INERTIAL MANIFOLDS AND STABILIZATION OF NONLINEAR BEAM EQUATIONS WITH BALAKRISHNAN-TAYLOR DAMPING

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ABSTRACT. In this paper we study a hinged, extensible, and elastic nonlinear beam equation with structural damping and Balakrishnan-Taylor damping with the full exponent $2(n+\beta)+1$. This strongly nonlinear equation, initially proposed by Balakrishnan and Taylor in 1989, is a very general and useful model for large aerospace structures. In this work, the existence of global solutions and the existence of absorbing sets in the energy space are proved. For this equation, the feature is that the exponential rate of the absorbing property is not a global constant, but which is uniform for the family of trajectories starting from any given bounded set in the state space. Then it is proved that there exists an inertial manifold whose exponentially attracting rate is accordingly non-uniform. Finally, the spillover problem with respect to the stabilization of this equation is solved by constructing a linear state feedback control involving only finitely many modes. The obtained results are robust in regard to the uncertainty of the structural parameters.

1. Introduction and Formulation

The objective of this paper is to study the following initial-boundary value problem for a nonlinear beam equation,

$$
\begin{aligned}
&u_{tt} + \alpha u_{xxxx} - \delta u_{xxt} \\
&\qquad - \left\{ a + b\int_0^1 |u_x(t,\xi)|^2 d\xi + q\left[\int_0^1 (u_x u_{xt})(t,\xi)d\xi\right]^{2(n+\beta)+1} \right\} u_{xx} \qquad (1)\\
&\quad = f, \quad \text{for } (t,x) \in \mathcal{R}^+ \times (0,1),\\
&\qquad u(t,0) = u_{xx}(t,0) = u(t,1) = u_{xx}(t,1) = 0, \quad \text{for } t \ge 0,\\
&\qquad u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \quad \text{for } x \in [0,1].
\end{aligned}
$$

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Here $u(t,x)$ is the transverse deflection of the beam. All the parameters α, δ, b, and q are assumed to be positive constants, but $a \in \mathcal{R}$. The term $-\delta u_{xxt}$ represents the structural damping, $[a + b\|u_x\|^2]u_{xx}$ is the tension due to the extensibility, and the last term on the left-hand side is called the Balakrishnan-Taylor damping. The parameter β satisfies $0 \leq \beta < 1/2$ and $n \geq 0$ is an integer. The function $f(t,x)$ stands for an external input which practically may be a control function. The boundary conditions correspond to hinged endpoints. This model equation of nonlinear beams was initially proposed by Balakrishnan and Taylor in 1989 (cf. [1] and [2]). The original motivation for studying this model seemed to solve the *spillover* problem, namely, to design a feedback control function f that involves only finitely many modes in order to achieve a high performance of the closed-loop system, such as a robust and exponential stabilization of the system when there might be some uncertainty in the values of the parameters.

In this paper we first analyze the global dynamics of the uncontrolled equation and prove the existence of inertial manifolds. Then, based on this analysis and result, we provide a solution to the spillover problem by constructing an implementable feedback control which involves only finitely many modes and is robust with respect to parameter uncertainty.

An initial result on the existence of a flat inertial manifold for the dynamics of a rotating beam was obtained in [3]. The long time behavior and global dynamics of a simpler version of Equation (1) with $q = 0$ and damping δu_t has been studied by many authors (cf. [6], [7] and [8]). This model with a simplified exponent assumption that $n = \beta = 0$ has been investigated and the affirmative results on the existence of inertial manifolds and on the finite-dimensional stabilization have been proved in [11] and [12]. More background materials in regard to infinite dimensional dynamical systems, especially the theory of global attractors, inertial manifolds, and approximate inertial manifolds, can be found in [5], [7], [9] and [10].

First of all, let us formulate this initial-boundary value problem for the uncontrolled equation

$$\begin{aligned} &u_{tt} + \alpha u_{xxxx} - \delta u_{xxt} \\ &\quad - \left\{ a + b\int_0^1 |u_x(t,\xi)|^2 d\xi + q\left[\int_0^1 (u_x u_{xt})(t,\xi)d\xi\right]^{2(n+\beta)+1} \right\} u_{xx} \\ &\quad = 0, \end{aligned} \tag{2}$$

into an abstract semilinear evolution equation and consider the existence, uniqueness, and regularity of local solutions.

Let $H = L^2(0,1)$ with norm and inner-product denoted by $|\cdot|$ and $\langle , \rangle$, respectively. Define a linear operator $A : D(A) \to H$ by

$$A\phi = \frac{d^4\phi}{dx^4}, \quad \text{for } \phi \in D(A), \text{ with}$$

$$D(A) = \{\phi \in H^4(0,1) \ : \ \phi(0) = \phi''(0) = \phi(1) = \phi''(1) = 0\},$$

where, in general, the derivatives are taken in the distributional sense. Here for $\phi \in D(A)$ these derivatives are consistent with the usual derivatives. This closed linear operator A is densely defined, self-adjoint, and positive definite. It has compact resolvent A^{-1}. The spectrum $\sigma(A)$ consists of the simple eigenvalues $\{\lambda_k = k^4\pi^4 \ : \ k \geq 1\}$, with eigenvectors

$$\{e_k = \sqrt{2}\sin(k\pi x) \ : \ k \geq 1\}.$$

By the approach of Fourier expansions, it can be shown that

$$A^{1/2}\phi = -d^2\phi/dx^2 \quad \text{end} \quad |\phi_x|^2 = |A^{1/4}\phi|^2. \tag{3}$$

Thus, Equation (2) can be formulated into a second-order semilinear evolution equation:

$$\begin{aligned} &\frac{d^2u}{du^2} + \alpha Au + \delta A^{1/2}\frac{du}{dt} \\ &\qquad + \left\{a + b|A^{1/4}u|^2 + q\langle A^{1/2}u, u_t\rangle^{2(n+\beta)+1}\right\} A^{1/2}u = 0, \\ &\qquad\qquad u(0) = u_0, \quad u_t(0) = u_1. \end{aligned} \tag{4}$$

Let $V = D(A^{1/2})$ with the norm $\|v\| = |A^{1/2}v|$. V is also a Hilbert space. Define the product Hilbert space $E = V \times H$, which can be called the energy space. Similarly, define $E_1 = D(A) \times V = D(A) \times D(A^{1/2})$ with the graph norm. Then define a linear operator G by

$$G = \begin{pmatrix} 0 & I_V \\ -\alpha A & -\delta A^{1/2} \end{pmatrix} : D(G) \to E, \text{ with } D(G) = D(A) \times V, \tag{5}$$

where I_V is the identity operator on V, and a nonlinear mapping R by

$$R\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ -\left[a + b|A^{1/4}\phi|^2 + q\langle A^{1/2}\phi, \psi\rangle^{2(n+\beta)+1}\right] A^{1/2}\phi \end{pmatrix}. \tag{6}$$

Then Equation (4) can be further formulated into a first-order semilinear evolution equation:

$$\begin{aligned} &\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = G\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + R\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad t \geq 0, \\ &\qquad\qquad \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in E, \end{aligned} \tag{7}$$

where $v_0 = u_1$. Let $w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ and $w_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$. Then Equation (7) can be rewritten as

$$\frac{dw}{dt} = Gw + R(w), \quad t \geq 0, \quad w(0) = w_0 \in E. \tag{8}$$

It can be shown that the operator $-G$ is a sectorial operator and that G generates an analytic contraction semigroup which will be denoted by $\{T(t) \ : \ t \geq 0\}$. Moreover, G has compact resolvent. The nonlinear mapping $R : E \to E$ is locally Lipschitz continuous and maps bounded sets into bounded sets. Therefore, by the standard semigroup theory, we have the following local existence and regularity result, whose proof is omitted.

Lemma 1. *For any $w_0 \in E$, there is a $\tau = \tau(w_0) > 0$ such that the mild solution $w(t)$ of Equation* (8) *with the initial condition $w(0) = w_0$ exists uniquely for $t \in [0, \tau]$ and*

$$w \in C([0,\tau];E) \cap C^1((0,\tau);E) \cap C((0,\tau);E_1). \tag{9}$$

If $w_0 \in E_1$, then this mild solution is a classical solution of Equation (8) *for $t \in [0, \tau]$.*

2. Global Existence of Solutions and Dissipativity

In this section we shall simultaneously prove the global existence of mild solutions of Equation (8) and the dissipativity of the dynamical system associated with the solutions semigroup. Let us introduce the relevant concepts.

A fixed bounded set $\mathbf{N}$ is called an *absorbing set* for the solution semigroup $S(t),\ t \geq 0$, generated by the mild solutions of Equation (8), if for any given bounded set Z of E, there exist constants $\varepsilon(Z) > 0$ and $K(Z) \geq 0$ such that, for any initial data $w_0 \in Z$,

$$\mathrm{dist}_E(w(t;w_0), \mathbf{N}) \leq K(Z)\exp\left(-\varepsilon(Z)t\right), \quad t \geq 0.$$

Theorem 2. *For any $w_0 \in E$, there exists a unique global mild solution $w(t)$ of Equation* (8), *$t \in [0, \infty)$, which has the regularity properties stated in Lemma 1. The associated solution semigroup $S(t),\ t \geq 0$, is dissipative in the sense that there exists an absorbing set in E.*

Proof. By taking the inner product in H of Equation (2) with $2u_t$, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(|u_t|^2 + \alpha|u_{xx}|^2\right) + 2\delta|u_{xt}|^2 \\ &\quad + \left\{a + b|u_x|^2 + q\left(\frac{1}{2}\frac{d}{dt}|u_x|^2\right)^{2(n+\beta)+1}\right\}\frac{d}{dt}|u_x|^2 \\ &= \frac{d}{dt}\left[|u_t|^2 + \alpha|u_{xx}|^2 + \frac{1}{2b}\left(a + b|u_x|^2\right)^2\right] \\ &\quad + 2\delta|u_{xt}|^2 + q\left(\frac{1}{2}\frac{d}{dt}|u_x|^2\right)^{2(n+\beta+1)} \\ &= 0. \end{aligned} \tag{10}$$

By integrating (10) over $[0, t]$, for any $t \geq 0$, we have

$$\begin{aligned} &|u_t|^2 + \alpha|u_{xx}|^2 + \frac{1}{2b}\left(a + b|u_x|^2\right)^2 \\ &\quad + \int_0^t \left[2\delta|u_{xt}|^2(s) + q\left(\frac{1}{2}\frac{d}{dt}|u_x|^2(s)\right)^{2(n+\beta+1)}\right] ds \\ &\leq |u_1|^2 + \alpha|A^{1/2}u_0|^2 + \frac{1}{2b}\left(a + b|A^{1/4}u_0|^2\right)^2 \\ &\leq C_0 + C_1\|(u_0, u_1)\|_E^4, \end{aligned} \tag{11}$$

where C_0 and $C_1 > 0$ are constants independent of the initial data (u_0, u_1). Note that each term on the left-hand side of the first inequality of (11) is uniformly bounded in t, if the initial data (u_0, u_1) belong to a given bounded subset Z of E.

Then take the inner product in H of Equation (2) with εu, where $\varepsilon > 0$ is an undetermined constant, to obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\varepsilon\langle u_t, u\rangle + \frac{\varepsilon\delta}{2}|u_x|^2\right) + \varepsilon\alpha|u_{xx}|^2 - \varepsilon|u_t|^2 \\
&\quad + \varepsilon\left\{a + b|u_x|^2 + q\left(\frac{1}{2}\frac{d}{dt}|u_x|^2\right)^{2(n+\beta)+1}\right\}|u_x|^2 \\
&= \frac{d}{dt}\left(\varepsilon\langle u_t, u\rangle + \frac{\varepsilon\delta}{2}|u_x|^2\right) + \varepsilon\alpha|u_{xx}|^2 - \varepsilon|u_t|^2 \\
&\quad + \varepsilon\left\{\frac{1}{2}ab^{-1/2} + b^{1/2}|u_x|^2\right\}^2 - \frac{1}{4}\varepsilon a^2 b^{-1} \\
&\quad + \varepsilon q\left(\frac{1}{2}\frac{d}{dt}|u_x|^2\right)^{2(n+\beta)+1}|u_x|^2 \\
&= 0.
\end{aligned} \tag{12}$$

Adding (10) and (12), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{|u_t|^2 + \alpha|u_{xx}|^2 + \frac{1}{2b}\left(a + b|u_x|^2\right)^2 + \varepsilon\langle u_t, u\rangle + \frac{\varepsilon\delta}{2}|u_x|^2\right\} \\
&\quad + \left\{2\delta|u_{xt}|^2 + q\left(\frac{1}{2}\frac{d}{dt}|u_x|^2\right)^{2(n+\beta+1)} + \varepsilon\alpha|u_{xx}|^2 - \varepsilon|u_t|^2\right. \\
&\quad \left. + \varepsilon\left[\frac{1}{2}ab^{-1/2} + b^{1/2}|u_x|^2\right]^2\right\} + \varepsilon q\left(\frac{1}{2}\frac{d}{dt}|u_x|^2\right)^{2(n+\beta)+1}|u_x|^2 \\
&= \frac{1}{4}\varepsilon a^2 b^{-1}.
\end{aligned} \tag{13}$$

Note that

$$q\left(\frac{1}{2}\frac{d}{dt}|u_x|^2\right)^{2(n+\beta+1)} \geq 0.$$

However,

$$\varepsilon q\left(\frac{1}{2}\frac{d}{dt}|u_x|^2\right)^{2(n+\beta)+1}|u_x|^2$$

may be negative. Define the functionals $N(t)$ and $L(t)$ by

$$
\begin{aligned}
N(t) &= 2\delta|u_{xt}|^2 + \varepsilon\alpha|u_{xx}|^2 - \varepsilon\,|u_t|^2 + \varepsilon\left[\frac{1}{2}ab^{-1/2} + b^{1/2}|u_x|^2\right]^2 \\
&\quad + q\left(\frac{1}{2}\frac{d}{dt}|u_x|^2\right)^{2(n+\beta)}\left[\frac{1}{2}\frac{d}{dt}|u_x|^2 + \frac{\varepsilon}{2}|u_x|^2\right]^2 \\
&= \varepsilon\left\{2\delta\varepsilon^{-1}|u_{xt}|^2 + \alpha|u_{xx}|^2 - |u_t|^2 + \left[\frac{1}{2}ab^{-1/2} + b^{1/2}|u_x|^2\right]^2\right\} \\
&\quad + q\left(\frac{1}{2}\frac{d}{dt}|u_x|^2\right)^{2(n+\beta)}\left[\frac{1}{2}\frac{d}{dt}|u_x|^2 + \frac{\varepsilon}{2}|u_x|^2\right]^2
\end{aligned} \tag{14}
$$

and

$$
L(t) = |u_t|^2 + \alpha|u_{xx}|^2 + \frac{1}{2b}\left(a + b|u_x|^2\right)^2 + \varepsilon\langle u_t, u\rangle + \frac{\varepsilon\delta}{2}|u_x|^2. \tag{15}
$$

By Poincaré's inequality and (3), we have $|u_{xt}| \geq |u_t|$ and $|u_{xx}| \geq |u_x| \geq |u|$. Then,

$$
\begin{aligned}
2\varepsilon^{-1}N(t) - L(t) &\geq \left(\frac{4\delta}{\varepsilon} - 3 - \varepsilon\right)|u_{xt}|^2 + \left(\alpha - \varepsilon - \frac{\varepsilon\delta}{2}\right)|u_{xx}|^2 \\
&\quad + \frac{1}{2b}\left[a + 2b|u_x|^2\right]^2 - \frac{1}{2b}\left(a + b|u_x|^2\right)^2 \\
&\geq \left(\frac{4\delta}{\varepsilon} - 3 - \varepsilon\right)|u_{xt}|^2 + \left(\alpha - \varepsilon - \frac{\varepsilon\delta}{2}\right)|u_{xx}|^2 \\
&\quad + |u_x|^2\left(a + b|u_x|^2\right).
\end{aligned} \tag{16}
$$

Now we can choose $\varepsilon > 0$ sufficiently small so that

$$
\frac{4\delta}{\varepsilon} - 3 - \varepsilon \geq 0, \quad \alpha - \varepsilon - \frac{\varepsilon\delta}{2} \geq 0, \quad \text{and } \frac{1}{2}\min\{1, \alpha\} \geq \varepsilon.
$$

By completing the square for the last term on the right-hand side of Inequality (16), we get

$$
2\varepsilon^{-1}N(t) - L(t) \geq -\frac{a^2}{4b}. \tag{17}
$$

Substitution of (17) into (13) gives rise to a differential inequality:

$$
\begin{aligned}
\frac{d}{dt}L(t) + \frac{\varepsilon}{2}L(t) &\leq \frac{\varepsilon a^2}{2b} + \frac{\varepsilon^2}{4}q\left(\frac{1}{2}\frac{d}{dt}|u_x|^2\right)^{2(n+\beta)}|u_x|^4 \\
&\leq \frac{\varepsilon a^2}{2b} + C(n+\beta)\varepsilon^2 q\left\{\left[\frac{1}{2}\frac{d}{dt}|u_x|^2\right]^{2(n+\beta+1)} + |u_x|^{4(n+\beta+1)}\right\} \\
&\leq \frac{\varepsilon a^2}{2b} + C(n+\beta)\varepsilon^2 q\left\{\left[\frac{1}{2}\frac{d}{dt}|u_x|^2\right]^{2(n+\beta+1)} + C_2(u_0, u_1)\right\}
\end{aligned} \tag{18}
$$

for $t \in I_{\max}$ (the maximal interval of existence), where $C(n+\beta)$ is a constant depending on $n+\beta$ and obtained by applying Young's inequality to split the product on the right-hand side of the first inequality of (18). Besides, the inequality

$$\frac{1}{2b}\left(a+b|u_x|^2\right)^2 \leq C_0 + C_1\|(u_0,u_1)\|_E^4 \tag{19}$$

is utilized to bound $|u_x|^{4(n+\beta+1)}$, with the constant $C_2(u_0,u_1)$ given by

$$C_2(u_0,u_1) = \left\{\frac{\left[2b\left(C_0+C_1\|(u_0,u_1)\|_E^4\right)\right]^{1/2}+|a|}{b}\right\}^{2(n+\beta+1)}. \tag{20}$$

Hence it follows that

$$\begin{aligned} L(t) &\leq L(0)\exp\left(-\frac{\varepsilon}{2}t\right)+\frac{a^2}{b} \\ &\quad + C(n+\beta)\varepsilon^2 q\int_0^t\left[\frac{1}{2}\frac{d}{dt}|u_x|^2(s)\right]^{2(n+\beta+1)}ds \\ &\quad + C(n+\beta)\varepsilon^2 q\int_0^t C_2(u_0,u_1)\exp\left(-\frac{\varepsilon}{2}(t-s)\right)ds \\ &\leq L(0)\exp\left(-\frac{\varepsilon}{2}t\right)+\frac{a^2}{b} \\ &\quad + C(n+\beta)\varepsilon^2\left(C_0+C_1\|(u_0,u_1)\|_E^4\right) \\ &\quad + 2C(n+\beta)\varepsilon q C_2(u_0,u_1), \end{aligned} \tag{21}$$

where the bound for the first integral term comes from the property (11). Note that

$$\begin{aligned} L(t) &= |u_t|^2+\alpha|u_{xx}|^2+\frac{1}{2b}\left(a+b|u_x|^2\right)^2+\varepsilon\langle u_t,u\rangle+\frac{\varepsilon\delta}{2}|u_x|^2 \\ &\geq (1-\varepsilon)|u_t|^2+(\alpha-\varepsilon)|u_{xx}|^2 \\ &\geq \frac{1}{2}\min\{1,\alpha\}\|(u(t),u_t(t))\|_E^2. \end{aligned} \tag{22}$$

Combining (22) with (21), we see that for any given bounded subset Z of E, there exists a constant $\varepsilon=\varepsilon(n+\beta,Z)>0$, which can be chosen small enough so that

$$C(n+\beta)\varepsilon^2\left(C_0+C_1\|(u_0,u_1)\|_E^4\right)+2C(n+\beta)\varepsilon q C_2(u_0,u_1)\leq 1. \tag{23}$$

It follows that

$$\frac{1}{2}\min\{1,\alpha\}\|(u(t),u_t(t))\|_E^2 \leq L(0)\exp\left(-\frac{\varepsilon}{2}t\right)+\frac{a^2}{b}+1, \tag{24}$$

for $t \in I_{\max}$. This inequality shows that any mild solution $w(t) = (u(t), u_t(t))$ cannot blow up. In other words, all the solutions exist globally for $t \in [0, \infty)$. In addition,

$$\limsup_{t\to\infty} \|w(t)\|_E^2 \leq 2\left(\min\{1,\alpha\}\right)^{-1}\left(a^2 b^{-1} + 1\right). \tag{25}$$

Hence the closed bounded ball

$$B_r = \{y \in E \ : \ \|y\|_E \leq r\}, \quad \text{with constant } r > \left[2\max\left\{1, \alpha^{-1}\right\}\left(a^2 b^{-1} + 1\right)\right]^{1/2}, \tag{26}$$

is an absorbing set for the solution semigroup $S(t)$. Here, as usual, the solution semigroup is defined by $S(t)w_0 = w(t; w_0)$ for $t \geq 0$ and $w_0 \in E$. ∎

Based on the dissipativity of this semiflow, one can explore the existence of a global attractor. However, since the spillover problem is not closely related to the global attractor even if it exists, we shall skip over the discussion about the global attractor and directly work on the existence of inertial manifolds in the next section.

3. The Existence of Inertial Manifolds

For the dynamical system ϑ in the space E defined by the solution semigroup $\{S(t) \ : \ t \geq 0\}$, a set $\mathbf{M} \subset E$ is called an *inertial manifold* for ϑ, if $\mathbf{M}$ is a Lipschitz continuous finite-dimensional manifold, positively invariant under $S(t)$, and attracts all the trajectories at a locally uniform exponential rate. That is, for any given bounded subset Z of E, there exist constants $\mu(Z) > 0$ and $C(Z) > 0$, such that

$$\mathrm{dist}_E(w(t; w_0), \mathbf{M}) \leq C(Z)\exp\left(-\mu(Z)t\right), \quad \text{for } t \geq 0, \quad \text{and any } w_0 \in Z.$$

Note that this definition is slightly generalized. Because usually it is required that an inertial manifold have a uniform exponential attracting rate $\mu > 0$, but the coefficient $C(Z)$ can depend on the given bounded set Z from which the trajectory is started (cf. [5]).

In this section, we shall prove that there exist inertial manifolds for the concerned dynamical system ϑ associated with this solution semigroup $\{S(t) \ : \ t \geq 0\}$.

Let $H_m = \mathrm{Span}\{e_1, \ldots, e_m\}$ and $P_m : H \to H_m$ be the orthogonal projection from H onto H_m. Then let

$$Q_m = I_H - P_m,$$

$$\Pi_m = \mathrm{diag}(P_m, P_m) : E \to H_m \times H_m,$$

$$\Theta_m = I_E - \Pi_m.$$

The decompositions $H = P_m H \oplus Q_m H$ and $E = (\Pi_m E) \oplus (\Theta_m E)$ are orthogonal direct sums of subspaces, in which $P_m H$ has a finite dimension m and $\Pi_m E$ has a finite dimension $2m$.

Accordingly, the H-valued function $u(t)$ admits an orthogonal decomposition $u(t) = p(t) + h(t)$, with $p(t) = P_m u(t)$ and $h(t) = Q_m u(t)$. The E-valued function $w(t) = \mathrm{col}(u(t), v(t))$ has an orthogonal decomposition $w(t) = \pi(t) + \vartheta(t)$, with $\pi(t) = \Pi_m w(t)$ and $\vartheta(t) = \Theta_m(t) w(t)$. Because of the commutativity between $A^{1/2}$ and P_m, the second-order evolution equation (4) can be decomposed into the following coupled equations:

$$(27)_p \qquad \frac{d^2 p}{dt^2} + \alpha A p + \delta A^{1/2} \frac{dp}{dt} + \left[a + b|A^{1/4}u|^2 + q\langle A^{1/2}u, u_t\rangle^{2(n+\beta)+1} \right] A^{1/2} p = 0,$$

$$(27)_h \qquad \frac{d^2 h}{dt^2} + \alpha A h + \delta A^{1/2} \frac{dh}{dt} + \left[a + b|A^{1/4}u|^2 + q\langle A^{1/2}u, u_t\rangle^{2(n+\beta)+1} \right] A^{1/2} h = 0,$$

with initial data $p(0) = P_m u_0,\ p_t(0) = P_m u_1$, and $h(0) = Q_m u_0,\ h_t(0) = Q_m u_1$, respectively. Define a functional $J_u(t)$ by

$$(28) \qquad J_u(t) = a + b|A^{1/4}u|^2 + q\langle A^{1/2}u, u_t\rangle^{2(n+\beta)+1}.$$

Theorem 3. *There exists a flat inertial manifold* $\mathbf{M}$ *in* E, *given by*

$$(29) \qquad \mathbf{M} = H_m \times H_m,$$

for the dynamical system ϑ *generated by the solution semigroup of Equation* (8), *where* $m > 0$ *is a suitably large number.*

Proof. Obviously, the set $\mathbf{M}$ given by (29) is a finite-dimensional subspace, hence it is a Lipschitz continuous, linear manifold. This $\mathbf{M}$ is positively invariant under the semigroup $\{S(t) : t \ge 0\}$, because of the commutativity between $A^{1/2}$ and P_m. In fact, if $w_0 = (u_0, u_1) \in \mathbf{M} \subset D(G)$, the mild solution $w(t) = (u(t), u_t(t)$ of Equation (8) is a classical solution, so that the first component $u(t) = p(t) + h(t)$ satisfies Equation (4).

Therefore, the functions $p(t)$ and $h(t)$ are respectively classical solutions of Equation $(27)_p$ and Equation $(27)_h$, with $h(0) = h_t(0) = 0$. By the uniqueness of solutions of Equation $(27)_h$, it follows that $h(t) = 0$, for $t \ge 0$, and consequently $p(t)$ is the solution of

$$(30) \qquad \frac{d^2 p}{dt^2} + \alpha A p + \delta A^{1/2} \frac{dp}{dt} + \left[a + b|A^{1/4}p|^2 + q\langle A^{1/2}p, p_t\rangle^{2(n+\beta)+1} \right] A^{1/2} p = 0,$$

$$p(0) = u_0 \in H_m, \quad p_t(0) = u_1 \in H_m.$$

This proves the positive invariance of **M**.

It remains to prove that **M** is exponentially attracting. By taking the inner product in H of Equation $(27)_h$ with $2h_t + \xi h$, where ξ is an undetermined constant, we obtain

$$\begin{aligned} &\frac{d}{dt}\left\{|h_t|^2 + \alpha|h_{xx}|^2 + \xi\langle h_t, h\rangle + (\xi\delta/2)|h_x|^2\right\} \\ &\quad + \left\{2\delta|h_{xt}|^2 - \xi|h_t|^2 + \xi\alpha|h_{xx}|^2\right\} \\ &\quad + \left\{2J_u(t)\langle h_x, h_{xt}\rangle + \xi J_u(t)|h_x|^2\right\} \\ &= 0. \end{aligned} \tag{31}$$

By Theorem 2 and its proof, we know that, for every given bounded set Z in E and for any initial point $w_0 \in Z$, the trajectory $w(t; w_0)$ will enter the fixed absorbing ball B_r in E (and stay in it forever) at an exponential rate $\varepsilon(n + \beta, Z)/2$ after a transient period $[0, t_0]$ with $t_0 = t_0(Z)$ also depending on Z. For this reason, below we assume that the trajectories have already been staying in the fixed absorbing ball B_r after the transient period. Then we have

$$\begin{aligned} |J_u(t)| &\le |a + b|u_x|^2 - q\langle u_{xx}, u_t\rangle^{2(n+\beta)+1}| \\ &\le |a| + b\|w(t; w_0)\|_E^2 + q\|w(t; w_0)\|_E^{4(n+\beta)+2} \\ &\le |a| + br^2 + qr^{4(n+\beta)+2}, \quad \text{for } t \ge t_0, \end{aligned} \tag{32}$$

where r is the radius of the ball B_r in (26). By using (32), we can estimate the last two terms in (31) as follows. Let

$$\Pi(r) = |a| + br^2 + qr^{4(n+\beta)+2}. \tag{33}$$

Then,

$$\begin{aligned} &|2J_u(t)\langle h_x, h_{xt}\rangle + \xi J_u(t)|h_x|^2| \\ &\qquad \le 2\Pi(r)|h_x||h_{xt}| + \xi\Pi(r)|h_x|^2 \\ &\qquad \le \left[[\Pi(r)]^2\,\delta^{-1} + \xi\Pi(r)\right]|h_x|^2 + \delta|h_{xt}|^2 \\ &\qquad \le \frac{[\Pi(r)]^2\,\delta^{-1} + \xi\Pi(r)}{\sqrt{\lambda_{m+1}}}|h_{xx}|^2 + \delta|h_{xt}|^2 \\ &\qquad \le K(r,\xi)(m+1)^{-2}\pi^{-2}|h_{xx}|^2 + \delta|h_{xt}|^2, \end{aligned} \tag{34}$$

where $K(r, \xi) = [\Pi(r)]^2\,\delta^{-1} + \xi\Pi(r)$. Substituting (34) into (31), we obtain

$$\begin{aligned} &\frac{d}{dt}\left\{|h_t|^2 + \alpha|h_{xx}|^2 + \xi\langle h_t, h\rangle + (\xi\delta/2)|h_x|^2\right\} \\ &\quad + \left\{\delta|h_{xt}|^2 - \xi|h_t|^2 + \xi\alpha|h_{xx}|^2 - K(r,\xi)(m+1)^{-2}\pi^{-2}|h_{xx}|^2\right\} \\ &\quad \le 0. \end{aligned} \tag{35}$$

Let

$$Y(t) = |h_t|^2 + \alpha|h_{xx}|^2 + \xi\langle h_t, h\rangle + (\xi\delta/2)|h_x|^2 \tag{36}$$

and

$$\Phi(t) = \delta|h_{xt}|^2 - \xi|h_t|^2 + \xi\alpha|h_{xx}|^2 - K(r,\xi)(m+1)^{-2}\pi^{-2}|h_{xx}|^2. \tag{37}$$

Then we have

$$\begin{aligned}\Phi(t)-(\xi/2)Y(t) &\geq \delta|h_{xt}|^2 - (3\xi/2)|h_t|^2 + (\xi\alpha/2)|h_{xx}|^2 \\ &\quad - K(r,\xi)(m+1)^{-2}\pi^{-2}|h_{xx}|^2 - (\xi/2)\langle h_t, h\rangle \\ &\geq (\delta - 2\xi)|h_{xt}|^2 \\ &\quad + \left\{(\xi\alpha/2) - \left[(\xi/2) + K(r,\xi)\right](m+1)^{-2}\pi^{-2}\right\}|h_{xx}|^2 \\ &\geq 0.\end{aligned} \tag{38}$$

Now choose ξ and fix it, such that

$$0 < \xi \leq \min\left\{1, \alpha(1+\delta)^{-1}, \delta/2\right\}, \tag{39}$$

and then take a positive integer m large enough so that

$$m \geq -1 + \sqrt{(\xi\alpha\pi^2)^{-1}\left[\xi + 2K(r,\xi)\right]}. \tag{40}$$

By these choices, it follows that

$$\frac{d}{dt}Y(t) + \frac{\xi}{2}Y(t) \leq 0, \quad \text{for } t \geq t_0, \tag{41}$$

so that

$$\begin{aligned}&\frac{1}{2}\min\{1,\alpha\}\left\|\begin{pmatrix} h(t) \\ h_t(t)\end{pmatrix}\right\|_E^2 \\ &= \frac{1}{2}\min\{1,\alpha\}\left(|h_t|^2 + |h_{xx}|^2\right) \\ &\leq \frac{1}{2}\min\{1,\alpha\}\left(|h_t|^2 + |h_{xx}|^2\right) + \frac{1-\xi}{2}|h_t|^2 + \frac{1}{2}(\alpha - \xi(1+\delta))|h_{xx}|^2 \\ &\leq |h_t|^2 + \alpha|h_{xx}|^2 + \xi\langle h_t, h\rangle + (\xi\delta/2)|h_x|^2 \\ &= Y(t) \\ &\leq Y(t_0)\exp\left(-\frac{\xi}{2}(t-t_0)\right) \\ &\leq [1+\alpha+\xi+(\xi\delta/2)]\left\|\begin{pmatrix} h(t_0) \\ h_t(t_0)\end{pmatrix}\right\|_E^2 \exp\left(-\frac{\xi}{2}(t-t_0)\right) \\ &\leq (2+\alpha+\delta)\left\|\begin{pmatrix} u(t_0) \\ u_t(t_0)\end{pmatrix}\right\|_E^2 \exp\left(-\frac{\xi}{2}(t-t_0)\right) \\ &\leq (2+\alpha+\delta)r^2\exp\left(-\frac{\xi}{2}(t-t_0)\right), \quad \text{for } t \geq t_0.\end{aligned} \tag{42}$$

Hence,

$$
\begin{aligned}
\|\vartheta(t)\|_E^2 &= \|\Theta_m w(t)\|_E^2 = \left\| \begin{pmatrix} h(t) \\ h_t(t) \end{pmatrix} \right\|_E^2 \\
&\le 2\min\{1,\alpha\}^{-1}(2+\alpha+\delta) \left\| \begin{pmatrix} u(t_0) \\ u_t(t_0) \end{pmatrix} \right\|_E^2 \exp\left(-\frac{\xi}{2}(t-t_0)\right) \\
&\le 2\min\{1,\alpha\}^{-1}(2+\alpha+\delta) r^2 \exp\left(-\frac{\xi}{2}(t-t_0)\right), \quad \text{for } t \ge t_0.
\end{aligned} \tag{43}
$$

The inequality (43) is valid for all solutions with initial data w_0 in E.

Note that, by condition (39), the exponentially decaying rate ξ depends only on the parameters α and δ. So ξ is independent of the specific bounded set Z in which the initial point w_0 lies.

Also observe that, by condition (40), the dimension m of the flat manifold H_m depends only on the parameters $\{\alpha, \delta, a, b, q, n, \beta\}$ and the radius r of the absorbing ball, where r in turn is determined by $\{\alpha, a, b\}$. Therefore, the dimension m is determined by the physical parameters of Equation (2).

Finally, (43) implies that, for $t \ge t_0$,

$$
\begin{aligned}
\operatorname{dist}_E(w(t;w_0), \mathbf{M}) &= \operatorname{dist}_E(\pi(t)+\vartheta(t), \mathbf{M}) \le \|\vartheta(t)\|_E \\
&\le 2\min\{1,\alpha\}^{-1}(2+\alpha+\delta)\|w(t_0)\|_E^2 \exp\left(-\frac{\xi}{2}(t-t_0)\right).
\end{aligned} \tag{44}
$$

From (24) we have

$$
\|w(t_0)\|_E^2 \le 2\min\{1,\alpha\}^{-1}\left\{K_1(Z)\exp\left(-\frac{\varepsilon(Z)}{2}t_0\right) + \frac{a^2}{b}\right\}, \tag{45}
$$

where $\varepsilon = \varepsilon(Z)$ depends on the specific bounded subset Z of E from which the trajectory is started, and $K_1(Z) = \sup\{L(0) : w(0) = w_0 \in Z\}$. Substituting (45) into (44) we have, for $t \ge t_0$,

$$
\begin{aligned}
\operatorname{dist}_E(w(t;w_0), \mathbf{M}) &\le 4\min\{1,\alpha\}^{-1}(2+\alpha+\delta) \\
&\quad \cdot \left\{K_1(Z)\exp\left(-\frac{\varepsilon(Z)}{2}t_0\right) + \frac{a^2}{b}\right\}\exp\left(-\frac{\xi}{2}(t-t_0)\right).
\end{aligned} \tag{46}
$$

In order to take into account the behavior of the solutions in the transient period, let

$$
\mu = \mu(Z) = \frac{1}{2}\min\{\varepsilon(Z), \xi\}, \tag{47}
$$

and

$$
K_2(Z, t_0(Z)) = 4\min\{1,\alpha\}^{-1}(2+\alpha+\delta)\left\{K_1(Z) + \frac{a^2}{b}\exp\left(\frac{\xi}{2}t_0\right)\right\}. \tag{48}
$$

Then we can conclude that

$$\text{dist}_E(w(t;w_0),\mathbf{M}) \le K_2(Z,t_0(Z))\exp(-\mu(Z)t), \quad \text{for } t \ge 0. \tag{49}$$

Thus, by definition, $\mathbf{M} = H_m \times H_m$ is an inertial manifold with nonuniform exponential attracting rate for this dynamical system. The proof is completed. ∎

We also obtain an explicit estimate for the dimension of such an inertial manifold. In fact,

$$\dim\mathbf{M} = 2m, \tag{50}$$

where m is an integer satisfying the condition (40) in which ξ satisfies (39). The following is a concrete estimate for this dimension in terms of the physical parameters involved in this nonlinear beam equation.

Corollary 4. *Let m be the smallest positive integer which satisfies*

$$\begin{aligned} m \ge -1 \\ + \frac{1}{\pi\alpha^{1/2}}\sqrt{1+2\rho(\alpha,a,b,q,n,\beta)\left[1+\delta^{-1}\rho(\alpha,a,b,q,n,\beta)\sigma(\alpha,\delta)\right]}, \end{aligned} \tag{51}$$

where

$$\begin{aligned} \rho(\alpha,a,b,q,n,\beta) = |a| + 2a^2\max\left\{1,\alpha^{-1}\right\} \\ +1+q\left[2a^2b^{-1}\max\left\{1,\alpha^{-1}\right\}+1\right]^{2(n+\beta)+1} \end{aligned} \tag{52}$$

and

$$\sigma(\alpha,\delta) = \max\left\{1,\alpha^{-1}(1+\delta),2\delta^{-1}\right\}. \tag{53}$$

Then there exists an inertial manifold $\mathbf{M}$ *given by* (29) *with* $\dim\mathbf{M} = 2m$.

Proof. Letting $r^2 = 2a^2b^{-1}\max\left\{1,\alpha^{-1}\right\}+1$ and $\xi = \min\{1,\alpha(1+\delta)^{-1}, \delta/2\}$ in $\Pi(r)$ and $K(r,\xi)$, we see that formula (40) can be written as (51). So the result holds. ∎

Remarks.

Here we have the following two remarks. First, from (51), we see that as the coefficient δ of the structural damping becomes smaller, the lower bound for the dimension of an inertial manifold will increase and its growth rate is roughly proportional to δ^{-1}, provided that there are no dramatic changes in the other parameters.

Second and more important, if one does not know each parameter exactly and must allow the value variations over some moderate range, then (51) also gives a conservative estimate of the lower bound for the dimension of an inertial manifold. Together with the result presented in the next section, this will be useful in providing a robust stabilization of this nonlinear beam system by a finite-dimensional, linear feedback control.

The governing equation for the dynamics on an inertial manifold is called *inertial form*, which is simply a system of ordinary differential equations.

Corollary 5. *For the inertial manifold* $\mathbf{M} = H_m \times H_m$ *the inertial form is the following equation in the subspace* H_m.

$$\begin{aligned} &\frac{d^2p}{dt^2} + \alpha Ap(t) \\ &\qquad + \delta A^{1/2}\frac{dp}{dt} + \left[a + b|A^{1/4}p(t)|^2 + q\langle A^{1/2}p(t), \frac{dp}{dt}\rangle\right] A^{1/2}p(t) \\ &= 0, \quad t \geq 0, \end{aligned} \tag{54}$$

$$p(0) = p_0 \in H_m, \quad p_t(0) = p_1 \in H_m.$$

Here $p(t)$ is a finite-dimensional vector.

4. Stabilization by Finite-dimensional Feedback Control

In the last section we shall resolve the *spillover problem* based on the existence of inertial manifolds. This approach for achieving stabilization is potentially applicable to other distributed control systems featuring nonlocal nonlinearities.

Now consider the full equation, Equation (1), with the control function $f(t, x)$ on the right-hand side. The stabilization problem for Equation (1) is to find a linear or nonlinear feedback mapping $F : E \to H$, in general it can be nonautonomous, such that the feedback control

$$f(t, \cdot) = F(u(t), u_t(t)), \quad t \geq 0,$$

makes the closed-loop system asymptotically stable in the sense that all the mild solutions $w(t)$ of the closed-loop equation converge to zero in E, as $t \to \infty$. If so, Equation (1) is said to be *strongly stabilizable* by the feedback control. Moreover, if in addition the convergence occurs at a uniform exponential rate, then Equation (1) is called *uniformly exponentially stabilizable.* If the convergence occurs at a nonuniform exponential rate, which means that the decay rate depends on the bounded set which the initial data belong to, then Equation (1) is said to be *nonuniformly exponentially stabilizable.*

Theorem 6. *The control system Equation (1) is nonuniformly exponentially stabilizable by a finite-dimensional linear feedback control*

$$f(t, \cdot) = aA^{1/2}P_m u(t), \quad t \geq 0, \tag{55}$$

where $P_m : H \to H_m$ *is the orthogonal projection, and* H_m *is the factor subspace associated with the inertial manifold* $\mathbf{M} = H_m \times H_m$ *for the uncontrolled Equation (2).*

Proof. By applying this feedback control (55) to Equation (1) and decomposing the equation into two component equations according to the decomposition of $H = H_m \oplus Q_m H$, we can get

$$p_{tt} + \alpha p_{xxxx} - \delta p_{xxt} - (J_u(t) - a)p_{xx} = 0, \tag*{$(56)_p$}$$

$$(56)_h \qquad h_{tt} + \alpha h_{xxxx} - \delta h_{xxt} - J_u(t)h_{xx} = 0,$$

where $J(u(t))$ is defined by (28), and $u(t) = p(t) + h(t)$ is a solution of the following closed-loop equation

$$\begin{aligned} (57) \qquad & u_{tt} + \alpha A u + \delta A^{1/2} u_t \\ & \qquad + \left[a + b|A^{1/4}u|^2 + q\langle A^{1/2}u, u_t\rangle^{2(n+\beta)+1}\right] A^{1/2}u \\ & = aA^{1/2}P_m u(t), \end{aligned}$$

$$u(0) = u_0, \quad u_t(0) = u_1.$$

Since this linear feedback (55) only partially cancels the term $-ap_{xx}$ on the left-hand side of the equation (56) or (57), an easy adaptation in the corresponding proof assures us that Theorem 2 remains valid and the ball B_r remains an absorbing set for the new closed-loop equation, Equation (57).

Since the linear feedback (55) does not affect Equation $(56)_h$, the same argument used in the proof of Theorem 3 for the exponential attraction, especially within the absorbing ball B_r, of the manifold $\mathbf{M}$ remains true without any change in the constants. Hence, it is true that

$$(58) \qquad \left\| \begin{pmatrix} h(t) \\ h_t(t) \end{pmatrix} \right\|_E^2 \leq K_2(Z, t_0(Z)) \exp\left(-\mu(Z)t\right), \quad t \geq 0,$$

where $h(t) = Q_m u(t)$ and the constants $K_2(Z, t_0(Z))$ and $\mu(Z)$ are the same as before.

It suffices to concentrate on Equation $(56)_p$. Specifically, we want to prove that the component $p(t) = P_m u(t)$ of the solution $u(t)$ of the closed-loop equation (57) also converges to zero at nonuniform exponential rate. By taking the inner product of $(56)_p$ in H with $2p_t + \kappa p$, where κ is an undetermined constant, we get

$$\begin{aligned} (59) \qquad & \frac{d}{dt}\left\{|p_t|^2 + \alpha|p_{xx}|^2 + \kappa\langle p_t, p\rangle + (\kappa\delta/2)|p_x|^2 + (b/2)|p_x|^4\right\} \\ & + \left\{2\delta|p_{xt}|^2 - \kappa|p_t|^2 + \kappa\alpha|p_{xx}|^2 + \kappa b|p_x|^4\right\} \\ & + \left\{2b\langle p_{xx}, p_t\rangle|h_x|^2 + \kappa b|p_x|^2|h_x|^2 \right. \\ & \left. + q\left[2\langle p_{xt}, p_x\rangle + \kappa|p_x|^2\right]\langle u_x, u_{xt}\rangle^{2(n+\beta)+1}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{dt}\left\{|p_t|^2 + \alpha |p_{xx}|^2 + \kappa\langle p_t, p\rangle + (\kappa\delta/2)|p_x|^2 + (b/2)|p_x|^4\right\} \\
&\quad + \left\{2\delta |p_{xt}|^2 - \kappa |p_t|^2 + \kappa\alpha |p_{xx}|^2 + \kappa b |p_x|^4\right. \\
&\quad \left. + \left[2q\langle p_x, p_{xt}\rangle^2 + \kappa q |p_x|^2 \langle p_x, p_{xt}\rangle\right] \langle u_x, u_{xt}\rangle^{2(n+\beta)}\right\} \\
&\quad + \left\{2b\langle p_{xx}, p_t\rangle |h_x|^2 + \kappa b |p_x|^2 |h_x|^2\right. \\
&\quad \left. + q\left[2\langle p_{xt}, p_x\rangle + \kappa |p_x|^2\right] \langle h_x, h_{xt}\rangle \langle u_x, u_{xt}\rangle^{2(n+\beta)}\right\} \\
&= 0.
\end{aligned} \tag{59}
$$

Let

$$
\Gamma(t) = |p_t|^2 + \alpha |p_{xx}|^2 + \kappa\langle p_t, p\rangle + (\kappa\delta/2)|p_x|^2 + (b/2)|p_x|^4, \tag{60}
$$

and

$$
\begin{aligned}
\Delta(t) &= 2\delta |p_{xt}|^2 - \kappa |p_t|^2 + \kappa\alpha |p_{xx}|^2 + \kappa b |p_x|^4 \\
&\quad + \left[2q\langle p_x, p_{xt}\rangle^2 + \kappa q |p_x|^2 \langle p_x, p_{xt}\rangle\right] \langle u_x, u_{xt}\rangle^{2(n+\beta)}.
\end{aligned} \tag{61}
$$

Using Young's inequality to treat the last term of $\Delta(t)$, we have

$$
\begin{aligned}
\Delta(t) - \frac{\kappa}{2}\Gamma(t) &\geq \delta |p_{xt}|^2 + \left(\delta - \frac{3\kappa}{2} - \frac{\kappa^2}{2}\right)|p_t|^2 \\
&\quad + \kappa\left(\frac{\alpha}{2} - \frac{\kappa}{2} - \frac{\kappa\delta}{4}\right)|p_{xx}|^2 \\
&\quad + \left[\frac{3\kappa b}{4}|p_x|^4 + 2q\langle p_x, p_{xt}\rangle^2 \langle u_x, u_{xt}\rangle^{2(n+\beta)}\right] \\
&\quad - \eta\kappa q |p_x|^4 \langle u_x, u_{xt}\rangle^{2(n+\beta)} - C(\eta)\kappa q |p_x|^2 |p_{xt}|^2 \langle u_x, u_{xt}\rangle^{2(n+\beta)} \\
&\geq \left(\delta - \frac{3\kappa}{2} - \frac{\kappa^2}{2}\right)|p_t|^2 + \kappa\left(\frac{\alpha}{2} - \frac{\kappa}{2} - \frac{\kappa\delta}{4} - \eta q r^{4(n+\beta)+2}\right)|p_{xx}|^2 \\
&\quad + \left[\delta - C(\eta)\kappa q r^{4(n+\beta)+2}\right]|p_{xt}|^2 \\
&\geq 0, \quad \text{for } t \geq 0,
\end{aligned} \tag{62}
$$

if we choose $\eta > 0$ and $\kappa > 0$ small enough, so that

$$
\frac{\alpha}{4} - \eta q r^{4(n+\beta)+2} \geq 0, \tag{63}
$$

and

$$
\begin{aligned}
\delta - \frac{3\kappa}{2} - \frac{\kappa^2}{2} \geq 0, \quad \frac{\alpha}{2} - \kappa - \frac{\kappa\delta}{2} \geq 0, \quad \delta - \kappa C(\eta) q r^{4(n+\beta)+2} \geq 0, \\
\min\{1, \alpha) - \kappa > 0.
\end{aligned} \tag{64}
$$

On the other hand, for the last $\{\dots\}$ portion on the left-hand side of the last equality in (59), we have the following estimate, which is valid within the given absorbing ball B_r.

$$\begin{aligned}&\Big|2b\langle p_{xx}, p_t\rangle \left|h_x\right|^2 + \kappa b \left|p_x\right|^2 \left|h_x\right|^2 \\ &\quad + q\left[2\langle p_{xt}, p_x\rangle + \kappa \left|p_x\right|^2\right] \langle h_x, h_{xt}\rangle \langle u_x, u_{xt}\rangle^{2(n+\beta)}\Big| \\ &\le (2+\kappa) b r^2 \left|h_x\right|^2 + (2+\kappa) q r^{4(n+\beta)+2} \left|h_{xx}\right| \left|h_t\right| \\ &\le (2+\kappa) r^2 \left(b + q r^{4(n+\beta)}\right) K_2\left(Z, t_0(Z)\right) \exp\left(-\mu(Z) t\right), \quad \text{for } t \ge t_0,\end{aligned} \tag{65}$$

where we used the fact that $|p_{xx}|,\ |p_x|,\ |p_t|,\ |u_{xx}|,\ |u_x|, |u_t| \le r$ in the fixed absorbing ball B_r and, in the last inequality of (65), $|h_x|^2$ and $|h_{xx}|\,|h_t|$ are replaced by the estimate(58) for the h-component.

We now substitute (62) and (65) into (59), then get

$$\frac{d}{dt}\Gamma(t) + \frac{\kappa}{2}\Gamma(t) \le K_3\left(Z, t_0(Z)\right) \exp\left(-\mu(Z) t\right), \quad t \ge t_0, \tag{66}$$

where

$$K_3\left(Z, t_0(Z)\right) = (2+\kappa) r^2 \left(b + q r^{4(n+\beta)}\right) K_2\left(Z, t_0(Z)\right), \tag{67}$$

with κ chosen and fixed as above. By integrating this inhomogeneous differential inequality (66) over the time interval $[t_0, t]$, we obtain

$$\begin{aligned}\Gamma(t) &\le \Gamma(t_0) \exp\left(-\frac{\kappa}{2}(t - t_0)\right) \\ &\qquad + \frac{K_3\left(Z, t_0(Z)\right)}{\left|\frac{\kappa}{2} - \mu(Z)\right|} \exp\left(-\min\left\{\frac{\kappa}{2}, \mu(Z)\right\}(t - t_0)\right), \quad t \ge t_0,\end{aligned} \tag{68$_a$}$$

if $\frac{\kappa}{2} \ne \mu(Z)$, or

$$\begin{aligned}\Gamma(t) &\le \Gamma(t_0) \exp\left(-\frac{\kappa}{2}(t - t_0)\right) \\ &\qquad + K_3\left(Z, t_0(Z)\right)(t - t_0) \exp\left(-\frac{\kappa}{2}(t - t_0)\right) \\ &\le \frac{4}{\kappa} K_3\left(Z, t_0(Z)\right) \exp\left(-\frac{\kappa}{4}(t - t_0)\right), \quad t \ge t_0,\end{aligned} \tag{68$_b$}$$

if $\frac{\kappa}{2} = \mu(Z)$. By (47), $\mu(Z) = (1/2)\min\{\varepsilon(Z), \xi\}$. Now let

$$\nu = \nu(Z) = \frac{1}{2}\min\left\{\frac{\kappa}{2}, \varepsilon(Z), \xi\right\}. \tag{69}$$

Since

$$\Gamma(t) \geq \frac{1}{2} \min\{1, \alpha\} \left(|p_t|^2 + |p_{xx}|^2\right) = \frac{1}{2} \min\{1, \alpha\} \left\| \begin{pmatrix} p(t) \\ p_t(t) \end{pmatrix} \right\|_E^2, \tag{70}$$

$$\Gamma(t_0) \leq \left(2 + \kappa + \frac{\delta}{2}\right) r^2 + \frac{b}{2} r^4, \tag{71}$$

where we can take $r^2 = 2a^2 b^{-1} \max\left\{1, \alpha^{-1}\right\} + 1$ in (26). Let

$$\begin{aligned} K_4(Z, t_0(Z)) = 2 \max\left\{1, \alpha^{-1}\right\} \Bigg[\left(2 + \alpha + \frac{\delta}{2}\right) r^2 + \frac{b}{2} r^4 \\ + \frac{K_3(Z, t_0(Z))}{\left|\frac{\kappa}{2} - \mu(Z)\right|} \Bigg] \exp(\nu(Z) t_0), \quad \text{if } \frac{\kappa}{2} \neq \mu(Z), \end{aligned} \tag{72$_a$}$$

or

$$\begin{aligned} K_4(Z, t_0(Z)) = 2 \max\left\{1, \alpha^{-1}\right\} \Bigg[\left(2 + \alpha + \frac{\delta}{2}\right) r^2 + \frac{b}{2} r^4 \\ + \frac{4}{\kappa} K_3(Z, t_0(Z)) \Bigg] \exp(\nu(Z) t_0), \quad \text{if } \frac{\kappa}{2} = \mu(Z). \end{aligned} \tag{72$_b$}$$

Then from $(68)_a$ and $(68)_b$ it follows that

$$\left\| \begin{pmatrix} p(t) \\ p_t(t) \end{pmatrix} \right\|_E^2 \leq K_4(Z, t_0(Z)) \exp(-\nu(Z) t), \quad t \geq t_0. \tag{73}$$

Next we combine this result with the exponential decay estimate (24) in the transient period $[0, t_0]$, which gives

$$\begin{aligned} &\left\| \begin{pmatrix} p(t) \\ p_t(t) \end{pmatrix} \right\|_E^2 \leq \left\| \begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} \right\|_E^2 \\ &\leq 2 \max\left\{1, \alpha^{-1}\right\} \left[L(0) \exp\left(-\frac{\varepsilon}{2} t\right) + \frac{a^2}{b} + 1 \right] \\ &\leq 2 \max\left\{1, \alpha^{-1}\right\} \left[K_1(Z) + \left(\frac{a^2}{b} + 1\right) \exp\left(\frac{\varepsilon}{2} t_0\right) \right] \exp\left(-\frac{\varepsilon}{2} t\right) \\ &\leq K_5(Z, t_0(Z)) \exp(-\nu(Z) t), \quad \text{for } t \in [0, t_0], \end{aligned} \tag{74}$$

where

$$K_5(Z, t_0(Z)) = 2 \max\left\{1, \alpha^{-1}\right\} \left[K_1(Z) + \left(\frac{a^2}{b} + 1\right) \exp\left(\frac{\varepsilon}{2} t_0\right) \right]. \tag{75}$$

Therefore, we have proved the nonuniform exponential decay of the p-component of the closed-loop solution, which is given by

$$\left\| \begin{pmatrix} p(t) \\ p_t(t) \end{pmatrix} \right\|_E^2 \leq \left[K_4\left(Z, t_0(Z)\right) + K_5\left(Z, t_0(Z)\right)\right] \exp\left(-\nu(Z)t\right), \quad t \geq 0. \tag{76}$$

Finally, (58) and (76) together imply that the solution of the closed-loop equation (57) has the following property of nonuniform exponential decay,

$$\left\| \begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix} \right\|_E^2 \leq K_6(Z) \exp\left(-\nu(Z)t\right), \quad t \geq 0, \tag{77}$$

where

$$K_6(Z) = \max\left\{K_2\left(Z, t_0(Z)\right), K_4\left(Z, t_0(Z)\right) + K_5\left(Z, t_0(Z)\right)\right\}. \blacksquare \tag{78}$$

Summary Remarks. As a conclusion, below we briefly summarize the important points in this paper.

First, Theorem 6 shows that, by using the finite-dimensional linear feedback (55), the exponential decay rate $\nu(Z)$ and the coefficient $K_6(Z)$ can be estimated in terms of the physical parameters in this beam equation and the radius of the initial bounded set Z in E.

Second, the number m of the modes involved in this stabilizing feedback can be estimated by Corollary 4. Note that m increases (roughly proportionally to $1/\delta$) as the coefficient δ of the structural damping decreases.

Third, it is noteworthy that if we replace the physical parameters $\{\alpha,\ \delta,\ a,\ b,\ q, n, \beta\}$ in the dimension formula (51)-(52) by their conservative bounds of uncertainty, then Theorem 3, Corollary 4, and Theorem 6 regarding the existence of inertial manifolds, the dimension estimate, and the exponential stabilization are all *robust*. Therefore, in other words, we have obtained a robust solution to the concerned spillover problem.

Fourth, the results on the existence of inertial manifolds and on the finite-dimensional stabilization hold also for the same nonlinear beam equation except that the structural damping $-\delta u_{xxt}$ is replaced by a strong structural damping δu_{xxxxt} without any substantial change in the argument.

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