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STRICT MODULES AND HOMOTOPY MODULES IN STABLE HOMOTOPY

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Abstract

Let R be any associative ring with unit and let HR denote the corresponding Eilenberg–Mac Lane spectrum. We show that the category of algebras over the monad $X \mapsto HR \wedge X$ on the homotopy category of spectra is equivalent to the homotopy category associated to a model category of HR-module spectra, if the ring R is a field or a subring of the rationals, but not for all rings.

1. Introduction

Classically, ring spectra and module spectra were defined as objects of the stable homotopy category equipped with suitable structure maps. The stable homotopy category, as described by Adams in [**1**], has a smash product which is associative and commutative up to homotopy. The structure maps that define ring spectra and module spectra give rise to diagrams that commute up to homotopy. For a given ring spectrum E, the E-modules in this sense, together with the E-module maps, form a subcategory which can be seen as the Eilenberg–Moore category associated with the monad defined by $X \mapsto E \wedge X$, i.e., the category of algebras over this monad. We call this category the *category of homotopy* E*-modules*.

The recent discovery of new structured model categories for stable homotopy, such as the categories of S-modules [**7**] or symmetric spectra [**9**], equipped with a strictly associative and commutative smash product, allows one to define strict ring spectra (the monoids in the category) and strict module spectra (modules over monoids). The structure maps for these objects give rise to diagrams that truly commute in the model category. Thus, for a strict ring spectrum E, we can as well consider the homotopy category of strict E-modules, by endowing the category of strict E-modules with a model structure as in [**7**] or in [**9**].

The categories of strict modules have better properties than the categories of homotopy modules. The fibre of an E-module map of strict E-modules is a strict E-module, yet this need not be true for homotopy E-modules. The model category of strict HR-modules is Quillen equivalent to the category $\mathrm{Ch}(R)$ of unbounded chain complexes of R-modules; see [**7**], [**12**], [**13**].

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The homotopy category of strict HR-modules is not equivalent to the category of homotopy HR-modules in general. However, they are equivalent in some special cases, for example when $R = \mathbb{Z}$ (the homotopy $H\mathbb{Z}$-modules are also called *stable GEMs*). We give a sufficient condition for a ring R in order that there is an equivalence between the homotopy category of strict HR-modules and the category of homotopy HR-modules. This condition is fulfilled by fields and by subrings of $\mathbb{Q}$.

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2. Monads and Eilenberg–Moore categories

In this section we recall the definition of a monad on a category, and collect some basic results about Eilenberg–Moore categories. These and other facts about monads can be found in [**3**, Ch. 3], [**4**, Ch. 4] or [**11**, Ch. VI].

A *monad* on a category $\mathcal{C}$ is a triple $\mathbf{T} = (T, \eta, \mu)$ where $T\colon \mathcal{C} \longrightarrow \mathcal{C}$ is a functor and $\eta\colon Id_{\mathcal{C}} \longrightarrow T$ and $\mu\colon TT \longrightarrow T$ are natural transformations such that the following diagrams commute:

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, equal] & TT \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} TTT \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"'] & T. \end{tikzcd}$$

Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. An *algebra over* $\mathbf{T}$ or a $\mathbf{T}$-*algebra* is a pair (M, m) where M is an object of $\mathcal{C}$ and $m\colon TM \longrightarrow M$ is a morphism such that the following diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[dr, equal] & TM \arrow[d, "m"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} TTM \arrow[r, "\mu_M"] \arrow[d, "Tm"'] & TM \arrow[d, "m"] \\ TM \arrow[r, "m"'] & M. \end{tikzcd}$$

If (N, n) is another $\mathbf{T}$-algebra, then a *morphism of* $\mathbf{T}$-*algebras* or a $\mathbf{T}$-*morphism* $f\colon (M, m) \longrightarrow (N, n)$ is a morphism $f\colon M \longrightarrow N$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "m"'] & TN \arrow[d, "n"] \\ M \arrow[r, "f"'] & N. \end{tikzcd}$$

Given a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathcal{C}$, we denote by $\mathcal{C}^{\mathbf{T}}$ the category whose objects are the $\mathbf{T}$-algebras and whose morphisms are the $\mathbf{T}$-morphisms. The category $\mathcal{C}^{\mathbf{T}}$ is called the *Eilenberg–Moore category* associated with $\mathbf{T}$.

There is a forgetful functor $U\colon \mathcal{C}^{\mathbf{T}} \longrightarrow \mathcal{C}$ defined by $U(M,m) = M$ and $U(f) = f$. This functor is faithful and has a left adjoint $F\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathbf{T}}$ defined by $F(M) = (TM, \mu_M)$ and $F(f) = T(f)$. This adjunction yields a bijection

$$\mathcal{C}^{\mathbf{T}}((TX,\mu_X),(Y,m)) \cong \mathcal{C}(X,Y) \tag{1}$$

for any $X \in \mathcal{C}$ and any $\mathbf{T}$-algebra (Y,m).

Given two monads $\mathbf{T} = (T,\eta,\mu)$ and $\mathbf{S} = (S,\eta',\mu')$ on a category $\mathcal{C}$, a *morphism of monads* $\mathbf{S} \longrightarrow \mathbf{T}$ is a natural transformation $\lambda\colon S \longrightarrow T$ such that the following diagrams commute:

$$\begin{tikzcd} S \arrow[r, "\lambda"] & T \\ Id_{\mathcal{C}} \arrow[u, "\eta'"] \arrow[ur, "\eta"'] & \end{tikzcd} \qquad \begin{tikzcd} SS \arrow[r, "\lambda\lambda"] \arrow[d, "\mu'"'] & TT \arrow[d, "\mu"] \\ S \arrow[r, "\lambda"'] & T. \end{tikzcd}$$

Remark 2.1. Any morphism of monads $\lambda\colon S \longrightarrow T$ yields a faithful functor between the categories of algebras $Q\colon \mathcal{C}^{\mathbf{T}} \longrightarrow \mathcal{C}^{\mathbf{S}}$, since any $\mathbf{T}$-algebra has an $\mathbf{S}$-algebra structure via the morphism λ. Thus, Q is defined as $Q(M,m) = (M, m \circ \lambda_M)$ and $Q(f) = f$. There is a commutative diagram

$$\begin{tikzcd} \mathcal{C}^{\mathbf{T}} \arrow[r, "Q"] \arrow[d, "U"'] & \mathcal{C}^{\mathbf{S}} \arrow[dl, "U"] \\ \mathcal{C} & \end{tikzcd}$$

where U is the forgetful functor. This shows that the functor Q is faithful.

Example 2.2. Let $\mathcal{A}b$ be the category of abelian groups and let R be a ring with unit. The functor $R \otimes -\colon \mathcal{A}b \longrightarrow \mathcal{A}b$ together with the product and the unit of R is a monad on the category of abelian groups. The Eilenberg–Moore category associated with this monad is the category of left R-modules.

3. Stable categories of E-modules

In this section we describe, for a ring spectrum E, the category of strict E-modules and the category of homotopy E-modules as particular cases of Eilenberg–Moore categories associated with monads. We will work in the category Sp^{Σ} of symmetric spectra [**9**] as a model category for the stable homotopy category. An object E of Sp^{Σ} is a *ring spectrum* if it is equipped with two maps $\mu\colon E \wedge E \longrightarrow E$ and $\eta\colon S \longrightarrow E$, where S is the sphere spectrum, such that the following diagrams commute:

$$\begin{tikzcd} S \wedge E \arrow[r, "\eta \wedge Id_E"] \arrow[dr, equal] & E \wedge E \arrow[d, "\mu"] & E \wedge S \arrow[l, "Id_E \wedge \eta"'] \arrow[dl, equal] \\ & E & \end{tikzcd} \qquad \begin{tikzcd} E \wedge E \wedge E \arrow[r, "\mu \wedge Id_E"] \arrow[d, "Id_E \wedge \mu"'] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"'] & E. \end{tikzcd} \tag{2}$$

It is said that E is *commutative* if $\mu \circ \tau = \mu$ where $\tau\colon E \wedge E \longrightarrow E \wedge E$ is the twist map. Given a ring spectrum $E \in Sp^\Sigma$, an *E-module spectrum* is a pair (M, m) with $M \in Sp^\Sigma$ and $m\colon E \wedge M \longrightarrow M$ such that the following diagrams commute:

$$\begin{tikzcd} S \wedge M \arrow[r, "\eta\wedge Id_M"] \arrow[dr, equal] & E \wedge M \arrow[d, "m"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} E \wedge E \wedge M \arrow[r, "\mu\wedge Id_M"] \arrow[d, "Id_E\wedge m"'] & E \wedge M \arrow[d, "m"] \\ E \wedge M \arrow[r, "m"'] & M. \end{tikzcd} \tag{3}$$

Example 3.1. If R is an associative ring with unit and M is a left R-module, then the Eilenberg–Mac Lane spectrum HR is a ring spectrum and the spectrum HM is an HR-module spectrum. The structure maps of HR and HM come from the product and the unit of R, and from the structure homomorphism of M as an R-module.

A *map of E-modules* or an *E-module map* $f\colon (M, m) \longrightarrow (N, n)$ is a map $f\colon M \longrightarrow N$ such that the following diagram commutes:

$$\begin{tikzcd} E \wedge M \arrow[r, "Id_E\wedge f"] \arrow[d, "m"'] & E \wedge N \arrow[d, "n"] \\ M \arrow[r, "f"'] & N. \end{tikzcd} \tag{4}$$

For any ring spectrum (E, η, μ) we can consider the functor $E \wedge -\colon Sp^\Sigma \longrightarrow Sp^\Sigma$. This functor sends any X to an E-module spectrum $E \wedge X$. The natural transformations $\eta \wedge Id\colon Id_{Sp^\Sigma} \longrightarrow E \wedge -$ and $\mu \wedge Id\colon E \wedge E \wedge - \longrightarrow E \wedge -$ form a monad on the category of symmetric spectra, by the commutativity of (3.1). The Eilenberg–Moore category associated with the monad $(E \wedge -, \eta \wedge Id, \mu \wedge Id)$ will be denoted by *E-mod* and called the *category of strict E-modules*. By [**13**], this category admits a model category structure. If $Ho(E\text{-}mod)$ is the corresponding homotopy category, and M and N are objects in this category, we denote by $[M, N]_{E\text{-}mod}$ the group of morphisms between M and N in $Ho(E\text{-}mod)$. Thus, (1) yields a bijection

$$E\text{-}mod(E \wedge M, N) \cong Sp^\Sigma(M, N)$$

for any E-module spectrum N and any M. This bijection does not induce a bijection of homotopy classes of maps in general, as shown in Corollaries 4.5 and 4.6.

Now, for a ring spectrum $E \in Sp^\Sigma$, consider the monad $(E \wedge -, \eta \wedge Id, \mu \wedge Id)$ on the homotopy category $Ho(Sp^\Sigma)$. The Eilenberg–Moore category associated with this monad will be called the *category of homotopy E-modules*, and denoted by *E-hmod*. If M and N are objects in *E-hmod*, we denote by $[M, N]_{E\text{-}hmod}$ the group of morphisms between them in the Eilenberg–Moore category. If N is a homotopy E-module and M is any spectrum, then the bijection (1) gives an isomorphism

$$[E \wedge M, N]_{E\text{-}hmod} \cong [M, N].$$

Note that the objects in *E-hmod* are E-module spectra in the traditional sense, i.e., endowed with structure maps for which the diagrams (3) and (4) commute up to homotopy. Thus, every strict E-module is a homotopy E-module.

4. Homotopy modules and derived categories

The categories *E-hmod* and $Ho(E\text{-}mod)$, defined in the previous section, are very different in general. In this section we compare these two categories in the case where E is the ring spectrum HR, for some associative ring R with unit. In what follows R-modules will be left modules.

The derived category $\mathcal{D}(R)$ of the ring R is defined as the homotopy category of $\mathrm{Ch}(R)$, the model category of unbounded chain complexes of R-modules; see [**8**]. The weak equivalences are the quasi-isomorphisms, i.e., the maps inducing isomorphisms in homology. If E is any R-module, we will denote by $E[k]$ the chain complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow E \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

where E is located in dimension k. If A and B are two R-modules, then the following holds:

$$\mathcal{D}(R)(A[0], B[k]) = \mathrm{Ext}^k_R(A, B);$$

see [**14**, Ch. 10] for a useful description of the derived category.

The *projective dimension* $\mathrm{pd}(A)$ of an R-module A is the minimum integer n (if it exists) such that there is a projective resolution of A of length n,

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0.$$

If no such integer exists, we say that $\mathrm{pd}(A) = \infty$. The *global dimension* of a ring R is defined as $\mathrm{gd}(R) = \sup_{A \in R\text{-}mod}\{\mathrm{pd}(A)\}$. For example, $\mathrm{gd}(\mathbb{Z}) = 1$, $\mathrm{gd}(\mathbb{Z}/p^2) = \infty$ if p is a prime, $\mathrm{gd}(R[x_1, \ldots, x_n]) = \mathrm{gd}(R) + n$.

The rings R with $\mathrm{gd}(R) = 0$ are called *semisimple*. All fields and finite direct products of fields are semisimple rings. In general, $\mathrm{gd}(R) = 0$ if and only if R is a finite direct product of matrix rings over division rings, by the Wedderburn–Artin Theorem; see [**2**, §13], for example. Thus, if R is commutative, then $\mathrm{gd}(R) = 0$ if and only if R is a finite direct product of fields.

The groups Ext^k_R are closely related with the global dimension of the ring R. A classical theorem in homological algebra states that $\mathrm{gd}(R) = k$ if and only if $\mathrm{Ext}^i_R(A, B) = 0$ for $i > k$ and all R-modules A and B; see [**14**].

Proposition 4.1. *If $\mathrm{gd}(R) \leqslant 1$, then any chain complex of R-modules*

$$\mathbf{C}: \ \cdots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow C_0 \xrightarrow{d_0} C_{-1} \longrightarrow \cdots$$

is weakly equivalent, and hence isomorphic in $\mathcal{D}(R)$, to $\oplus_{k \in \mathbb{Z}} H_k(\mathbf{C})[k]$.

Proof. If $\mathrm{gd}(R) = 0$, then there exists a map $\widetilde{p}_k \colon H_k(\mathbf{C}) \longrightarrow \ker d_k \subset C_k$ such that $\pi \circ \widetilde{p}_k = id$, where π denotes the projection $\ker d_k \longrightarrow H_k(\mathbf{C})$, since every R-module is projective. Hence, for each $k \in \mathbb{Z}$ we have a map

$$\begin{tikzcd} \cdots \arrow[r] & C_k \arrow[r] & C_{k-1} \arrow[r] & \cdots \\ & H_k(\mathbf{C}) \arrow[u, "\widetilde{p}_k"] \arrow[r] & 0 \arrow[u] & \end{tikzcd}$$

and this yields a map of chain complexes $\phi\colon \oplus_{k\in\mathbb{Z}} H_k(\mathbf{C})[k] \longrightarrow \mathbf{C}$ inducing an isomorphism in homology. For the case $\mathrm{gd}(R) = 1$, let $R_k \xrightarrow{i_k} F_k \xrightarrow{p_k} H_k(\mathbf{C})$ be a projective resolution of the k-th homology group $H_k(\mathbf{C})$. Take $\mathbf{A}_k$ to be the complex $\cdots \to 0 \to R_k \to F_k \to 0 \to \cdots$ with F_k in dimension k and R_k in dimension $k+1$. Now we construct a map from $\oplus_{k\in\mathbb{Z}}\mathbf{A}_k$ to $\mathbf{C}$ inducing an isomorphism in homology. For each $k \in \mathbb{Z}$, we have the following diagram:

$$\begin{tikzcd}
 & & C_{k+1} \arrow[d, two heads] \\
 & & \operatorname{Im} d_{k+1} \arrow[d, tail] \\
 & & \ker d_k \arrow[d, two heads, "\pi"] \\
R_k \arrow[r, tail, "i_k"'] \arrow[uurr, "\widetilde{p}_k\circ i_k"] \arrow[uuurr, dotted, bend left, "\exists\widetilde{q}_k"] & F_k \arrow[r, two heads, "p_k"'] \arrow[ur, dotted, "\exists\widetilde{p}_k"] & H_k(\mathbf{C}).
\end{tikzcd}$$

Since F_k is projective and π is surjective, there exists a map $\widetilde{p}_k\colon F_k \longrightarrow \ker d_k \subset C_k$ closing the diagram. The map $\widetilde{p}_k \circ i_k$ lifts to $\operatorname{Im} d_{k+1}$ because $\widetilde{p}_k \circ i_k \subset \ker \pi = \operatorname{Im} d_{k+1}$. Again, R_k is projective and the map $C_{k+1} \to \operatorname{Im} d_{k+1}$ is surjective, hence there exists a map $\widetilde{q}_k\colon R_k \longrightarrow C_{k+1}$ closing the diagram. For each $k \in \mathbb{Z}$, we have defined maps $\widetilde{p}_k$ and $\widetilde{q}_k$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{k+1} \arrow[r] & C_k \arrow[r] & \cdots \\
 & R_k \arrow[r] \arrow[u, "\widetilde{q}_k"] & F_k \arrow[u, "\widetilde{p}_k"'] &
\end{tikzcd}$$

and this yields a map $\phi\colon \oplus_{k\in\mathbb{Z}} \mathbf{A}_k \longrightarrow \mathbf{C}$ that is a quasi-isomorphism. Since the complex $\mathbf{A}_k$ is quasi-isomorphic to $H_k(\mathbf{C})[k]$, we have that $\oplus_{k\in\mathbb{Z}}H_k(\mathbf{C})[k]$ and $\mathbf{C}$ are quasi-isomorphic. □

Remark 4.2. Note that Proposition 4.1 does not hold if $\mathrm{gd}(R) > 1$. Suppose that $\mathrm{gd}(R) = k > 1$ and consider a nonzero element ξ in $\mathrm{Ext}^k_R(M, N)$, where M and N are R-modules. This element ξ can be represented by an extension of modules

$$0 \longrightarrow N \longrightarrow E_k \longrightarrow \cdots \longrightarrow E_1 \longrightarrow M \longrightarrow 0,$$

where $E_1, \ldots, E_k$ are free [**10**, Corollary III.6.5]. Take now the chain complex

$$\mathbf{E}: \ \cdots \longrightarrow 0 \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow 0 \longrightarrow \cdots$$

where E_1 is in dimension 0. This complex has homology only in dimensions 0 and $k-1$, namely $H_0(\mathbf{E}) = M$ and $H_{k-1}(\mathbf{E}) = N$. But if this complex is quasi-isomorphic to $M[0] \oplus N[k-1]$ then $\xi = 0$, because $E_1, \ldots, E_k$ are free and hence there exists a quasi-isomorphism from the first complex to the second one, and therefore a commutative diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d, "\cong"'] & E_k \arrow[r] \arrow[d] & E_{k-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E_2 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & M \arrow[d, "\cong"] \\ N \arrow[r, "id"'] & N \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & M \arrow[r, "id"'] & M. \end{tikzcd}$$

The following result was first proved in [**12**]. A recent generalization can be found in [**13**].

Theorem 4.3. *For any ring R there is a Quillen equivalence between the model category of unbounded chain complexes of R-modules and the model category of (strict) HR-modules. This equivalence induces an equivalence between the homotopy categories $\mathcal{D}(R)$ and $Ho(HR\text{-}mod)$ that sends each HR-module M to a chain complex $\mathbf{C}$ such that $H_k(\mathbf{C}) \cong \pi_k(M)$ for every $k \in \mathbb{Z}$.* □

The objects of the category HR-*hmod* are precisely the stable R-GEMs and have been studied in [**6**, Section 5]. Recall that a spectrum $E \in Ho(Sp^\Sigma)$ is a stable R-GEM if $E \simeq \vee_{k\in\mathbb{Z}}\Sigma^k HA_k$ where each A_k is an R-module. In the case $R = \mathbb{Z}$, by [**6**, Proposition 5.3], any $H\mathbb{Z}$-module is isomorphic in $H\mathbb{Z}$-*hmod* to $\vee_{k\in\mathbb{Z}}\Sigma^k HA_k$, where $A_k \cong \pi_k(M)$. Hence, if $M, N \in H\mathbb{Z}$-*hmod*, and $M \simeq \vee_{k\in\mathbb{Z}}\Sigma^k HA_k$ and $N \simeq \vee_{j\in\mathbb{Z}}\Sigma^j HB_j$, then

$$[M,N]_{H\mathbb{Z}\text{-}hmod} = \prod_{k\in\mathbb{Z}}\prod_{j\in\mathbb{Z}}[HA_k, \Sigma^{j-k}HB_j]_{H\mathbb{Z}\text{-}hmod} \tag{5}$$

since the natural map $\vee_{k\in\mathbb{Z}}\Sigma^k HA_k \longrightarrow \prod_{k\in\mathbb{Z}}\Sigma^k HA_k$ is an equivalence in this particular case. Thus, the study of morphisms in $H\mathbb{Z}$-*hmod* amounts to the study of $[HA, \Sigma^k HB]_{H\mathbb{Z}\text{-}hmod}$. These abelian groups have already been described in [**6**, Section 5], as follows:

Proposition 4.4. *For all abelian groups A and B, the following holds:*

$$[HA, HB]_{H\mathbb{Z}\text{-}hmod} \cong \mathrm{Hom}(A,B),$$
$$[HA, \Sigma HB]_{H\mathbb{Z}\text{-}hmod} \cong \mathrm{Ext}(A,B), \text{ and}$$
$$[HA, \Sigma^k HB]_{H\mathbb{Z}\text{-}hmod} = 0 \text{ if } k \neq 0,1.$$

Proof. The left adjoint of the monad given by $H\mathbb{Z} \wedge -$ yields an isomorphism $[MA, \Sigma^k HB] \cong [HA, \Sigma^k HB]_{H\mathbb{Z}\text{-}hmod}$ because $HA \simeq H\mathbb{Z} \wedge MA$ where MA denotes a Moore spectrum for the abelian group A; see [**1**]. Now use the exact sequence

$$0 \to \mathrm{Ext}(A, \pi_{k+1}X) \to [\Sigma^k MA, X] \to \mathrm{Hom}(A, \pi_k X) \to 0$$

in the case $X = HB$. □

Corollary 4.5. *Given any ring R and R-modules A and B, if $k \neq 0, 1$, then $[HA, \Sigma^k HB]_{HR\text{-}hmod} = 0$.*

Proof. The inclusion $\mathbb{Z} \hookrightarrow R$ provides a natural transformation $H\mathbb{Z}\wedge- \longrightarrow HR\wedge-$ and a morphism of monads. The result follows from Remark 2.1. □

Corollary 4.6. *There is no equivalence of categories between the categories $Ho(HR\text{-}mod)$ and $HR\text{-}hmod$ if* $\mathrm{gd}(R) > 1$.

Proof. Suppose that there exists an equivalence between the categories $HR\text{-}hmod$ and $Ho(HR\text{-}mod)$. Then for any R-modules A, B and any $k \in \mathbb{Z}$, we have that

$$[HA, \Sigma^k HB]_{HR\text{-}hmod} \cong [HA, \Sigma^k HB]_{HR\text{-}mod} = \mathrm{Ext}^k_R(A, B)$$

by Theorem 4.3. But $[HA, \Sigma^k HB]_{HR\text{-}hmod} = 0$ for $k \neq 0, 1$ by Corollary 4.5, and this is a contradiction since $\mathrm{gd}(R) > 1$. □

We will now discuss $[HA, \Sigma^k HB]_{HR\text{-}hmod}$ in the cases $k = 0$ and $k = 1$, for any ring R. The following proposition generalizes Proposition 4.4 for any ring R in the case $k = 0$.

Proposition 4.7. *For any ring R and all R-modules A and B, the correspondence $f \mapsto \pi_0(f)$ yields a natural isomorphism*

$$[HA, HB]_{HR\text{-}hmod} \cong \mathrm{Hom}_R(A, B).$$

Proof. Let $f\colon HA \longrightarrow HB$ be any map. Recall that HA and HB are HR-modules because A and B are R-modules. The map f will be a map in $HR\text{-}hmod$ if the diagram

$$\begin{tikzcd} HR \wedge HA \arrow[r, "Id_{HR} \wedge f"] \arrow[d, "m_{HA}"'] & HR \wedge HB \arrow[d, "m_{HB}"] \\ HA \arrow[r, "f"'] & HB \end{tikzcd}$$

commutes up to homotopy. We can define a map $\Phi\colon [HA, HB] \longrightarrow [HR \wedge HA, HB]$ by $\Phi(f) = f \circ m_{HA} - m_{HB} \circ (Id_{HR} \wedge f)$. Then f is a map in $HR\text{-}hmod$ if and only if $f \in \ker \Phi$. But $[HA, HB] \cong \mathrm{Hom}(A, B)$ and $[HR \wedge HA, HB] \cong \mathrm{Hom}(R \otimes A, B)$. The map f is in $\ker \Phi$ if and only if $f(ra) = rf(a)$ for all $r \in R$ and $a \in A$, and this is the same as stating that $f \in \mathrm{Hom}_R(A, B)$. □

The case $k = 1$ is more involved. Although we can give a description of $[HA, \Sigma HB]_{HR\text{-}hmod}$ as the kernel of a map $\Phi\colon [HA, \Sigma HB] \longrightarrow [HR \wedge HA, \Sigma HB]$, as in the proof of Proposition 4.7, and

$$[HA, \Sigma HB] \cong \mathrm{Ext}(A, B) \text{ and } [HR \wedge HA, \Sigma HB] \cong \mathrm{Ext}(R \otimes A, B),$$

it turns out that $[HA, \Sigma HB]_{HR\text{-}hmod} \ncong \mathrm{Ext}_R(A, B)$ in general. Indeed, suppose that $[HA, \Sigma HB]_{HR\text{-}hmod} \cong \mathrm{Ext}_R(A, B)$. Then, by Remark 2.1 and Proposition 4.4, there would be an injective map

$$\mathrm{Ext}_R(A, B) \rightarrowtail \mathrm{Ext}(A, B)$$

for any ring R and all R-modules A and B, and this is not true. The following counterexample for a ring R of global dimension one was pointed out to us by Jérôme Scherer.

Example 4.8. Let $R = \mathbb{Q}[x]$, $A = \mathbb{Q}[x]/(x^n)$ and $B = \mathbb{Q}$. Then $\mathrm{Ext}_R(A, B) \neq 0$ because the exact sequence

$$\mathbb{Q} \rightarrowtail \mathbb{Q}[x]/(x^{n+1}) \twoheadrightarrow \mathbb{Q}[x]/(x^n)$$

does not split. If this splitting did exist, then $x^n = 0$ in $\mathbb{Q}[x]/(x^{n+1})$, which is a contradiction. On the other hand, $\mathrm{Ext}(A, B) = 0$ as abelian groups because $\mathbb{Q}$ is divisible.

Note that this example shows that, for the ring $R = \mathbb{Q}[x]$, which has global dimension 1, there is no possible equivalence of categories between $Ho(HR\text{-}mod)$ and $HR\text{-}hmod$.

5. An equivalence of categories

In this section we study for which rings R there is an equivalence of categories between $Ho(HR\text{-}mod)$ and $HR\text{-}hmod$. As we have seen, Corollary 4.6 states that there is no possible equivalence if $\mathrm{gd}(R) > 1$. But not all rings of global dimension one yield such an equivalence, as illustrated by Example 4.8. However, as we next show, the equivalence holds if the ring R is a field or R is a subring of the rationals.

Proposition 5.3 of [**6**] can be extended to the case of HR-modules when R is a field or R is a torsion free solid ring. For these rings, equality (5) also holds if one replaces $\mathbb{Z}$ by R. If R is a field, then every R-module splits as a direct sum of copies of R, and hence

$$[HA, \Sigma HB]_{HR\text{-}hmod} \cong [\vee_i HR, \Sigma HB]_{HR\text{-}hmod} \cong \prod_i [S, \Sigma HB] = 0.$$

A ring R is *solid* if the multiplication induces an isomorphism $R \otimes R \cong R$, where the tensor product is over $\mathbb{Z}$. Solid rings were introduced in [**5**]. If R is solid, then in particular $R \otimes A \cong A$ whenever A is an R-module. This implies that, if R is solid, then

$$\mathrm{Hom}_R(A, B) \cong \mathrm{Hom}(A, B) \text{ and } \mathrm{Ext}_R(A, B) \cong \mathrm{Ext}(A, B).$$

If R is torsion-free, then $HR \wedge MA \simeq H(R \otimes A)$ for any R-module A. If a ring R satisfies these two conditions, then $[HA, \Sigma HB]_{HR\text{-}hmod} = \mathrm{Ext}_R(A, B)$.

Lemma 5.1. *If R is a torsion-free solid ring of global dimension one, then R is a subring of the rationals.*

Proof. This follows from the classification of solid rings (see [**5**]). □

Theorem 5.2. *If R is a field or R is a subring of the rationals, then there is an equivalence of categories between $Ho(HR\text{-}mod)$ and $HR\text{-}hmod$.*

Proof. We construct a functor $\Phi\colon HR\text{-}hmod \longrightarrow \mathcal{D}(R)$ that is an equivalence of categories. It will be enough to define the functor on objects of the form $\Sigma^i HA$, since any $M \in HR\text{-}hmod$ is isomorphic to $\vee_{k\in\mathbb{Z}} \Sigma^k HA_k$ in $HR\text{-}hmod$, and on morphisms of the form $f\colon HA \longrightarrow \Sigma^k HB$ in the cases $k = 0$ and $k = 1$, by equality (5). If

ϕ is an equivalence of categories, $Ho(HR\text{-}mod) \simeq HR\text{-}hmod$ by Theorem 4.3. We consider separately the case of a field and of a subring of the rationals.

If R is a field, then $\mathrm{gd}(R) = 0$ and hence $\mathrm{Ext}_R(A, B) = 0$. We define $\Phi(\Sigma^k HA) = A[k]$ and, thus, if $M \in HR\text{-}hmod$ is such that $M \simeq \vee_{k\in\mathbb{Z}}\Sigma^k HA_k$, then $\Phi(M) = \oplus_{k\in\mathbb{Z}} A_k[k]$. Thus, for a map $f\colon HA \longrightarrow HB$ we define $\Phi(f) = \pi_0(f)$, the corresponding map between $A[0]$ and $B[0]$. Now, Φ is a functor and it is full and faithful. Moreover, every object in $\mathcal{D}(R)$ lies in the image of Φ up to isomorphism by Proposition 4.1, so it is an equivalence of categories.

If R is a subring of $\mathbb{Q}$, we define $\Phi(\Sigma^k HA) = \mathbf{P}_k(A)$ where $\mathbf{P}_k(A)$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow R_k \longrightarrow F_k \longrightarrow 0 \longrightarrow \cdots$$

with F_k in dimension k, and $R_k \to F_k \to A$ is a projective resolution of A. If $M \simeq \vee_{k\in\mathbb{Z}}\Sigma^k HA_k$, then $\Phi(M) = \oplus_{k\in\mathbb{Z}}\mathbf{P}_k(A_k)$. A map $f \in [HA, HB]_{HR\text{-}hmod}$ corresponds to a morphism of R-modules from A to B and hence lifts to a map $\widetilde{f}$ between the projective resolutions of A and B

$$\begin{array}{ccc} R_A & \longrightarrow & F_A \\ \downarrow & & \downarrow \\ R_B & \longrightarrow & F_B. \end{array}$$

This yields a map from the complex $\mathbf{P}_0(A) = \Phi(HA)$ to $\mathbf{P}_0(B) = \Phi(HB)$. We define $\Phi(f) = \widetilde{f}$.

Similarly, a map $g \in [HA, \Sigma HB]_{HR\text{-}hmod}$ lifts to a map $\widetilde{g}$ between the complexes $\mathbf{P}_0(A) = \Phi(HA)$ and $\mathbf{P}_1(B) = \Phi(\Sigma HB)$,

$$\begin{array}{cccc} & R_A & \longrightarrow & F_A \\ & \downarrow & & \\ R_B \longrightarrow & F_B. & & \end{array}$$

We define $\Phi(g) = \widetilde{g}$. The Yoneda product

$$\mathrm{Ext}^i_R(B, C) \otimes \mathrm{Ext}^j_R(A, B) \xrightarrow{Y} \mathrm{Ext}^{i+j}_R(A, C)$$

makes Φ a functor. This functor is full and faithful by Proposition 4.7 and because $[HA, \Sigma HB]_{HR\text{-}hmod} = \mathrm{Ext}_R(A, B)$. By Proposition 4.1, every object in $\mathcal{D}(R)$ lies in the image of Φ up to isomorphism, so it is an equivalence of categories. □

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