Homology, Homotopy and Applications, vol.4(1), 2002, pp.163–173

BRAVE NEW HOPF ALGEBROIDS AND EXTENSIONS OF MU-ALGEBRAS

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(*communicated by J. D. Stasheff*)

Abstract

We apply recent work of A. Lazarev which develops an obstruction theory for the existence of R-algebra structures on R-modules, where R is a commutative S-algebra. We show that certain MU-modules have such A_∞ structures. Our results are often simpler to state for the related BP-modules under the currently unproved assumption that BP is a commutative S-algebra. Part of our motivation is to clarify the algebra involved in Lazarev's work and to generalize it to other important cases. We also make explicit the fact that BP admits an MU-algebra structure as do $E(n)$ and $\widehat{E(n)}$, in the latter case rederiving and strengthening older results of U. Würgler and the first author.

Introduction

Recent work of A. Lazarev [**11**] has developed an obstruction theory for the existence of R-algebra structures on R-modules, where R is a commutative S-algebra in the sense of [**8**]. In [**4**], 'brave new Hopf algebroids' were discussed and related to the Adams Spectral Sequence for R-modules, thus generalizing the classical homotopy theoretic version described in [**14**]. In the present work we again consider some of the main examples of that paper and apply Lazarev's techniques to show that certain MU-modules have such A_∞ structures. In fact, our results are often simpler to state for the related BP-modules under the assumption that BP is a commutative S-algebra. However this currently seems to remain unproved, a preprint by I. Kriz showing this apparently has so far unfilled gaps. We often state BP analogues but normally work over MU. Part of our motivation is to clarify the algebra involved in [**11**] and to show how it generalizes to some other important cases. We also make explicit the fact that BP admits an MU-algebra structure as do $E(n)$ and $\widehat{E(n)}$, in the latter case rederiving and strengthening results of [**2, 5**].

As a matter of history we remark that most of the material described here originated during the summer of 2000; subsequent preprints by P. Goerss, M. Hopkins

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The first author would like to thank the Mathematisches Institut, Universität Bern for providing a stimulating environment in which much of this work was carried out. We would also like to thank Christian Ausoni for helpful comments.
Received April 23, 2002, revised July 23, 2002; published on November 25, 2002.
2000 Mathematics Subject Classification: 55N20, 55P43, 55N45, 55S35, 55T25.
Key words and phrases: R-algebra, Hopf algebroid, obstruction theory.

and A. Lazarev have contained stronger results on the realizability of BP and other MU-algebras using more developed machinery. However, we feel that the present approach provides a good illustration of the power of the obstruction theory implicit in [**11**] applied to some examples of fundamental importance to homotopy theorists.

1. Brave new Hopf algebroids

Throughout, we will work in a good category of spectra $\mathscr{S}$ such as $\mathscr{S}[\mathbb{L}]$ of [**8**]. Associated to this is the category of S-modules $\mathscr{M}_S$ and its derived homotopy category $\mathscr{D}_S$. If R is a commutative S-algebra in the sense of [**8**], there is an associated category of R-modules $\mathscr{M}_R$ and its derived category $\mathscr{D}_R$.

The following notions were introduced in [**8**]. Let A be an R-module with a *unit* $\eta\colon R \longrightarrow A$ and *product* $\varphi\colon A\underset{R}{\wedge}A \longrightarrow A$. Then A, or more precisely, (A,φ,η), is an *R-algebra* if the following diagrams commute in $\mathscr{M}_R$.

$$\begin{tikzcd}
A\underset{R}{\wedge}A\underset{R}{\wedge}A \arrow[r, "\varphi\wedge\mathrm{id}"] \arrow[d, "\mathrm{id}\wedge\varphi"'] & A\underset{R}{\wedge}A \arrow[d, "\varphi"] \\
A\underset{R}{\wedge}A \arrow[r, "\varphi"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
R\underset{R}{\wedge}A \arrow[r, "\eta\wedge\mathrm{id}"] \arrow[dr, "\mu"'] & A\underset{R}{\wedge}A \arrow[d, "\varphi"] & A\underset{R}{\wedge}R \arrow[l, "\mathrm{id}\wedge\eta"'] \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}
\tag{1.1a}$$

A is *commutative* if the following diagram commutes in $\mathscr{M}_R$.

$$\begin{tikzcd}
A\underset{R}{\wedge}A \arrow[rr, "\tau"] \arrow[dr, "\varphi"'] & & A\underset{R}{\wedge}A \arrow[dl, "\varphi"] \\
& A &
\end{tikzcd}
\tag{1.1b}$$

There are also weaker conditions for such a product. If the diagrams of (1.1a) commute in $\mathscr{D}_R$ then A is an *R-ring spectrum*, and if (1.1b) also commutes in $\mathscr{D}_R$ then A is a *commutative R-ring spectrum*. In that case the smash product $A\underset{R}{\wedge}A$ is also a commutative R-ring spectrum and is also naturally an A-algebra spectrum in two different ways induced from the left and right units

$$A \xrightarrow{\cong} A\underset{R}{\wedge}R \longrightarrow A\underset{R}{\wedge}A \longleftarrow R\underset{R}{\wedge}A \xleftarrow{\cong} A.$$

As discussed in [**4**], we have

Theorem 1.1. *Let A be a commutative R-ring spectrum. If A^R_*A is flat as a left, or equivalently as a right, A_*-module, then*

(i) (A_*, A^R_*A) is a Hopf algebroid over R_*;

(ii) for any R-module M, A^R_*M is a left A^R_*A-comodule.

Such Hopf algebroids are commonly encountered and we will meet important examples in the rest of this paper.

2. Some examples

Recall that the Thom spectrum MU is a commutative S-algebra. At the time of writing it seems not to be known whether the Brown-Peterson spectrum BP for a prime p is a commutative S-algebra. We will state many of our results both in terms of MU-algebras, and also in parallel in terms of BP-algebras on the assumption that these will eventually prove to be of interest if BP is indeed shown to be a commutative S-algebra.

We will assume that a choice of polynomial generators $x_r \in MU_{2r}$ $(r \geqslant 1)$ for MU_* has been made. We also assume that a rational prime $p = x_0 > 0$ has been chosen.

For any subset $S \subseteq \{x_r : r \geqslant 0\}$, the sequence of elements of S conventionally ordered by increasing degree is regular. By successively killing the homotopy elements by forming mapping cones in the category of MU-modules we can form a MU-module spectrum MU/S. More precisely, this is a CW-cell MU-module whose cells are indexed by the monomial basis of the exterior algebra $\Lambda_{MU_*}(\tau_r : x_r \in S)$ in which τ_r has bidegree $(1, 2r)$. The cell corresponding to τ_r has dimension $2r+1$ and an attaching map $S^{2r}_{MU} \longrightarrow (MU/S)^{(2r)}$ in the homotopy class of $x_r \in \pi_{2r}(MU/S)^{(2r)}$. We make this exterior algebra into an MU_*-dga with differential d for which $\mathrm{d}\,\tau_r = x_r$. Of course, $\Lambda_{MU_*}(\tau_r : x_r \in S)_*$ is a Koszul complex providing a free resolution of the MU_*-module $\pi_* MU/S = MU_*/(S)$,

$$\Lambda_{MU_*}(\tau_r : x_r \in S)_* \longrightarrow MU_*/(S) \to 0.$$

Recall from [**8**], the Künneth Spectral Sequence

$$\mathrm{E}_2^{r,s} = \mathrm{Tor}^{MU_*}_{r,s}(MU_*/(S), MU_*/(S)) \Longrightarrow MU/S^{MU}_{r+s} MU/S^{\mathrm{op}}. \tag{2.1}$$

We will need to consider situation where MU/S is an MU-ring spectrum as well as the *opposite MU-ring spectrum* MU/S^{op}. By [**4**, lemma 1.3], (2.1) is then multiplicative. The following result is proved in [**12**, lemma 2.6].

Theorem 2.1. *Suppose that MU/S is an MU-ring spectrum. Then the Künneth Spectral Sequence* (2.1) *for $MU/S^{MU}_* MU/S^{\mathrm{op}}$ collapses at E_2 to give*

$$MU/S^{MU}_* MU/S^{\mathrm{op}} = \Lambda_{MU_*/S}(\tau_r : x_r \in S),$$

where the exterior generators satisfy $\tau_r \in MU/S^{MU}_{2r+1} MU/S^{\mathrm{op}}$.

If MU/S is commutative then of course $MU/S^{\mathrm{op}} = MU/S$ as MU-ring spectra; Theorem 2.1 then follows directly from [**4**, lemma 1.3]. See Strickland [**16**] for details on when this commutativity condition on MU/S is satisfied. In particular, when $x_0 = p$ is an odd prime, all the standard MU-module spectra of this form such as $MU\langle n\rangle$, $MU\langle n\rangle$, BP and $BP\langle n\rangle$ are commutative MU-ring spectra, while when $x_0 = 2$, care needs to be taken so that $MU\langle n\rangle$ and $BP\langle n\rangle$ need to be replaced by spectra constructed using non-standard polynomial generators for MU_*.

As a particular case, let us assume that we have chosen S to contain a complete set of generators of MU_* except for x_{p^k-1} $(0 \leqslant k \leqslant n)$. We then have an MU-module $MU\langle n\rangle$. Here we allow the possibility of $x_0 = p > 0$ corresponding to the case $MU\langle -1\rangle$.

Proposition 2.2.

$$MU\langle n\rangle^{MU}_{*}\, MU\langle n\rangle^{\mathrm{op}} = \Lambda_{MU\langle n\rangle_*}(\tau_r : r \neq p^k - 1,\ 0 \leqslant k \leqslant n).$$

Let $p > 0$ be a prime. If BP is a commutative S-algebra then we can form the p-localization of $MU\langle n\rangle$ as a BP-module, denoted $BP\langle n\rangle$, for which the following holds.

Proposition 2.3.

$$BP\langle n\rangle^{BP}_{*}\, BP\langle n\rangle^{\mathrm{op}} = \Lambda_{BP\langle n\rangle_*}(\tau_{p^k-1} : k \geqslant n+1).$$

Notice that for the special (commutative) case $BP\langle 0\rangle = H\mathbb{Z}_{(p)}$,

$$H\mathbb{Z}_{(p)}{}^{BP}_{*} H\mathbb{Z}_{(p)} = \Lambda_{\mathbb{Z}_{(p)}}(\tau_{p^k-1} : k \geqslant 1).$$

Similarly, in the case $BP\langle -1\rangle = H\mathbb{F}_p$ we have

$$H\mathbb{F}_p{}^{BP}_{*} H\mathbb{F}_p = \Lambda_{\mathbb{F}_p}(\tau_{p^k-1} : k \geqslant 0)$$

and the natural map

$$H\mathbb{F}_p{}^{S}_{*} H\mathbb{F}_p \longrightarrow H\mathbb{F}_p{}^{BP}_{*} H\mathbb{F}_p$$

corresponds to the quotient of the dual Steenrod algebra by the Hopf ideal generated by the Milnor elements ζ_i $(i \geqslant 1)$; this quotient is well known to be a primitively generated exterior algebra.

3. Extending MU-algebra structures

As a starting point we recall from [**8**] that for any commutative ring R, the natural homomorphism $MU_0 = \mathbb{Z} \longrightarrow \mathrm{R}$ lifts to a morphism of S-algebras $MU \longrightarrow H\mathrm{R}$, where $H\mathrm{R}$ is an Eilenberg-MacLane MU-module. $H\mathrm{R}$ is known to be a commutative MU-algebra. We set $MU\langle 0\rangle = H\mathbb{Z}$ since the latter realizes the spectrum discussed above.

Theorem 3.1. *Let $n \geqslant 0$ and suppose that $MU\langle n\rangle$ admits the structure of an MU-algebra. Then $MU\langle n+1\rangle$ admits an MU-algebra structure so there is a morphism of MU-algebras*

$$MU\langle n+1\rangle \longrightarrow MU\langle n\rangle$$

realizing the natural ring homomorphism $MU\langle n+1\rangle_* \longrightarrow MU\langle n\rangle_*$.

If BP is a commutative S-algebra and $BP\langle n\rangle$ admits the structure of a BP-algebra, then $BP\langle n+1\rangle$ admits a BP-algebra structure so that there is a morphism of BP-algebras

$$BP\langle n+1\rangle \longrightarrow BP\langle n\rangle$$

realizing the natural ring homomorphism $BP\langle n+1\rangle_* \longrightarrow BP\langle n\rangle_*$.

Proof. Set $MU\langle n+1; 1\rangle = MU\langle n\rangle$. We will prove by induction that the following holds for each $m \geqslant 1$:

- If $MU\langle n+1;m\rangle$ is an MU-algebra which as an MU_*-module satisfies
$$\pi_* MU\langle n+1;m\rangle = MU\langle n+1\rangle_* /((x_{p^{n+1}-1})^m),$$
then there is an MU-algebra $MU\langle n+1;m+1\rangle$ for which
$$\pi_* MU\langle n+1;m+1\rangle = MU\langle n+1\rangle_* /((x_{p^{n+1}-1})^{m+1})$$
and a morphism of MU-algebras
$$MU\langle n+1;m+1\rangle \longrightarrow MU\langle n+1;m\rangle$$
realizing the evident homomorphism
$$MU\langle n+1;m+1\rangle_* \longrightarrow MU\langle n+1;m\rangle_* .$$

There is a short exact sequence of MU_*-modules
$$0 \to \Sigma^{2(p^{n+1}-1)m} MU\langle n\rangle_* \longrightarrow MU\langle n+1\rangle_* /((x_{p^{n+1}-1})^{m+1}) \longrightarrow MU\langle n+1\rangle_* /((x_{p^{n+1}-1})^{m}) \to 0,$$
so we need to show that this is realized by an extension of MU-algebras, for which the fibre of the map $MU\langle n+1;m+1\rangle \longrightarrow MU\langle n+1;m\rangle$ is $\Sigma^{2(p^{n+1}-1)m} MU\langle n\rangle$.

Following [**11**], we need to determine the Hochschild cohomology
$$\mathrm{HH}^{**}_{MU_*}(MU\langle n+1;m\rangle_*, MU\langle n\rangle_*).$$
We begin by calculating $MU\langle n+1;m\rangle^{MU}_* MU\langle n+1;m\rangle^{\mathrm{op}}$. There is a Koszul resolution
$$\Lambda_{MU_*}(\tau[n+1;m]_r : 1 \leqslant r \neq p^k-1,\, 1 \leqslant k \leqslant n+1)_* \longrightarrow MU\langle n+1;m\rangle_* \to 0$$
which is an MU_*-dga with
$$\mathrm{d}\,\tau[n+1;m]_r = \begin{cases} (x_{p^{n+1}-1})^m & \text{if } r = p^{n+1}-1, \\ x_r & \text{otherwise.} \end{cases}$$
Tensoring over MU_* with $MU\langle n+1;m\rangle_*$ and taking homology, we find that the Künneth Spectral Sequence of (2.1) collapses to give
$$MU\langle n+1;m\rangle^{MU}_* MU\langle n+1;m\rangle^{\mathrm{op}} = \Lambda_{MU\langle n+1;m\rangle}(\tau[n+1;m]_r : 1 \leqslant r \neq p^k-1,\, 1 \leqslant k \leqslant n+1). \tag{3.1}$$
Similarly, using a standard divided power complex we find that
$$\mathrm{HH}^{**}_{MU_*}(MU\langle n+1;m\rangle_*, MU\langle n\rangle_*) = MU\langle n\rangle_* [y[n+1;m]_r : 1 \leqslant r \neq p^k-1,\, 1 \leqslant k \leqslant n+1]. \tag{3.2}$$
Finally we require the fact that
$$\mathrm{Ext}^{**}_{MU_*}(MU\langle n+1;m\rangle_*, MU\langle n\rangle_*) = \Lambda_{MU\langle n\rangle_*}(Q[n+1;m]^r : 1 \leqslant r \neq p^k-1,\, 1 \leqslant k \leqslant n+1), \tag{3.3}$$

which is obtained using the above Koszul resolution. Here

$$Q[n+1;m]^r \in \operatorname{Ext}^{1,*}_{MU_*}(MU\langle n+1;m\rangle_*\,, MU\langle n\rangle_*)$$

is the element 'dual' to $\tau[n+1;m]_r$ with respect to the $MU\langle n\rangle_*$-basis for

$$\Lambda_{MU_*}(\tau[n+1;m]_r : 1 \leqslant r \neq p^k-1,\ 1 \leqslant k \leqslant n+1)_{1,*}$$

consisting of the τ_r's. Then

$$\operatorname{bideg} Q[n+1;m]^r = \begin{cases} (1, 2(p^{n+1}-1)m) & \text{if } r = p^{n+1}-1, \\ (1, 2r) & \text{otherwise.} \end{cases}$$

Consider the element $y[n+1;m]_{p^{n+1}-1}$. By similar arguments to those of [**11**], this element gives rise to an element of

$$\operatorname{Der}^{2(p^{n+1}-1)m+1}_{MU}(MU\langle n+1;m\rangle\,, MU\langle n\rangle)$$

and moreover it corresponds to $Q[n+1;m]^{p^{n+1}-1}$ under the map

$$\operatorname{Der}^{2(p^{n+1}-1)m+1}_{MU}(MU\langle n+1;m\rangle\,, MU\langle n\rangle) \longrightarrow \operatorname{Ext}^{1,2(p^{n+1}-1)m}_{MU_*}(MU\langle n+1;m\rangle_*\,, MU\langle n\rangle_*)$$

defined in that paper. Using the standard identification of elements of Ext^1 with extensions, it is easy to see that $Q[n+1;m]^r$ specifies the explicit extension of MU_*-modules

$$0 \to \Sigma^{2(p^{n+1}-1)m} MU\langle n\rangle_* \longrightarrow MU\langle n+1;m+1\rangle_* \longrightarrow MU\langle n+1;m\rangle_* \to 0$$

corresponding to the fibration sequence

$$\Sigma^{2(p^{n+1}-1)m} MU\langle n\rangle \longrightarrow MU\langle n+1;m+1\rangle \longrightarrow MU\langle n+1;m\rangle$$

of MU-modules. By Lazarev's algebra extension machinery, there is a morphism of MU-algebras $MU\langle n+1;m+1\rangle \longrightarrow MU\langle n+1;m\rangle$ which realizes the natural map in homotopy. We require the following lemma which is essentially a generalization of a result to be found in [**7**], see [**6, 10, 17**]. To apply this we may need to replace maps by homotopic maps which are fibrations.

Lemma 3.2. *Let R be an S-algebra.*

(a) If

$$M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_n \longrightarrow \cdots$$

is a directed system of cofibrations in $\mathscr{M}_R$ then $\operatorname{hocolim}_n M_n$ is weakly equivalent to $\operatorname{colim}_n M_n$.

(b) If

$$N_0 \longleftarrow N_1 \longleftarrow \cdots \longleftarrow N_n \longleftarrow \cdots$$

is a directed system of fibrations in $\mathscr{M}_R$ then $\operatorname{holim}_n M_n$ is weakly equivalent to $\lim_n M_n$.

Let

$$MU\langle n+1;\infty\rangle = \lim_m MU\langle n+1;m\rangle,$$

which is equivalent in $\mathscr{D}_R$ to $\operatorname{holim}_m MU\langle n+1;m\rangle$. We obtain an MU-algebra structure on $MU\langle n+1;\infty\rangle$ and the Milnor exact sequence

$$0 \to \lim_m{}^1 MU\langle n+1;m\rangle_* \longrightarrow MU\langle n+1;\infty\rangle_* \longrightarrow \lim_m MU\langle n+1;m\rangle_* \to 0,$$

gives

$$\lim_m MU\langle n+1;m\rangle_* = MU\langle n+1\rangle_*, \quad \lim_m{}^1 MU\langle n+1;m\rangle_* = 0,$$

since each map $MU\langle n+1;m+1\rangle_* \longrightarrow MU\langle n+1;m\rangle_*$ is surjective. Hence

$$MU\langle n+1;\infty\rangle_* = MU\langle n+1\rangle_*.$$

An obstruction theory argument based on the fact that $MU\langle n+1\rangle$ is a cell MU-module provides an MU-module map $MU\langle n+1\rangle \longrightarrow MU\langle n+1;\infty\rangle$ which is a weak equivalence.

The BP version follows by a parallel argument. □

Theorem 3.3. *There is a tower of MU-algebras*

$$H\mathbb{Z} = MU\langle 0\rangle \longleftarrow MU\langle 1\rangle \longleftarrow \cdots \longleftarrow MU\langle n+1\rangle \longleftarrow \cdots$$

whose limit $MU\langle\infty\rangle$ is an MU-algebra with $MU\langle\infty\rangle_* = MU_*/(x_r : 1 \leqslant r \neq p^k-1)$.

If BP is a commutative S-algebra, there is a tower of BP-algebras

$$H\mathbb{Z}_{(p)} = BP\langle 0\rangle \longleftarrow BP\langle 1\rangle \longleftarrow \cdots \longleftarrow BP\langle n\rangle \longleftarrow BP\langle n+1\rangle \longleftarrow \cdots$$

whose limit $BP\langle\infty\rangle$ is a BP-algebra and is equivalent to BP as a BP-ring spectrum.

Proof. The starting point is the observation of [**8**] that the unit $MU \longrightarrow H\mathbb{Z} = MU\langle 0\rangle$ is an MU-algebra morphism so that realizes the augmentation $MU_* \longrightarrow MU_0 = \mathbb{Z}$. Theorem 3.1 and an induction on n shows that the tower exists as claimed.

In the Milnor exact sequence

$$0 \to \lim_n{}^1 MU\langle n\rangle_* \longrightarrow MU\langle\infty\rangle_* \longrightarrow \lim_n MU\langle n\rangle_* \to 0,$$

we have

$$\lim_n MU\langle n\rangle_* = MU_*/(x_r : 1 \leqslant r \neq p^k-1), \quad \lim_n{}^1 MU\langle n\rangle_* = 0$$

since each map $MU\langle n+1\rangle_* \longrightarrow MU\langle n\rangle_*$ is surjective. Hence

$$MU\langle\infty\rangle_* = MU_*/(x_r : 1 \leqslant r \neq p^k-1).$$

In the BP case, the unit map $BP \longrightarrow BP\langle\infty\rangle$ is a weak equivalence and so is an equivalence of BP-ring spectra. □

Let $\varepsilon\colon MU \longrightarrow BP$ be the map inducing the algebraic projection $(MU_*)_{(p)} \longrightarrow BP_*$ due to Quillen and described in [**1**]. It is well known that the image of each Hazewinkel generator $v_n \in BP_{2(p^n-1)}$ in $(MU_*)_{(p)}$ actually lies in MU_* and we refer to this image as $v_n \in MU_{2(p^n-1)}$. Recall also that there is an MU-module map $\eta\colon MU \longrightarrow BP$ where BP can be given the structure of MU-ring spectrum with unit η [**8, 16**].

Theorem 3.4. *Choose the generators x_r so that $x_r \in \ker\varepsilon$ for $r \neq p^k-1$ and $x_{p^k-1} = v_k$. Then there is a map of MU-ring spectra $\theta\colon BP \longrightarrow MU\langle\infty\rangle_{(p)}$ factoring the unit $MU \longrightarrow MU\langle\infty\rangle_{(p)}$ as*

$$MU \longrightarrow BP \xrightarrow{\theta} MU\langle\infty\rangle_{(p)},$$

where the first map is the unit for BP. θ is an equivalence of MU-ring spectra or equivalently of $MU_{(p)}$-ring spectra $BP \longrightarrow MU\langle\infty\rangle_{(p)}$. Hence the ring spectrum BP can be realized as an MU-algebra or equivalently as an $MU_{(p)}$-algebra.

Proof. Since BP is a cell $MU_{(p)}$-module with cells corresponding to the generators in the Koszul resolution of $(MU_{(p)})_*/(x_r : 1 \leqslant r \neq p^k-1)$, the map θ can be constructed by induction to satisfy the required properties up to homotopy. $\square$

Corollary 3.5. *Each spectrum $BP\langle n\rangle$ $(1 \leqslant n)$ can be realized as an MU-algebra or equivalently as an $MU_{(p)}$-algebra and the natural map $BP\langle n+1\rangle \longrightarrow BP\langle n\rangle$ can be realized as a morphism of algebras.*

Proof. Obstruction theory gives an MU-module map $BP\langle n\rangle \longrightarrow MU\langle n\rangle_{(p)}$ which is visibly a weak equivalence. $\square$

4. Some MU-algebras obtained by localization

From [**8, 18**] we know that on inverting an element $u \in MU_{2d}$ we obtain the Bousfield localization at $MU[u^{-1}]$ as

$$M \longrightarrow \mathrm{L}^{MU}_{MU[u^{-1}]} M = MU[u^{-1}] \underset{MU}{\wedge} M$$

for any MU-module M. Furthermore, if A is an MU-algebra then

$$A \longrightarrow \mathrm{L}^{MU}_{MU[u^{-1}]} A = MU[u^{-1}] \underset{MU}{\wedge} A$$

is a morphism of MU-algebras. Similar considerations would apply if BP were a commutative S-algebra. We will use the notation

$$MU(n) = \mathrm{L}^{MU}_{MU[x^{-1}_{p^n-1}]} MU\langle n\rangle, \quad BP(n) = \mathrm{L}^{BP}_{BP[v_n^{-1}]} BP\langle n\rangle,$$

where

$$MU(n)_* = MU\langle n\rangle_*[x^{-1}_{p^n-1}], \quad BP(n)_* = BP\langle n\rangle[v_n^{-1}].$$

Proposition 4.1. *For each prime p and $n \geqslant 1$, there is a localization morphism of MU-algebras*

$$MU\langle n\rangle \longrightarrow \mathrm{L}^{MU}_{MU[x^{-1}_{p^n-1}]} MU\langle n\rangle = MU(n)$$

and an equivalence of MU-ring spectra $MU(n)_{(p)} \longrightarrow E(n)$. Hence also $E(n)$ admits the structure of an MU-algebra. Therefore $E(n)$ also admits the structure of an S-algebra which is a commutative S-ring spectrum.

If BP is a commutative S-algebra, then there is a localization morphism of BP-algebras

$$BP\langle n\rangle \longrightarrow \mathrm{L}^{BP}_{BP[v_n^{-1}]} BP\langle n\rangle = BP(n)$$

and an equivalence of BP-ring spectra $BP(n) \longrightarrow E(n)$. Hence $E(n)$ admits the structure of a BP-algebra.

Given $E(n)$ as an MU or BP-algebra, we can form the quotient module $E(n)/I_n^k$ obtained by killing all the monomials $v_0^{r_0}v_1^{r_1}\cdots v_{n-1}^{r_{n-1}}$ with $\sum_{i=0}^{n-1} r_i = k$. In [**2, 5**], these spectra were constructed as $\widehat{E(n)}$-modules for an appropriate S-algebra structure; here $\widehat{E(n)}$ is the I_n-adic completion of the S-module $E(n)$. Our present approach shows that each of the natural maps $E(n)/I_n^{k+1} \longrightarrow E(n)/I_n^k$ can be realized as a map of $E(n)$-modules. We obtain $K(n)$ as $E(n)/I_n$. By [**2, 5**], the tower

$$E(n)/I_n \longleftarrow E(n)/I_n^2 \longleftarrow \cdots \longleftarrow E(n)/I_n^k \longleftarrow E(n)/I_n^{k+1} \longleftarrow \cdots \tag{4.1}$$

has $\widehat{E(n)}$ for its limit as an S-ring spectrum, however here we are working with MU, BP or $E(n)$-modules. This gives a new proof of the existence of an S-algebra structure on $\widehat{E(n)}$. We also have $\widehat{E(n)} \simeq \mathrm{L}^S_{K(n)} E(n)$, so this result can also be proved by another application of the Bousfield localization theory of S-algebras.

Proposition 4.2. *The natural map $E(n) \longrightarrow \widehat{E(n)}$ is a morphism of MU-algebras.*

Proof. We need to make use of [**8**, XIII.1.8]; actually, in the statement of this result it is assumed that A is a *commutative* R-algebra but this is unnecessary and the following correct formulation occurs as [**13**, XXIII.6.5].

Lemma 4.3. *Let R be a commutative S-algebra, A be an R-algebra and E be an R-module and M an A-module. Then the $(A\underset{R}{\wedge}E)^A$-localization map $\lambda\colon M \longrightarrow \mathrm{L}^A_{A\underset{R}{\wedge}E}M$ is an E^R-localization map for M. Hence there is a weak equivalence of R-modules $\mathrm{L}^R_E M \longrightarrow \mathrm{L}^A_{A\underset{R}{\wedge}E}M$.*

Now take $R = S$, $A = MU$, $E = K(n)$ and $M = E(n)$. Then we are done since there is an equivalence of MU-modules

$$\widehat{E(n)} \simeq \mathrm{L}^{MU}_{MU\underset{S}{\wedge}K(n)} E(n).$$

For reference we determine the Bousfield class of the MU-module $MU\underset{S}{\wedge}K(n)$. Making use of ideas similar to those in the Appendix of [**3**] (see especially Corollary A.2) we find that $\pi_* MU\underset{S}{\wedge}K(n) = MU_*K(n)$ is a free $MU_*/I_n[v_n^{-1}]$-module. By a standard argument, there is a weak equivalence of MU-modules

$$\bigvee_\alpha \Sigma^{2d_\alpha} MU/I_n[v_n^{-1}] \simeq MU\underset{S}{\wedge}K(n),$$

hence $MU\underset{S}{\wedge}K(n)$ is Bousfield equivalent to $MU/I_n[v_n^{-1}]$.

Actually, Theorem 6.4 of [**4**] also tells us that there is an equivalence of MU-modules

$$\mathrm{L}^{MU}_{MU/I_n[v_n^{-1}]}\,E(n) \simeq \widehat{E(n)};$$

indeed the tower of (4.1) is constructed in [**4**] as a tower of MU-modules and its homotopy limit is shown to be $\mathrm{L}^{MU}_{MU/I_n[v_n^{-1}]}\,E(n)$. $\square$

Having obtained $\widehat{E(n)}$ as an MU-algebra, ideas of [**12, 15**] can be used to show that there is an MU-algebra $\widehat{E(n)}\mathrm{W}(\mathbb{F}_{p^n})$ obtained by adjoining a primitive (p^n-1)-st root of unity to $\widehat{E(n)}_*$. The 2-periodic version of this spectrum is known to be a commutative S-algebra by work of Hopkins, Miller and Goerss [**9**].

References

[**1**] J. F. Adams, Stable Homotopy and Generalised Homology, University of Chicago Press (1974).

[**2**] A. Baker, A_∞ stuctures on some spectra related to Morava K-theory, Quart. J. Math. Oxf. **42** (1991), 403–419.

[**3**] A. Baker, On the Adams E_2-term for elliptic cohomology, Contemp. Math. **271** (2001), 1–15.

[**4**] A. Baker & A. Lazarev, On the Adams Spectral Sequence for R-modules, Algebraic & Geometric Topology **1** (2001), 173-99.

[**5**] A. Baker & U. Würgler, Bockstein operations in Morava K-theory, Forum Math. **3** (1991), 543–60.

[**6**] J. M. Boardman, Conditionally convergent spectral sequences, Contemp. Math. **239** (1999) 49–84.

[**7**] A. K. Bousfield & D. M. Kan, Homotopy limits, completions and localizations, Lecture Notes in Mathematics, **304** (1972).

[**8**] A. Elmendorf, I. Kriz, M. Mandell & J. P. May, Rings, modules, and algebras in stable homotopy theory, Mathematical Surveys and Monographs **47** (1999).

[**9**] P. G. Goerss & M. J. Hopkins, Realizing commutative ring spectra as E_∞ ring spectra, preprint.

[**10**] M. Hovey, Model Categories, Mathematical Surveys and Monographs **63** (1999).

[**11**] A. Lazarev, Homotopy theory of A_∞ ring spectra and applications to MU-modules, K-theory **24** (2001), 243–81.

[**12**] A. Lazarev, Towers of MU-algebras and the generalized Hopkins-Miller theorem, preprint `math.AT/0209386`.

[**13**] J. P. May, Equivariant homotopy and cohomology theory, CBMS Regional Conference Series in Mathematics **91** (1996).

[14] D. C. Ravenel, Complex Cobordism and the Stable Homotopy Groups of Spheres, Academic Press (1986).

[15] R. Schwänzl, R. M. Vogt & F. Waldhausen, Adjoining roots of unity to E_∞ ring spectra in good cases– a remark, Contemp. Math. **239** (1999), 245–249,

[16] N. P. Strickland, Products on MU-modules, Trans. Amer. Math. Soc. **351** (1999), 2569–2606.

[17] R. M. Vogt, Homotopy limits and colimits, Math. Zeit. **134** (1973), 11–52.

[18] J. J. Wolbert, Classifying modules over K-theory spectra, J. Pure Appl. Algebra **124** (1998), 289–323.

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