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ON SIGNATURES AND A SUBGROUP OF A CENTRAL EXTENSION TO THE MAPPING CLASS GROUP

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Abstract

Atiyah's work [1] describes the relationship between multiplication in a central extension of the mapping class group of a surface of genus n and the signatures of 4-dimensional manifolds. This work studies a subgroup of the central extension, which comes from the image of a representation of the pure framed braid group on n-strands found in [5], and the signatures of corresponding 4-manifolds via a split exact sequence. We construct a splitting map to prove the sequence is split exact, and we use the splitting to give a topological description of homology classes in 4-dimensional manifolds with non-zero intersection. We conclude with a description of multiplication in the subgroup.

1. Introduction

In his paper [1], M. Atiyah uses 2-framings to place Witten's 3-manifold invariant in the context of algebraic topology. Witten's invariant relies on 2-framings, which are related to a central extension $\overline{\Gamma_n}$ of the mapping class group, Γ_n, of a compact oriented surface of genus n. In [1] Atiyah proves the sequence

$$0 \to \mathbb{Z} \to \overline{\Gamma_n} \to \Gamma \to 0$$

is an exact sequence. $\overline{\Gamma_n}$ consists of pairs (f, α) with $f \in \Gamma_n$, and α a 2-framing on a 3-manifold.

Given an oriented compact 3-manifold Y with tangent bundle T_Y, Atiyah defined a 2-framing as a homotopy class of trivializations of $T_Y \oplus T_Y$ viewed as a Spin(6) bundle of the diagonal embedding $SO(3) \to SO(3) \times SO(3) \to SO(6)$. He then showed the correspondence between a 2-framing α and an integer via

$$\sigma(\alpha) = Sign(Z) - \frac{1}{6} p_1(T_Y \oplus T_Y),$$

where Z is a 4-manifold with boundary Y, and p_1 is the relative Pontrjagin class $p_1 \in H^4(Y \times I, Y \times \partial I)$. Atiyah proved σ does not depend on the choice of Z.

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In this paper we use a representation Φ, see section 2, from the pure framed braid group $\mathbb{Z}^n \oplus P_n$ to the mapping class group Γ_n, and study the pullback of the image subgroup in $\overline{\Gamma_n}$, which we denote $\overline{\mathbb{Z}^n \oplus P_n}$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \overline{\Gamma_n} & \longrightarrow & \Gamma_n & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \Phi} & & \uparrow \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \overline{\mathbb{Z}^n \oplus P_n} & \longrightarrow & \mathbb{Z}^n \oplus P_n & \longrightarrow & 0
\end{array}$$

The 4-manifolds in [1] are constructed using mapping tori X_f and X_g, where X is a surface (see definition 3). To explain restricting to $\overline{\mathbb{Z}^n \oplus P_n}$, in [5] it was shown that any compact orientable 3-manifold can be represented as X_f, where $f \in im(\Phi)$ and X is compact oriented surface of genus n. Furthermore, the multiplication in $\overline{\mathbb{Z}^n \oplus P_n}$ is computed via linking matrices coming from the closures of pure framed braids, see theorem 2 of section 5.

We use the above correspondence between 2-framings and integers to consider elements of $\overline{\mathbb{Z}^n \oplus P_n}$ to be pairs (h, c), consisting of mapping class elements h, and integers c. The group multiplication is given by $(f, a) \cdot (g, b) = (fg, a + b + 3 \cdot Sign(W^4_{f,g}))$, where $Sign(W^4_{f,g})$ is the signature of the intersection form

$$H_2(W^4_{f,g}, \mathbb{Z}) \times H_2(W^4_{f,g}, \mathbb{Z}) \to \mathbb{Z}.$$

$H_2(W^4_{f,g})$ is computed using a Mayer Vietoris sequence, see equation 1 of section 3. We construct a splitting map τ of section 4 to show the Mayer Vietoris sequence is split exact. In section 4 we use τ to construct specific examples of classes in $H_2(W^4_{f,g})$ with non zero intersection. Note: we compute homology groups over the integers.

2. The representation $\Phi : \mathbb{Z}^n \oplus P_n \to \Gamma_n$

Let B_n, $n > 1$, be the group given by generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| \geqslant 2$
2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

B_n is called the *Braid Group on n-strands*. Let Σ_n denote the symmetric group acting on $\{1, 2, \cdots, n\}$, and $\pi : B_n \to \Sigma_n$ denote the homomorphism sending σ_i to the transposition $(i, i + 1)$. Let P_n denote the kernel of π.

Definition 1. *The pure framed braid group on n-stands* is the direct sum $\mathbb{Z}^n \oplus P_n$, where $\mathbb{Z}^n$ denotes the group of $n - tuples$ of integers under addition.

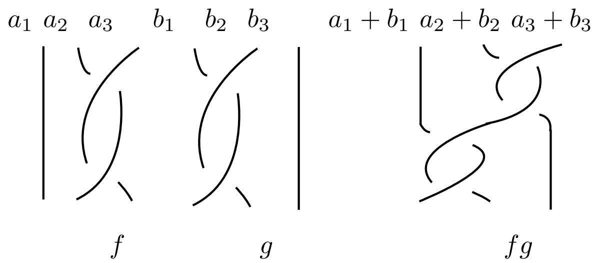

Figure 1: Multiplication of pure framed braids.

We will use a geometric description of pure framed braids. The equivalence between the algebraic and geometric definitions is found in [2]. Consider a cylinder $D^2 \times [0,1]$ with projection $\pi_1 : D^2 \times [0,1] \to D^2$ defined $\pi_1(x,t) = x$. Choose n non-intersecting (smooth) curves in $D^2 \times [0,1]$, where each curve intersects $D^2 \times \{t\}$ in exactly one point, for $0 \leqslant t \leqslant 1$. If x_i and y_i denote the points the i^{th} curve intersects $D^2 \times \{0\}$ and $D^2 \times \{1\}$ respectively, then $\pi_1(x_i) = \pi_1(y_i)$ for $1 \leqslant i \leqslant n$. We require the x_i to be evenly spaced along a fixed diameter in the interior of $D^2 \times \{0\}$. In Figure 1 we see the projection of pure framed braids on 2 strands into the plane. We use breaks in the curves to indicate one strand passing over another. The framed pure braid has an integer assigned to each curve called the framing number. Two pure framed braids are equivalent if they are isotopic via an isotopy of $D^2 \times [0,1]$ which fixes the end-points of the curves and preserves the framings. We illustrate the multiplication in figure 1 with $(a_1,\, a_2,\, a_3)f$ and $(b_1,\, b_2,\, b_3)g \in Z^3 \oplus P_3$. Their product $(a_1+b_1,\, a_2+b_2,\, a_3+b_3)fg$ is formed by stacking f on top of g and adding the corresponding framings. Note: we omit a framing number if it is zero.

Notation 1. By D_n we mean a (unit) disc D^2 with n open disjoint discs removed from the interior. The removed discs were chosen so that their centers were evenly spaced on a fixed diameter of D^2.

Let $pfb(D_n)$ be the group of isotopy classes of homeomorphisms from D_n to D_n which fix the ∂D_n pointwise.

Theorem 1. (Prasolov, V. V. and Sossinsky, A. B.)
$pfb(D_n)$ is isomorphic to $\mathbb{Z}^n \oplus P_n$.

The proof appears in [6]. To give a description of the isomorphism, let $[g] \in pfb(D_n)$, and g_0 be a representative. To obtain a pure framed braid corresponding to $[g]$, we extend the map g_0 to $g : D^2 \to D^2$ by $g(x) = x$ for x not in the interior of D_n. This extension is possible because $g_0(x) = x$ for $x \in \partial D_n$. Now define an isotopy $I : D^2 \times [0,1] \to D^2$,

$$I(x,t) = \begin{cases} x, & \text{if } t = 0; \\ x, & \text{if } |x| > t; \\ tg(t^{-1}x), & \text{otherwise.} \end{cases}$$

Note $I(x,0) = x$ and $I(x,1) = g(x)$. In the complement of the interior of D_n in D^2 there are n closed discs. If c_i is the center of the i^{th} disc, then $(I(c_i,t),t)$ for

$0 \leqslant t \leqslant 1$ is an arc in $D^2 \times [0,1]$. The n arcs are the n strands of the pure braid. Choose any point b_i on the S^1 boundary of the disc with center c_i. The linking number of $(I(b_i,t),t)$ with $c_i \times [0,1]$ is an integer, and this will be the framing corresponding to the i^{th} strand. For example, in figure 2 we have a Dehn twist about the dashed curve, which is an element of $pfb(D_3)$. The pure framed braid is formed by following the image of the interior points c_1, c_2, c_3 under the isotopy I.

Definition 2. The *mapping class group of a (compact oriented) genus n surface,* denoted here by Γ_n, is the group of isotopy classes of orientation preserving homeomorphisms from the surface to itself.

An element of $pfb(D_n)$ is an isotopy class of maps of $D_n \to D_n$, or equivalently $D_n \times \{1\} \to D_n \times \{1\}$. Each representative of the isotopy class, fixes $\partial(D_n \times \{1\})$ pointwise, and may be extended by the identity map to $\partial(D_n \times [0,1])$. This extension is a homomorphism $pfb(D_n) \to \Gamma_n$. Composing this map with the isomorphism $\mathbb{Z}^n \oplus P_n \to pfb(D_n)$ yields $\Phi : \mathbb{Z}^n \oplus P_n \to \Gamma_n$. Our viewpoint for the construction of Φ was inspired by [3] in which the authors discussed representations of platts.

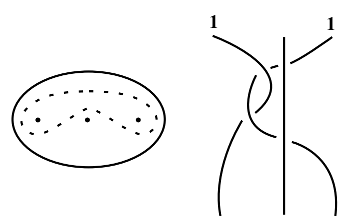

Figure 2: A Dehn twist and corresponding pure framed braid.

3. The 4-manifold $W^4_{f,g}$

The construction of $W^4_{f,g}$ from [1] requires the definition of a mapping torus.

Definition 3. For a homeomorphism f of a surface X to itself, define the *mapping torus* X_f as $\frac{X\times[0,1]}{(p,0)\sim(f(p),1)}$

We are interested in the case where $X = \partial(D_n \times [0,1])$, a surface of genus n, and f a representative of $[f]$ in the image of $\Phi : \mathbb{Z}^n \oplus P_n \to \Gamma_n$. We can view X_f as a fiber bundle over S^1, with fiber $\partial(D_n \times [0,1])$, and $X_f \times [0,1]$ as a fiber bundle over an annulus. The 4-manifold $W^4_{f,g}$ is defined as the quotient space of $(X_f \times [0,1]) \cup (X_g \times [0,1])$ where, for $t \in [0.5, 0.75]$, we identify $(\partial(D_n \times [0,1]) \times \{t\}) \times \{1\}$ of $X_f \times [0,1]$ with $(\partial(D_n \times [0,1]) \times \{t\}) \times \{1\}$ of $X_g \times [0,1]$ by the identity map $\partial(D_n \times [0,1]) \to \partial(D_n \times [0,1])$. Viewing $X_f \times [0,1]$ and $X_g \times [0,1]$ as fiber bundles over annuli, we have identified arcs on the outer boundary circles of the annuli base spaces and their corresponding fiber. We obtain homeomorphic 4-manifolds if different representatives of $[f]$ and $[g]$ are used. The result is a fiber bundle with base D_2. Restricting $W^4_{f,g}$ to the interior boundary circles of D_2 we have X_f and X_g. Restricting to the exterior boundary circle of D_2 we have X_{fg}.

It is shown in [1] that 4-manifolds Z with $\partial Z = X_f + X_g - X_{fg}$ have the same signature.

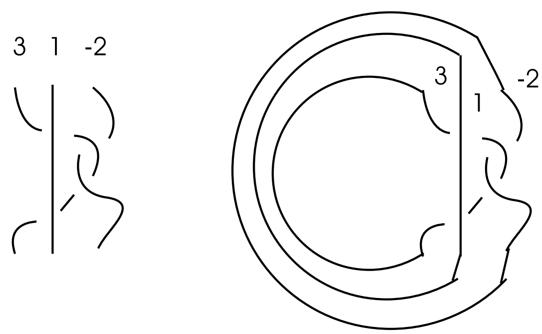

Figure 3: A framed braid and its closure to a framed link.

In order to obtain the signature of $W^4_{f,g}$, we compute the homology via a Mayer Vietoris sequence. This sequence is defined with spaces X_0 and X_1, whose union is $W^4_{f,g}$. The space X_1 is a fiber bundle over D_2 with fiber D_n. It is constructed in the same way as $W^4_{f,g}$, except that we replace the surface fiber $\partial(D_n \times [0,1])$ with $D_n \times \{1\}$. The fiber in X_1 over the two interior boundary circles of the base space D_2 is homeomorphic to complements of tubular neighborhoods of $\hat{f}$ and of $\hat{g}$ in solid tori, where $\hat{f}$ and $\hat{g}$ denote the closures of the pure braids corresponding to f and g. The closure of a framed braid is formed by connecting the two ends of each strand using a simple closed arc. The result is a link with each path component given the framing coming from the corresponding strand, see figure 3. The space X_0 is the product space $D_2 \times (D_n \times \{0\})$. Gluing two copies of D_n along their boundary is homeomorphic to a genus n surface. If at each point in D_2 we glue the $D_n \times \{1\}$ fiber of X_1 to the $D_n \times \{0\}$ fiber of X_0, the result is $W^4_{f,g}$. Using collar neighborhoods of the D_n fiber so that X_0 and X_1 are open, we obtain the Mayer Vietoris sequence.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_2(X_0) \oplus H_2(X_1) & \xrightarrow{\sigma} & H_2(W^4_{fg}) & & \\ & \xrightarrow{\delta} & H_1(X_0 \cap X_1) & \xrightarrow{j} & H_1(X_0) \oplus H_1(X_1) & \longrightarrow & \cdots \end{array} \tag{1}$$

The map $j : H_1(X_0 \cap X_1) \to H_1(X_0) \oplus H_1(X_1)$ takes $a \to (a, a)$ and $\sigma(b, c) = b - c$. By the exactness of the sequence, $im(\sigma) = ker(\delta)$ and $ker(j) = im(\delta)$. In Lemma 2 of section 5 we show that the above sequence is split exact. It will then follow that a basis for $H_2(W^4_{fg})$ is isomorphic to a direct sum of a basis for $im(\sigma)$ with a basis for $ker(j)$.

4. Constructing intersecting classes.

For $[f]$ and $[g]$ in the image of $\Phi : \mathbb{Z}^n \oplus P_n \to \Gamma_n$ we will define a splitting $\tau : ker(j) \to H_2(W^4_{f,g})$. In lemma 3 we show that only elements in the image of τ can have non-zero intersection. To describe $ker(j)$ we label curves in $X_0 \cap X_1$, which is a fiber bundle with base D_2 and fiber ∂D_n; recall D_n is a disc D^2 with interior open discs $int(e_1), \cdots, int(e_n)$ removed. Label the boundary circles of D_n by s_0, $s_1, \cdots, s_n$, with $s_0 = \partial D^2$. Let D_2 be the disc D^2 with the interior of two discs, $int(d_1)$ and $int(d_2)$, removed.

Notation 2. Let y_i be a point of s_i, and $x \in D_2$. Set $p_i = \partial d_1 \times \{y_i\}$, $q_i = \partial d_2 \times \{y_i\}$ and $m_i = s_i \times \{x\}$.

Using the Mayer Vietoris sequence in equation (1), along with the Künneth formula for computing the homology of a product space, one obtains homology bases for $H_1(X_0 \cap X_1)$ and $H_1(X_0) \oplus H_1(X_1)$. Write $j : H_1(X_0 \cap X_1) \to H_1(X_0) \oplus H_1(X_1)$ as $j = j_0 \oplus j_1$, so $j_1 : H_1(X_0 \cap X_1) \to H_1(X_1)$ and $j_0 : H_1(X_0 \cap X_1) \to H_1(X_0)$. An ordered basis for $H_1(X_0 \cap X_1)$ is
$\{[p_0], \cdots, [p_n], [q_0], \cdots [q_n], [m_0], \cdots, [m_n]\}$. An ordered basis for
$H_1(X_0) \oplus H_1(X_1)$ is $\{j_0([p_0]), j_0([s_0]), \cdots j_0([s_n]), j_1([q_0]), j_1([s_0]), \cdots j_1([s_n])\}$. The following lemma is proved with a standard analysis of the matrix of j.

Lemma 1. *Let $[f], [g]$ be in the image of $\Phi : \mathbb{Z}^n \oplus P_n \to \Gamma_n$. Let $F = [F_{ij}]$ and $-G = -[G_{ij}]$ be the $n \times n$ linking matrices of $\hat{f}$ and -1 times the linking matrix of $\hat{g}$, respectively. Set B equal to the submodule of $H_1(X_0 \cap X_1)$ with ordered basis $\{[p_1] - [p_0], \cdots, [p_n] - [p_0], [q_1] - [q_0], \cdots, [q_n] - [q_0]\}$. Define $(F \quad -G) : B \to H_1(X_0) \oplus H_1(X_1)$ to be multiplication by the matrix formed by juxtaposing the linking matrices of F and $-G$.*

$$\textit{Then, } ker(j) \textit{ is isomorphic to } ker\begin{pmatrix}F & -G\end{pmatrix} \oplus < [\eta] >,$$

where $[\eta] = [m_0] - (\sum_{i=1}^{n} [m_i])$.

Example:

Consider the example of representatives $(0, 0, 0)f$ and $(-1, 0, 0)g$ in the image of $\Phi : \mathbb{Z}^3 \oplus P_3 \to \Gamma_3$ corresponding to the pure framed braids in figure 1, where the framings $a_1 = a_2 = a_3 = b_2 = b_3 = 0$ and $b_1 = -1$. From $(0, 0, 0)f$ and $(-1, 0, 0)g$ we obtain the matrix

$$\begin{pmatrix}F & -G\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Using the symbol T to stand for transpose, two elements in the kernel are

$$\alpha = \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 1\end{pmatrix}^T \quad \text{and} \quad \beta = \begin{pmatrix}0 & 1 & 0 & -1 & 0 & 0\end{pmatrix}^T.$$

To construct the classes $\tau(\alpha)$ and $\tau(\beta)$ we decompose $W^4_{(0,0,0)f,(-1,0,0)g}$ into X_1 and X_0. In this case both are fiber bundles with base space D_2, and fiber D_3. For $i = 1, 2, 3$, $j_1([p_i])$ is a longitude on a tubular neighborhood of the i^{th} strand of the closed braid $\hat{f}$ in the fiber over ∂d_1. Similarly, $j_1([q_i])$ corresponds to the i^{th} strand of $\hat{g}$ in the fiber over ∂d_2. The classes $j_0([p_i])$ and $j_0([q_i])$ are longitudes on tubular neighborhoods of the i^{th} strands of the closed identity braids in the fiber over ∂d_1 and ∂d_2, respectively, in X_0. With y_0 a point on the exterior boundary circle of D_3, the classes $j_1([p_0])$ and $j_1([q_0])$ represent $S^1 \times \{y_0\}$ in the $S^1 \times D_3$ fiber over ∂d_1 and ∂d_2 in X_1, respectively.

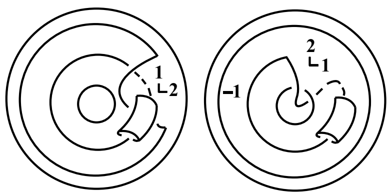

Figure 4: $\tau(\alpha)$ in X_1

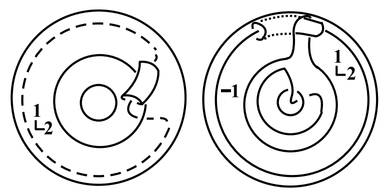

Figure 5: $\tau(\beta)$ in X_1

Since $\alpha = ([p_1]-[p_0])+([q_3]-[q_0])$, attach two annular regions. One annular region has boundary $j_1([p_1]) - j_1([p_0])$, and the second has boundary $j_1([q_3]) - j_1([q_0])$. The first annular region is in the fiber over ∂d_1 in X_1, see the left side of figure 4, corresponding to $\hat{f}$. To allow the tubular neighborhood with longitude $j_1([p_2])$ to pass through, we remove a disc and attach an $S^1 \times [0,1]$ along the $S^1 \times \{0\}$ boundary. The annular region with boundary $j_1([q_3]) - j_1([q_0])$ is in the fiber over ∂d_2, corresponding to $\hat{g}$. We remove a disc and attach a tube so that a tubular neighborhood with longitude $j_1([q_2])$ may pass through, see the right side of figure 4. The tubes about $j_1([p_2])$ and $j_1([q_2])$ are then joined by passing through fibers over D_2.

In the fiber over ∂d_1, the annular region in X_1 is glued to the boundary of an annulus in X_0 between $j_0([p_1])$ and $j_0([p_0])$. In the fiber over ∂d_2, the annular region in X_1 is glued to an annulus in X_0 between $j_0([q_3])$ and $j_0([q_0])$. The surface $\tau(\alpha)$ is two tori joined by a tube, and with an orientation is a class of $H_2(W^4_{f,g})$. The projection of this class to the base space D_2 is two circles, concentric to the interior boundary circles joined together with a simple non-intersecting arc, see figure 6. The arc being the projection of the tube joining the annular regions in X_1.

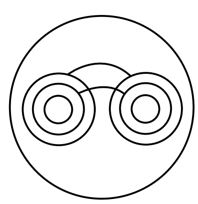

Figure 6: The projection of $\tau(\alpha)$ and $\tau(\beta)$ to D_2 in general position.

We construct a second surface $\tau(\beta)$ in a similar fashion. In the fiber over ∂d_1 in X_1, there is an annular region between $j_1([p_2])$ and $j_1([p_0])$, and a tube is created about a tubular neighborhood with $j_1([p_1])$. In the fiber over ∂d_2 there is an annular region with boundary $j_1([q_0])$ and a circle linking $j_1([q_1])$ with linking number -1.

This is due to the framing number -1. For the tubular neighborhood with $j_1([q_1])$ to pass through the annular region, a disc is removed and a tube is attached, see figure 5. These tubes are joined together. After attaching the boundary to the corresponding annuli in X_0 we obtain a surface, denoted $\tau(\beta)$. The projection of $\tau(\beta)$ to D_2 is two circles joined by an arc. $\tau(\alpha)$ and $\tau(\beta)$ have an intersection of 1. Without the tubes, the projections to the base space D_2, in general position, would be two pairs of concentric circles. Such surfaces would not intersect.

5. The intersection form on $H_2(W^4_{f,g})$

Lemma 2. *There is a splitting $\tau : ker(j) \to H_2(W^4_{fg})$.*

Proof

As in section 4, $\tau([p_i]-[p_0])$ is an annular region in X_1 over ∂d_1 with boundary $j_1([p_i])-j_1([p_0])$ glued to an annulus in X_0 with boundary $j_0([p_i])-j_0([p_0])$. Define $\tau([q_i]-[q_0])$ to be an annular region in X_1 over ∂d_2 with boundary $j_1([q_i])-j_1([q_0])$ glued to an annulus in X_0 with boundary $j_0([q_i])-j_0([q_0])$. $\tau([\eta])$ is defined to be the surface fiber over a point in D_2. To define τ on a general element of $ker(j)$, extend linearly. In the construction of $\tau(a_1([p_1]-[p_0])+\cdots+a_n([p_n-p_0])+b_1([q_1]-[q_0])+\cdots+b_n([q_n]-[q_0]))$, for each $a_k \neq 0$, there are $|a_k|$ copies of annular regions between $j_1([p_k])$ and $j_1([p_0])$. The algebraic intersection of $j_1([p_i])$ with these regions is a_kF_{ik}. Similarly, for $b_k \neq 0$ there are $|b_k|$ copies of annular regions between $j_1([q_k])$ and $j_1([q_0])$, and the algebraic intersection with $j_1([q_i])$ is b_kG_{ik}. If $(a_1,\cdots,a_n,\, b_1,\cdots,b_n)$ is in $ker(F \quad -G)$, then for $1 \leqslant i \leqslant n$, $\sum_{k=1}^n a_kF_{ik} = \sum_{k=1}^n b_kG_{ik}$. Therefore, we can glue $|\sum_{k=1}^n a_kF_{ik}|$ tubes about $j_1([p_i])$ to $|\sum_{k=1}^n b_kG_{ik}|$ tubes about $j_1([q_i])$. By examining the definition of δ one sees that τ is a splitting. Q.E.D.

Lemma 3. *Let $\alpha,\, \beta \in H_2(W^4_{f,g})$ and $\alpha = a_1\kappa_1+a_2\kappa_2$, where $a_1,\, a_2 \in \mathbb{Z}$, $\kappa_1 \in im(\tau)$ and $\kappa_2 \in im(\sigma)$ of equation (1). If $a_1 = 0$, then the intersection $\alpha \cdot \beta = 0$.*

Proof

From equation (1) and lemma 2 we have $H_2(W^4_{f,g}) \approx im(\sigma) \oplus \tau(ker(j))$. A Mayer Vietoris sequence shows that a class in image of σ can be divided into a class in the fiber over ∂d_1 of the base D_2 and a class in the fiber over ∂d_2. In general position, the projection of two such classes to D_2 is a finite set of circles concentric to ∂d_1 and to ∂d_2. Such classes do not intersect. A class in the image of τ projected to the base space D_2 is a finite set of circles concentric to ∂d_1 and ∂d_2, possibly with some simple arcs between them. In general position, one can arrange the class in the image of σ so its projection to D_2 is a set of circles interior to the circles coming from the class in the image of τ. Thus, there is no intersection. Q.E.D.

Remark 1. In general position, the fiber over a point does not intersect a surface. Thus, $\tau([\eta])$ does not contribute to signature computations.

Next we relate the algebraic computation of intersections in $W^4_{f,g}$ to our topological description of section 4. In the computation of $\tau(\alpha)\cdot\tau(\beta)$,

$$\begin{pmatrix} F & 0_{3\times 3} \\ 0_{3\times 3} & -G \end{pmatrix} \cdot \alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}^T$$

indicates the tubes created about $j_1([p_2])$ and $j_1([q_2])$ in the fiber over ∂d_1 and ∂d_2, respectively. The multiplication

$$\beta^T \cdot \left(\begin{pmatrix} F & 0_{3\times 3} \\ 0_{3\times 3} & -G \end{pmatrix} \cdot \alpha \right) =$$
$$\begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}^T \right) = 1$$

reflects that the tube about $j_1([p_2])$ intersects the annular region in the fiber over ∂d_1 with boundary components $j_1([p_0])$ and $j_1([p_2])$. More generally, we have the next theorem. Recall the map $\delta : H_2(W^4) \to H_1(X_1 \cap X_2)$ from equation (1).

Theorem 2. *Let $[f], [g]$ be in the image of $\Phi : \mathbb{Z}^n \oplus P_n \to \Gamma_n$, and α, $\beta \in H_2(W^4_{f,g})$. Set $F = [F_{ij}]$ to be the linking matrix of $\hat{f}$, and $-G = -[G_{ij}]$ to be -1 times the linking matrix for $\hat{g}$. Suppose the projection of $\delta(\alpha)$to $ker(F \quad -G)$ has coordinates $(a_1, \ldots, a_n, b_1, \ldots, b_n)$, and the projection of $\delta(\beta)$ has coordinates $(c_1, \ldots, c_n, d_1, \ldots, d_n)$. Then, the intersection $(\alpha)\cdot(\beta) =$*

$$\begin{pmatrix} c_1 & \ldots & c_n & d_1 & \ldots & d_n \end{pmatrix} \begin{pmatrix} F & 0_{n\times n} \\ 0_{n\times n} & -G \end{pmatrix} \begin{pmatrix} a_1 & \ldots & a_n & b_1 & \ldots & b_n \end{pmatrix}^T$$

Proof

By lemma 3 and remark 1, the only non-zero intersections are from classes in the image of τ restricted to $ker(F \quad -G)$. Consider the sum $c_k(a_k F_{1k} + \ldots + a_k F_{nk})$. When $c_k \neq 0$ we have $|c_k|$ annular regions between with boundary components $j_1([p_k])$ and $j_1([p_0])$. If $a_k \neq 0$, there will be $|a_k F_{ik}|$ number of tubes around a tubular neighborhood of $j_1([p_k])$, formed by avoiding intersections with annular regions between $j_1([p_i])$ and $j_1([p_0])$. Summing for $i = 1 \ldots n$ yields an algebraic intersection of $c_k(a_k F_{1k} + a_k F_{2k} + a_k F_{nk})$. Accounting for all $j_1([p_k])$, $1 \leqslant k \leqslant n$, we obtain

$$\begin{pmatrix} c_1, & \ldots & , c_n \end{pmatrix} \begin{pmatrix} F \end{pmatrix} \begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix}^T.$$

A similar analysis yields the intersection number coming from $j_1([q_k])$, $1 \leqslant k \leqslant n$, is

$$-\begin{pmatrix} d_1, & \ldots & , d_n \end{pmatrix} \begin{pmatrix} G \end{pmatrix} \begin{pmatrix} b_1 & \ldots & b_n \end{pmatrix}^T. \text{ Q.E.D.}$$

Using the map $\Phi : \mathbb{Z}^n \oplus P_n \to \Gamma_n$ we obtain the commutative diagram below.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] & \overline{\Gamma_n} \arrow[r] & \Gamma_n \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u] & \overline{\mathbb{Z}^n \oplus P_n} \arrow[r] \arrow[u] & \mathbb{Z}^n \oplus P_n \arrow[r] \arrow[u, "\Phi"'] & 0 \arrow[u]
\end{tikzcd}$$

The group $\overline{\Gamma_n}$ is a central extension of the mapping class group Γ_n studied in [1], and $\overline{\mathbb{Z}^n \oplus P_n}$ is the pullback of Φ. An element of the subgroup $\overline{\mathbb{Z}^n \oplus P_n}$ is a pair (h, a), where a is an integer and h is in the image of Φ. The multiplication in $\overline{\mathbb{Z}^n \oplus P_n}$ is computed using linking matrices.

In conclusion, we multiply $([(0,0,0)f],\, a) \cdot ([(-1,0,0)g],\, b)$ in $\overline{\mathbb{Z}^3 \oplus P_3}$. The intersection pairing on W^4_{fg} is given by the quadratic form $x^T Ax$ where

$$A = \begin{pmatrix} F & 0_{3\times 3} \\ 0_{3\times 3} & -G \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

The signature of the matrix A is 1, and therefore
$([(0,0,0)f], a) \cdot ([(-1,0,0)g], b) = ([(-1,0,0)fg], a + b + 3Sign(W^4_{fg}))$
$= ([(-1,0,0)fg], a + b + 3)$.

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