5.18}$$

Hence,

$$\begin{aligned}\int_0^T (\Psi, Sy - Sz)dt &\le \liminf \int_0^T \{V(Sy, Sz_n'; \beta) - V(Sz_n, Sz_n'; \beta)\}dt \\ &\le \int_0^T V(Sy, Sz'; \beta)dt - \limsup \int_0^T V(Sz_n, Sz_n'; \beta)dt \\ &\le \int_0^T V(Sy, Sz'; \beta)dt - \limsup \int_0^T \mathcal{J}(x_n; Sz_n, Sz_n'; \beta)dt \\ &\le \int_0^T V(Sy, Sz'; \beta)dt - \int_0^T \mathcal{J}(x; Sz, Sz'; \beta)dt \\ &\le \int_0^T \{V(Sy, Sz'; \beta) - V(Sz, Sz'; \beta)\}dt\end{aligned} \tag{5.19}$$

for all $y \in L^2(0,T;L^2(\Omega))$. Hence $\Psi \in \partial_0 \mathcal{V}(z_0, z_1; \beta, t)$. We have applied Theorem 4.1, and note that

$$V(Sz_n, Sz_n'; \beta) = \mathcal{J}(x_n; u_n, v_n(u_n); Sz_n, Sz_n'; \beta). \tag{5.20}$$

(4) We now show that Ψ is the unique element of $\partial_0 \mathcal{V}(z, z'; \beta)$ with minimum $L^2(0,T;$ $L^2(\Omega))$-norm. Let

$$\mathcal{B}^\varepsilon(z) = \Big\{z_\varepsilon : z_\varepsilon \in \mathcal{B}_{\overline{C}}, ||z_\varepsilon - z_n||_{L^\infty(0,T;L^2(\Omega))} + ||z_\varepsilon' - z_n'||_{L^\infty(0,T;H^{-1}(\Omega))} \le \varepsilon\Big\}. \tag{5.21}$$

Then,

$$\bigcap_\varepsilon \{\partial_0 \mathcal{V}(z_\varepsilon, z_\varepsilon'; \beta) : z_\varepsilon \in \mathcal{B}^\varepsilon(z)\} \subset \partial_0 \mathcal{V}(z, z'; \beta) \tag{5.22}$$

as $z_n \in \mathcal{B}^\varepsilon(z)$ for $n \ge n_0$. Thus,

$$||p(z_n, z_n'; \beta)||_{L^2(0,T;L^2(\Omega))} \le \|p(z, z'; \beta)\|_{L^2(0,T;L^2(\Omega))}, \quad \forall p(z, z'; \beta) \in \partial_0 \mathcal{V}(z, z'; \beta). \tag{5.23}$$

Hence

$$\|\Psi\|_{L^2(0,T;L^2(\Omega))} \le \|p(z, z'; \beta)\|_{L^2(0,T;L^2(\Omega))}, \quad \forall p \in \partial_0 \mathcal{V}(z, z'; \beta). \tag{5.24}$$

Since $\partial_0 \mathcal{V}(z, z'; \beta)$ is a closed, convex subset of $L^2(0,T;L^2(\Omega))$, there exists a unique element of minimum norm of the set and the lemma is proved. □

Proof of Theorem 5.1. It follows from Lemmas 5.3-5.4 that the nonlinear mapping $\mathcal{A}$ satisfies all the hypotheses of the Schauder fixed point theorem. Hence, there exists $\tilde{y} \in \mathcal{B}_{\tilde{C}}$ such that

$$\mathcal{A}\tilde{y} = \tilde{y}. \tag{5.25}$$

The theorem is proved. □

The following theorem gives us the feedback laws.

Theorem 5.5. *Let $\{\alpha,\beta\}$ be in $L^2(\Omega)\times H^{-1}(\Omega)$, let f be as in Assumption 2.2, and let*

$$\tilde{u} = \mathcal{P}p(\tilde{y},\tilde{y}';\beta), \qquad \tilde{v} = v(\tilde{y},\tilde{y}';\beta), \tag{5.26}$$

where $\{\tilde{y},\tilde{v}\}$ is as in Theorem 5.1. Then

$$V(\alpha,\beta) = \mathcal{J}(\tilde{y};\tilde{u},\tilde{v};\alpha,\beta). \tag{5.27}$$

Proof. Let $\tilde{y}$ be as in Theorem 5.1 and consider the problem

$$\begin{gathered} y'' - \Delta y + f(y) = u\chi_\omega \quad \text{in } \Omega\times(t,T), \\ y(x,s) = 0 \quad \text{on } \Gamma_0\times(t,T), \qquad y(x,s) = v(u) \quad \text{on } \Gamma_1\times(t,T), \\ y(x,t) = \tilde{y}(x,t), \qquad y'(x,s)|_{s=t} = \tilde{y}(x,t) \quad \text{in } \Omega, \\ y(x,T) = \beta_0, \qquad y'(x,T) = \beta_1 \quad \text{in } \Omega. \end{gathered} \tag{5.28}$$

Applying Theorem 4.1, we obtain

$$\begin{aligned} V(\tilde{y}(\cdot,t),\tilde{y}'(\cdot,t);\beta;t) &= \inf\left\{\int_t^T\int_\Omega |y(x,s)|\,dx\,ds\ y \text{ solution of (5.28)}, \forall u\in\mathcal{U}\right\} \\ &= \mathcal{J}(y_*;u_*,v_*;\beta;t), \end{aligned} \tag{5.29}$$

where y_* is the solution of (5.28) for some $\{u_*,v_*\}$ in $L^2(0,T;L^2(\Omega))\times L^2(0,T;L^2(\Gamma_1))$. The solution y_* depends on t through its interval of definition.

The dynamic programming principle gives

$$\begin{aligned} &V(\tilde{y}(t),\tilde{y}'(t);\beta;t) \\ &\quad = \inf\Big\{V(y(t+h);y'(t+h);\beta;t+h) \\ &\qquad + \int_t^{t+h}\int_\Omega |y(x,s)|\,dx\,ds\ y \text{ solution of (5.28)}, \forall u\in\mathcal{U}\Big\}. \end{aligned} \tag{5.30}$$

It follows that

$$\begin{aligned} V(\tilde{y}(t),\tilde{y}'(t);\beta,t) &= \mathcal{J}(y_*;u_*,v_*;\beta,t) \\ &\le V(y_*(t+h),y_*'(t+h);\beta,t+h) \\ &\quad + \int_t^{t+h}\int_\Omega |y_*(x,s)|\,dx\,ds. \end{aligned} \tag{5.31}$$

Therefore,

$$\int_{t+h}^T\int_\Omega |y_*(x,s)|\,dx\,ds \le V(y_*(t+h),y_*'(t+h);\beta;t+h). \tag{5.32}$$

From the definition of the value function, we deduce that

$$V\big(y_*(t+h), y'_*(t+h);\beta, t+h\big) = \int_{t+h}^{T}\int_{\Omega} |\, y_*(x,s)\,|\, dx\, ds. \tag{5.33}$$

Since $\{y_*(\cdot,t), y'_*(\cdot,t)\} = \{\tilde{y}(\cdot,t), \tilde{y}'(\cdot,t)\}$ in Ω, we get

$$V(\tilde{y}(t), \tilde{y}'(t);\beta, t) = V\big(y_*(t), y'_*(t);\beta, t\big). \tag{5.34}$$

Hence,

$$\begin{aligned} &V\big(y_*(t), y'_*(t);\beta;t\big) - V\big(y_*(t+h), y'_*(t+h);\beta, t+h\big) \\ &\quad = \int_{t}^{t+h}\int_{\Omega} |\, y_*(x,s)\,|\, dx\, ds. \end{aligned} \tag{5.35}$$

Thus,

$$\begin{aligned} &\lim_{h\longrightarrow 0^+} h^{-1}\Big\{V\big(y_*(t), y'_*(t);\beta, t\big) - V\big(y_*(t+h), y'_*(t+h), \beta, t+h\big)\Big\} \\ &\quad = \lim_{h\longrightarrow 0^+} h^{-1}\int_{t}^{t+h}\int_{\Omega} |\, y_*(x,s)\,|\, dx\, ds \end{aligned} \tag{5.36}$$

It follows that

$$-\frac{d}{dt}\big\{V\big(y_*(t), y'_*(t);\beta;t\big)\big\} = \int_{\Omega} |y_*(x,t)|\, dx = \int_{\Omega} |\tilde{y}(x,t)|\, dx. \tag{5.37}$$

Hence,

$$V\big(y_*(t), y'_*(t);\beta;t\big) = V(\tilde{y}(t), \tilde{y}'(t);\beta;t) = \int_{t}^{T}\int_{\Omega} |\tilde{y}(x,s)|\, dx\, ds. \tag{5.38}$$

Thus,

$$V(\alpha_0, \alpha_1;\beta) = \int_{0}^{T}\int_{\Omega} |\tilde{y}(x,t)|\, dx\, dt. \tag{5.39}$$

The theorem is proved. □

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