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On Some New Paranormed Sequence Spaces

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Abstract

The sequence spaces c_0^λ, c^λ *and* ℓ_∞^λ *have been recently introduced and studied by Mursaleen and Noman [*On the spaces of $\lambda-$ convergent and bounded sequences, *Thai J. Math.* ***8****(2)(2010),311-329]. The main purpose of the present paper is to extend the results of Mursaleen and Noman to the paranormed case and is to work the spaces* $c_0^\lambda(u,p), c^\lambda(u,p)$ *and* $\ell_\infty^\lambda(u,p)$*. Let* μ *denote any of the spaces* c_0, c *and* ℓ_∞*. We prove that* $\mu^\lambda(u,p)$ *is linearly paranorm isomorphic to* $\mu(p)$ *and determine the* $\alpha-, \beta-$ *and* $\gamma-$ *duals of the* $\mu^\lambda(u,p)$*. Furthermore, the basis of* $c_0^\lambda(u,p)$ *and* $c^\lambda(u,p)$ *are constructed. Finally, we characterize the matrix transformations from the spaces* $c_0^\lambda(u,p), c^\lambda(u,p)$ *and* $\ell_\infty^\lambda(u,p)$ *to the spaces* $c_0(q), c(q), \ell(q)$ *and* $\ell_\infty(q)$*.*

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1 Introduction

By ω, we shall denote the space of all real valued sequences. Any vector subspace of ω is called as a *sequence space*. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p ; we denote the spaces of all bounded, convergent, absolutely and $p-$ absolutely convergent series, respectively; $1 < p < \infty$.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g : X \to \mathbb{R}$ such that $g(\theta) = 0, g(x) = g(-x), g(x + y) \leq g(x) + g(y)$ and scalar multiplication is continuous,i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Assume here and after that $u = (u_k)$ be a sequence such that $u_k \neq 0$ for all $k \in \mathbb{N}$ and $(q_k), (p_k)$ be the bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $L = \max\{1, H\}$, where $\mathbb{N} = \{0, 1, 2, ...\}$. Then, the linear spaces $\ell_\infty(p), c(p), c_0(p)$ and $\ell(p)$ were defined by Maddox [11, 12] (see also Simons [18] and Nakano [9]) as follows:

$$\begin{aligned}
\ell_\infty(p) &= \left\{x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} |x_k|^{p_k} < \infty\right\}, \\
c(p) &= \left\{x = (x_k) \in \omega : \lim_{k \to \infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{C}\right\}, \\
c_0(p) &= \left\{x = (x_k) \in \omega : \lim_{k \to \infty} |x_k|^{p_k} = 0\right\},
\end{aligned}$$

and

$$\ell(p) = \left\{x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty\right\},$$

which are the complete paranormed spaces by

$$g_1(x) = \sup_{k \in \mathbb{N}} |x_k|^{p_k/L} \iff \inf p_k > 0 \text{ and } g_2(x) = (\sum_k |x_k|^{p_k})^{1/L}, \tag{1}$$

respectively. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $\mathcal{F}$ and $\mathbb{N}_k$, we shall denote the collection of all finite subsets of $\mathbb{N}$ and the set of all $n \in \mathbb{N}$ such that $n \geq k$.

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk},where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = ((Ax)_n)$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \quad (n \in \mathbb{N}). \tag{2}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right-hand side of (2) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A- summable to α if Ax converges to α which is called the A- limit of x.

Let $\lambda = (\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive reals tending to infinity, that is

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < ... \quad \text{and} \quad \lambda_k \to \infty \quad \text{as} \quad k \to \infty.$$

We say that a sequence $x = (x_k) \in \omega$ is $\lambda-$ convergent to the number $l \in \mathbb{C}$, called the $\lambda-$ limit of x, if $\Lambda_n(x) \to l$ as $n \to \infty$ where

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k; \quad (n \in \mathbb{N}). \tag{3}$$

In particular, we say that x is a $\lambda-$ null sequence if $\Lambda_n(x) \to 0$ as $n \to \infty$. Further, we say that x is $\lambda-$ bounded if $\sup_{n\in\mathbb{N}} |\Lambda_n(x)| < \infty$, [16].

The main purpose of this paper is to introduce the sequence spaces $c_0^\lambda(u,p)$, $c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ of non-absolute type which are the set of all sequences whose Λ^u- transforms are in the spaces $c_0(p), c(p)$ and $\ell_\infty(p)$, respectively; where Λ^u denotes the matrix $\Lambda^u = (\lambda_{nk}^u)$ defined by

$$\lambda_{nk}^u = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n} u_k, & (0 \le k \le n) \\ 0, & (k > n). \end{cases}$$

Besides this, we have constructed the basis of the spaces $c_0^\lambda(u,p)$ and $c^\lambda(u,p)$ and computed the $\alpha-, \beta-$ and $\gamma-$ duals of the spaces $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$. Finally, a basic theorem is given and some matrix mappings from the spaces $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ to the sequence spaces of Maddox are characterized.

2 The Sequence Spaces $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ of non-absolute type

In this section, we define the sequence spaces $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ and prove that $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ are the complete paranormed linear spaces.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined by

$$X_A = \{x = (x_k) \in \omega : \quad Ax \in X\}. \tag{4}$$

By using the matrix domain of a triangular infinite matrix, the new sequence spaces have been defined by many authors. For example see [2, 3, 7, 8, 4, 14, 15] and the others.

Quite recently, Demiriz and Çakan have studied the sequence spaces $e_0^r(u,p)$ and $e_c^r(u,p)$ in [17]. With the notation of (4), the spaces $e_0^r(u,p)$ and $e_c^r(u,p)$ may be redefined as

$$e_0^r(u,p) = [c_0(p)]_{E^{r,u}}, \quad e_c^r(u,p) = [c(p)]_{E^{r,u}},$$

where the matrix $E^{r,u} = (e_{nk}^r(u))$ is defined by

$$e_{nk}^r(u) = \begin{cases} \binom{n}{k}(1-r)^{n-k}r^k u_k, & (0 \leq k \leq n) \\ 0, & (k > n). \end{cases}$$

The sequence spaces c_0^λ, c^λ and ℓ_∞^λ of non-absolute type have been introduced by Mursaleen and Noman [16] as follows:

$$\begin{aligned} c_0^\lambda &= \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k = 0 \right\} \\ c^\lambda &= \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \quad \text{exists} \right\} \end{aligned}$$

and

$$\ell_\infty^\lambda = \left\{ x = (x_k) \in \omega : \sup_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \right| < \infty \right\}.$$

Following Choudhary and Mishra [4], Başar and Altay [7], Aydın and Başar [5], Demiriz and Çakan [17], we define the sequence spaces $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ as the set of all sequences such that Λ^u-transforms of them are in the spaces $c_0(p), c(p)$ and $\ell_\infty(p)$, respectively, that is

$$\begin{aligned} c_0^\lambda(u,p) &= \left\{ x = (x_k) \in \omega : \lim_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) u_k x_k \right|^{p_n} = 0 \right\} \\ c^\lambda(u,p) &= \left\{ x = (x_k) \in \omega : \lim_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) u_k x_k \right|^{p_n} \text{exists} \right\} \end{aligned}$$

and

$$\ell_\infty^\lambda(u,p) = \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) u_k x_k \right|^{p_n} < \infty \right\}.$$

In the case $(u_k) = (p_k) = e = (1,1,1,...)$, the sequence spaces $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ are , respectively, reduced to the sequence spaces c_0^λ, c^λ and ℓ_∞^λ which are introduced by Mursaleen and Noman [16]. With the notation of (4), we may redefine the spaces $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ as follows:

$$c_0^\lambda(u,p) = [c_0(p)]_{\Lambda^u},\ c^\lambda(u,p) = [c(p)]_{\Lambda^u} \quad \text{and} \quad \ell_\infty^\lambda(u,p) = [\ell_\infty(p)]_{\Lambda^u}.$$

Define the sequence $y = \{y_n(\lambda)\}$, which will be frequently used, as the Λ^u-transform of a sequence $x = (x_k)$, i.e.

$$y_n(\lambda) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})u_k x_k; \quad (n \in \mathbb{N}). \tag{5}$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1 *$c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ are the complete linear metric spaces paranormed by g, defined by*

$$g(x) = \sup_{k\in\mathbb{N}}\left|\frac{1}{\lambda_k}\sum_{j=0}^{k}(\lambda_j - \lambda_{j-1})u_j x_j\right|^{p_k/M}.$$

g is a paranorm for the spaces $\ell_\infty^\lambda(u,p)$ and $c^\lambda(u,p)$ only in the trivial case $\inf p_k > 0$ when $\ell_\infty^\lambda(u,p) = \ell_\infty^\lambda$ and $c^\lambda(u,p) = c^\lambda$.

Proof. We prove the theorem for the space $c_0^\lambda(u,p)$. The linearity of $c_0^\lambda(u,p)$ with respect to the co-ordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $x, z \in c_0^\lambda(u,p)$ (see [10, p.30])

$$\begin{aligned}\sup_{k\in\mathbb{N}}\left|\frac{1}{\lambda_k}\sum_{j=0}^{k}(\lambda_j - \lambda_{j-1})u_j(x_j + z_j)\right|^{p_k/M} &\le \sup_{k\in\mathbb{N}}\left|\frac{1}{\lambda_k}\sum_{j=0}^{k}(\lambda_j - \lambda_{j-1})u_j x_j\right|^{p_k/L} \\ &+ \sup_{k\in\mathbb{N}}\left|\frac{1}{\lambda_k}\sum_{j=0}^{k}(\lambda_j - \lambda_{j-1})u_j z_j\right|^{p_k/L} \end{aligned} \tag{6}$$

and for any $\alpha \in \mathbb{R}$ (see [12])

$$|\alpha|^{p_k} \le \max\{1, |\alpha|^M\}. \tag{7}$$

It is clear that $g(\theta) = 0$ and $g(x) = g(-x)$ for all $x \in c_0^\lambda(u,p)$. Again the inequalities (6) and (7) yield the subadditivity of g and

$$g(\alpha x) \le \max\{1, |\alpha|\}g(x).$$

Let $\{x^n\}$ be any sequence of the points $x^n \in c_0^\lambda(u,p)$ such that $g(x^n - x) \to 0$ and (α_n) also be any sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality

$$g(x^n) \leq g(x) + g(x^n - x)$$

holds by the subadditivity of g, $\{g(x^n)\}$ is bounded and we thus have

$$\begin{aligned} g(\alpha^n x^n - \alpha x) &= \sup_{k\in\mathbb{N}} \left| \frac{1}{\lambda_k} \sum_{j=0}^{k} (\lambda_j - \lambda_{j-1}) u_j (\alpha^n x_j^n - \alpha x_j) \right|^{p_k/M} \\ &\leq |\alpha_n - \alpha| \;\; g(x^n) + |\alpha| \;\; g(x^n - x) \end{aligned}$$

which tends to zero as $n \to \infty$. This means that the scalar multiplication is continuous. Hence, g is a paranorm on the space $c_0^\lambda(u,p)$.

It remains to prove the completeness of the space $c_0^\lambda(u,p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $c_0^\lambda(u,p)$, where $x^i = \{x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, ...\}$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$g(x^i - x^j) < \frac{\varepsilon}{2}$$

for all $i, j > n_0(\varepsilon)$. By using the definition of g we obtain for each fixed $k \in \mathbb{N}$ that

$$|(\Lambda^u x^i)_k - (\Lambda^u x^j)_k|^{p_k/M} \leq \sup_{k\in\mathbb{N}} |(\Lambda^u x^i)_k - (\Lambda^u x^j)_k|^{p_k/M} < \frac{\varepsilon}{2} \tag{8}$$

for every $i, j \geq n_0(\varepsilon)$ which leads us to the fact that $\{(\Lambda^u x^0)_k, (\Lambda^u x^1)_k, (\Lambda^u x^2)_k, ...\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(\Lambda^u x^i)_k \to (\Lambda^u x)_k$ as $i \to \infty$. Using these infinitely many limits $(\Lambda^u x)_0, (\Lambda^u x)_1, (\Lambda^u x)_2, ...$, we define the sequence $\{(\Lambda^u x)_0, (\Lambda^u x)_1, (\Lambda^u x)_2, ...\}$. From (8) with $j \to \infty$, we have

$$|(\Lambda^u x^i)_k - (\Lambda^u x)_k|^{p_k/M} \leq \frac{\varepsilon}{2} \qquad (i \geq n_0(\varepsilon)) \tag{9}$$

for every fixed $k \in \mathbb{N}$. Since $x^i = \{x_k^{(i)}\} \in c_0^\lambda(u,p)$ for each $i \in \mathbb{N}$, there exists $k_0(\varepsilon) \in \mathbb{N}$ such that

$$|(\Lambda^u x^i)_k|^{p_k/M} < \frac{\varepsilon}{2}$$

for every $k \geq k_0(\varepsilon)$ and for each fixed $i \in \mathbb{N}$. Therefore, taking a fixed $i \geq n_0(\varepsilon)$ we obtain by (9) that

$$|(\Lambda^u x)_k|^{p_k/M} \leq |(\Lambda^u x)_k - (\Lambda^u x^i)_k|^{p_k/M} + |(\Lambda^u x^i)_k|^{p_k/M} < \varepsilon$$

for every $k \geq k_0(\varepsilon)$. This shows that $x \in c_0^\lambda(u,p)$. Since $\{x^i\}$ was an arbitrary Cauchy sequence, the space $c_0^\lambda(u,p)$ is complete and this concludes the proof.

Note that the absolute property does not hold on the spaces $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$, since there exists at least one sequence in the spaces $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ such that $g(x) \neq g(|x|)$; where $|x| = (|x_k|)$. This says that $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ are the sequence spaces of non-absolute type.

Theorem 2.2 *The sequence spaces $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ of non-absolute type are linearly isomorphic to the spaces $c_0(p), c(p)$ and $\ell_\infty(p)$, respectively; where $0 < p_k \leq H < \infty$.*

Proof. To avoid the repetition of the similar statements, we give the proof only for $c_0^\lambda(u,p)$. We should show the existence of a linear bijection between the spaces $c_0^\lambda(u,p)$ and $c_0(p)$. With the notation of (5), define the transformation T from $c_0^\lambda(u,p)$ and $c_0(p)$ by $x \mapsto y = Tx$. The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y \in c_0(p)$ and define the sequence $x = \{x_k(\lambda)\}$ by

$$x_k(\lambda) = \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j}{(\lambda_k - \lambda_{k-1})u_k} y_j; \qquad (k \in \mathbb{N}).$$

Then, we have

$$\begin{aligned} g(x) &= \sup_{k\in\mathbb{N}} \left| \frac{1}{\lambda_k} \sum_{j=0}^{k} (\lambda_j - \lambda_{j-1}) u_j x_j \right|^{p_k/M} \\ &= \sup_{k\in\mathbb{N}} \left| \sum_{j=0}^{k} \delta_{kj} y_j \right|^{p_k/M} \\ &= \sup_{k\in\mathbb{N}} |y_k|^{p_k/M} = g_1(y) < \infty. \end{aligned}$$

Thus, we have that $x \in c_0^\lambda(u,p)$ and consequently T is surjective and paranorm preserving. Hence, T is a linear bijection and this says us that the spaces $c_0^\lambda(u,p)$ and $c_0(p)$ are linearly isomorphic, as was desired.

3 The basis for the spaces $c_0^\lambda(u,p)$ and $c^\lambda(u,p)$

In the present section, we give two sequences of the points of the spaces $c_0^\lambda(u,p)$ and $c^\lambda(u,p)$ which form the basis for those spaces.

Firstly, we give the definition of the Schauder basis of a paranormed space and later give the theorem exhibiting the basis of the spaces $c_0^\lambda(u,p)$ and $c^\lambda(u,p)$. Let (λ, h) be a paranormed space. A sequence (b_k) of the elements of λ is called a basis for λ if and only if, for each $x \in \lambda$, there exists a unique sequence (α_k) of scalars such that

$$h\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) \to 0 \quad as \quad n \to \infty.$$

The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$.

Because of the isomorphism T is onto, defined in the Proof of Theorem 2.2, the inverse image of the basis of those spaces $c_0(p)$ and $c(p)$ are the basis of the new spaces $c_0^\lambda(u,p)$ and $c^\lambda(u,p)$, respectively. Therefore, we have the following:

Theorem 3.1 *Let $\nu_k(\lambda) = (\Lambda^u x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \leq H < \infty$. Define the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n\in\mathbb{N}}$ of the elements of the space $c_0^\lambda(u,p)$ by*

$$b_n^{(k)}(\lambda) = \begin{cases} (-1)^{k-n} \dfrac{\lambda_n}{(\lambda_k - \lambda_{k-1})u_k} & (n \leq k \leq n+1), \\ 0 & (n < k \text{ or } n > k+1) \end{cases} \tag{10}$$

for every fixed $k \in \mathbb{N}$. Then,
(a) The sequence $\{b^{(k)}(\lambda)\}_{k\in\mathbb{N}}$ is a Schauder basis for the space $c_0^\lambda(u,p)$ and any $x \in c_0^\lambda(u,p)$ has a unique representation of the form

$$x = \sum_k \nu_k(\lambda) b^{(k)}(\lambda). \tag{11}$$

(b) The set $\{b, b^{(1)}(\lambda), b^{(2)}(\lambda), ...\}$ is a basis for the space $c^\lambda(u,p)$ and any $x \in c^\lambda(u,p)$ has a unique representation of the form

$$x = lz + \sum_k [\nu_k(\lambda) - l] b^{(k)}(\lambda); \tag{12}$$

where $b = \{\frac{1}{u_k}\}_{k=0}^{\infty}$ and

$$l = \lim_{k\to\infty} (\Lambda^u x)_k. \tag{13}$$

Proof. It is clear that $\{b^{(k)}(\lambda)\} \subset c_0^\lambda(u,p)$, since

$$\Lambda^u b^{(k)}(\lambda) = e^{(k)} \in c_0(p), \qquad (k \in \mathbb{N}) \tag{14}$$

for $0 < p_k \leq H < \infty$; where $e^{(k)}$ is the sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$.

Let $x \in c_0^\lambda(u,p)$ be given. For every non-negative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \nu_k(\lambda) b^{(k)}(\lambda). \tag{15}$$

Then, we obtain by applying Λ^u to (15) with (14) that

$$\Lambda^u x^{[m]} = \sum_{k=0}^{m} \nu_k(\lambda)\Lambda^u b^{(k)}(\lambda) = \sum_{k=0}^{m} (\Lambda^u x)_k e^{(k)}$$

and

$$\{\Lambda^u(x - x^{[m]})\}_i = \begin{cases} 0, & (0 \leq i \leq m), \\ (\Lambda^u x)_i, & (i > m), \end{cases}$$

where $i, m \in \mathbb{N}$. Given $\varepsilon > 0$, then there is an integer m_0 such that

$$\sup_{i \geq m} |(\Lambda^u x)_i|^{p_k/M} < \frac{\varepsilon}{2}$$

for all $m \geq m_0$. Hence,

$$g(x - x^{[m]}) = \sup_{i \geq m} |(\Lambda^u x)_i|^{p_k/M} \leq \sup_{i \geq m_0} |(\Lambda^u x)_i|^{p_k/M} < \frac{\varepsilon}{2} < \varepsilon$$

for all $m \geq m_0$ which proves that $x \in c_0^\lambda(u,p)$ is represented as in (11).

Let us show the uniqueness of the representation for $x \in c_0^\lambda(u,p)$ given by (11). Suppose, on the contrary, that there exists a representation $x = \sum_k \mu_k(\lambda) b^{(k)}(\lambda)$. Since the linear transformation T from $c_0^\lambda(u,p)$ to $c_0(p)$, used in Theorem 2.2, is continuous we have at this stage that

$$(\Lambda^u x)_n = \sum_k \mu_k(\lambda)\{\Lambda^u b^{(k)}(\lambda)\}_n = \sum_k \mu_k(\lambda) e_n^{(k)} = \mu_n(\lambda); \qquad (n \in \mathbb{N})$$

which contradicts the fact that $(\Lambda^u x)_n = \nu_n(\lambda)$ for all $n \in \mathbb{N}$. Hence, the representation (11) of $x \in c_0^\lambda(u,p)$ is unique. This completes the proof of Part (a) of Theorem.

(b) Since $\{b^{(k)}(\lambda)\} \subset c_0^\lambda(u,p)$ and $b \in c_0(p)$, the inclusion $\{b, b^{(k)}(\lambda)\} \subset c^\lambda(u,p)$ is obviously true. Let us take $x \in c^\lambda(u,p)$. Then there uniquely exists an l satisfying (13). We thus have $z \in c_0^\lambda(u,p)$ whenever we set $z = x - lb$. Therefore, we deduce by Part (a) of the present theorem that the representation of z is unique. This implies that the representation of x given by (12) is unique, which concludes the proof.

Proposition 3.2 *[1, Remark 2.4] The matrix domain X_A of a normed sequence space X has basis if and only if X has a basis.*

Since it is known that $\ell_\infty(p)$ has no basis, we can deduce from this proposition the following corollary.

Corollary 3.3 *$\ell_\infty^\lambda(u,p)$ has no Schauder basis.*

4 The $\alpha-,\beta-$ and $\gamma-$ duals of the spaces $c_0^\lambda(u,p)$, $c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$

In this section, we state and prove the theorems determining the $\alpha-,\beta-$ and $\gamma-$ duals of the sequence spaces $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ of non-absolute type.

We shall firstly give the definition of $\alpha-,\beta-$ and $\gamma-$ duals of a sequence spaces and later quote the lemmas which are needed in proving the theorems given in Section 4.

For the sequence spaces λ and μ, define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu)=\{z=(z_k): xz=(x_kz_k)\in\mu \ \ for \ \ all \ \ x\in\lambda\} \tag{16}$$

With the notation of (16), the $\alpha-,\beta-$ and $\gamma-$ duals of a sequence space λ, which are respectively denoted by $\lambda^\alpha,\lambda^\beta$ and λ^γ, are defined by

$$\lambda^\alpha=S(\lambda,\ell_1),\quad \lambda^\beta=S(\lambda,cs)\ and\ \ \lambda^\gamma=S(\lambda,bs).$$

Lemma 4.1 *[13, Theorem 5.1.3 with $q_n=1$] $A\in(\ell_\infty(p):\ell_1)$ if and only if*

$$\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}a_{nk}B^{1/p_k}\right|<\infty \quad \text{for all integers} \quad B>1. \tag{17}$$

Lemma 4.2 *[13, Theorem 5.1.9] $A\in(c_0(p):c(q))$ if and only if*

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|B^{-1/p_k}<\infty \qquad (\exists B\in\mathbb{N}_2) \tag{18}$$

$$\exists(\alpha_k)\subset\mathbb{R}\ni\lim_{n\to\infty}|a_{nk}-\alpha_k|^{q_n}=0 \quad \text{for all } k\in\mathbb{N}. \tag{19}$$

$$\exists(\alpha_k)\subset\mathbb{R}\ni\sup_{n\in\mathbb{N}}N^{1/q_n}\sum_k|a_{nk}-\alpha_k|B^{-1/p_k}<\infty \quad (\exists B\in\mathbb{N}_2 \text{ and } \forall N\in\mathbb{N}_1). \tag{20}$$

Lemma 4.3 *[6, Theorem 3] Let $p_k>0$ for every k. Then $A\in(\ell_\infty(p):\ell_\infty)$ if and only if*

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|B^{1/p_k}<\infty \quad \text{for all integers} \quad B>1. \tag{21}$$

Theorem 4.4 *Let $K^* = \{k \in \mathbb{N} : n-1 \leq k \leq n\} \cap K$ for $K \in \mathcal{F}$ and $B \in \mathbb{N}_2$. Define the sets $\Lambda_1(u,p), \Lambda_2(u)$ and $\Lambda_3(u,p)$ as follows:*

$$\begin{aligned}
\Lambda_1(u,p) &= \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K^*} (-1)^{n-k} \frac{\lambda_k}{(\lambda_n - \lambda_{n-1}) u_n} a_n B^{-1/p_k} \right| < \infty \right\} \\
\Lambda_2(u) &= \left\{ a = (a_k) \in \omega : \sum_n \left| \frac{a_n}{u_n} \right| < \infty \right\} \\
\Lambda_3(u,p) &= \bigcap_{B>1} \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K^*} (-1)^{n-k} \frac{\lambda_k}{(\lambda_n - \lambda_{n-1}) u_n} a_n B^{1/p_k} \right| < \infty \right\}
\end{aligned}$$

Then

$$\left[c_0^\lambda(u,p)\right]^\alpha = \Lambda_1(u,p), \quad \left[c^\lambda(u,p)\right]^\alpha = \Lambda_1(u,p) \cap \Lambda_2(u) \quad \text{and } \left[\ell_\infty^\lambda(u,p)\right]^\alpha = \Lambda_3(u,p).$$

Proof. We give the proof only for the space $\ell_\infty^\lambda(u,p)$. Let us take any $a = (a_k) \in \omega$ and define the matrix $C^\lambda = (c_{nk}^\lambda)$ via the sequence $a = (a_n)$ by

$$c_{nk}^\lambda = \begin{cases} (-1)^{n-k} \dfrac{\lambda_k}{(\lambda_n - \lambda_{n-1}) u_n} a_n & (n-1 \leq k \leq n), \\ 0 & (0 \leq k < n-1 \text{ or } k > n) \end{cases}$$

where $n, k \in \mathbb{N}$. Bearing in mind (5) we immediately derive that

$$a_n x_n = \sum_{k=n-1}^{n} (-1)^{n-k} \frac{\lambda_k}{(\lambda_n - \lambda_{n-1}) u_n} a_n y_k = (C^\lambda y)_n, \quad (n \in \mathbb{N}). \tag{22}$$

We therefore observe by (22) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in \ell_\infty^\lambda(u,p)$ if and only if $C^\lambda y \in \ell_1$ whenever $y \in \ell_\infty(p)$. This means that $a = (a_n) \in \left[\ell_\infty^\lambda(u,p)\right]^\alpha$ whenever $x = (x_n) \in \ell_\infty^\lambda(u,p)$ if and only if $C^\lambda \in (\ell_\infty(p) : \ell_1)$. Then, we derive by Lemma 4.1 for all $n \in \mathbb{N}$ that

$$\left[\ell_\infty^\lambda(u,p)\right]^\alpha = \Lambda_3(u,p).$$

Theorem 4.5 *Define the sets $\Lambda_4(u,p), \Lambda_5(u,p), \Lambda_6(u), \Lambda_7(u), \Lambda_8(u,p)$ and $\Lambda_9(u,p)$ as follows:*

$$\begin{aligned}
\Lambda_4(u,p) &= \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sum_k \left| \tilde{\Delta} \left[\frac{a_k}{(\lambda_k - \lambda_{k-1}) u_k} \right] \lambda_k \right| B^{-1/p_k} < \infty \right\} \\
\Lambda_5(u,p) &= \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \left\{ \frac{\lambda_k}{(\lambda_k - \lambda_{k-1}) u_k} a_k B^{-1/p_k} \right\} \in \ell_\infty \right\}
\end{aligned}$$

$$\begin{aligned}
\Lambda_6(u) &= \left\{a=(a_k)\in\omega : \sum_k \left|\tilde{\Delta}\left[\frac{a_k}{(\lambda_k-\lambda_{k-1})u_k}\right]\lambda_k\right| < \infty\right\} \\
\Lambda_7(u) &= \left\{a=(a_k)\in\omega : \lim_{k\to\infty}\left\{\frac{\lambda_k}{(\lambda_k-\lambda_{k-1})u_k}a_k\right\} \quad \text{exists}\right\} \\
\Lambda_8(u,p) &= \bigcap_{B>1}\left\{a=(a_k)\in\omega : \sum_k \left|\tilde{\Delta}\left[\frac{a_k}{(\lambda_k-\lambda_{k-1})u_k}\right]\lambda_k\right| B^{1/p_k} < \infty\right\} \\
\Lambda_9(u,p) &= \bigcap_{B>1}\left\{a=(a_k)\in\omega : \left\{\frac{\lambda_k}{(\lambda_k-\lambda_{k-1})u_k}a_k B^{1/p_k}\right\}\in c_0\right\}.
\end{aligned}$$

Then,

$$\left[c_0^\lambda(u,p)\right]^\beta = \Lambda_4(u,p)\cap\Lambda_5(u,p),\ \left[c^\lambda(u,p)\right]^\beta = \left[c_0^\lambda(u,p)\right]^\beta\cap\Lambda_6(u)\cap\Lambda_7(u)$$

and

$$\left[\ell_\infty^\lambda(u,p)\right]^\beta = \Lambda_8(u,p)\cap\Lambda_9(u,p).$$

Proof. We give the proof only for the space $c_0^\lambda(u,p)$. Consider the equation

$$\begin{aligned}
\sum_{k=0}^n a_k x_k &= \sum_{k=0}^n\left[\sum_{j=k-1}^k (-1)^{k-j}\frac{\lambda_j}{(\lambda_k-\lambda_{k-1})u_k}y_j\right]a_k \\
&= \sum_{k=0}^{n-1}\tilde{\Delta}\left[\frac{a_k}{(\lambda_k-\lambda_{k-1})u_k}\right]\lambda_k y_k + \frac{\lambda_n}{(\lambda_n-\lambda_{n-1})u_n}a_n y_n = (D^\lambda y)_n; \quad (23)
\end{aligned}$$

where $D^\lambda = (d_{nk}^\lambda)$ is defined by

$$d_{nk}^\lambda = \begin{cases} \tilde{\Delta}\left[\frac{a_k}{(\lambda_k-\lambda_{k-1})u_k}\right]\lambda_k & (0\le k\le n-1), \\ \frac{\lambda_n}{(\lambda_n-\lambda_{n-1})u_n}a_n & (k=n), \\ 0, & (k>n), \end{cases}$$

and

$$\tilde{\Delta}\left[\frac{a_k}{(\lambda_k-\lambda_{k-1})u_k}\right]\lambda_k = \left[\frac{a_k}{(\lambda_k-\lambda_{k-1})u_k} - \frac{a_{k+1}}{(\lambda_{k+1}-\lambda_k)u_{k+1}}\right]\lambda_k.$$

Thus, we deduce from Lemma 4.2 with $q_n = 1$ for all $n\in\mathbb{N}$ and (23) that $ax = (a_k x_k)\in cs$ whenever $x = (x_k)\in c_0^\lambda(u,p)$ if and only if $D^\lambda y\in c$ whenever $y=(y_k)\in c_0(p)$. This means that $a=(a_n)\in\left[c_0^\lambda(u,p)\right]^\beta$ whenever $x=(x_n)\in c_0^\lambda(u,p)$ if and only if $D^\lambda\in(c_0(p):c)$. Therefore we derive from (18) with $q_n=1$ for all $n\in\mathbb{N}$ and some $B\in\mathbb{N}_2$ that

$$\sum_k\left|\tilde{\Delta}\left[\frac{a_k}{(\lambda_k-\lambda_{k-1})u_k}\right]\lambda_k\right| B^{-1/p_k} < \infty$$

and

$$\left\{\frac{\lambda_k}{(\lambda_k-\lambda_{k-1})u_k}a_kB^{-1/p_k}\right\}\in \ell_\infty.$$

This shows that $\left[c_0^\lambda(u,p)\right]^\beta=\Lambda_4(u,p)\cap\Lambda_5(u,p)$.

Theorem 4.6 *Define the sets* $\Lambda_{10}(u)$ *and* $\Lambda_{11}(u,p)$ *as follows:*

$$\Lambda_{10}(u)=\left\{a=(a_k)\in\omega:\left\{\frac{\lambda_k}{(\lambda_k-\lambda_{k-1})u_k}a_k\right\}\in bs\right\}$$

and

$$\Lambda_{11}(u,p)=\bigcap_{B>1}\left\{a=(a_k)\in\omega:\left\{\frac{\lambda_k}{(\lambda_k-\lambda_{k-1})u_k}a_kB^{1/p_k}\right\}\in\ell_\infty\right\}.$$

Then,

$$\left[c_0^\lambda(u,p)\right]^\gamma=\Lambda_4(u,p)\cap\Lambda_5(u,p),\quad \left[c^\lambda(u,p)\right]^\gamma=\left[c_0^\lambda(u,p)\right]^\gamma\cap\Lambda_{10}(u)$$

and

$$\left[\ell_\infty^\lambda(u,p)\right]^\gamma=\Lambda_8(u,p)\cap\Lambda_{11}(u,p).$$

Proof. This may be obtained by proceedings as in Theorems 4.4 and 4.5, above. So we omit the details.

5 Certain Matrix Mappings on the spaces $c_0^\lambda(u,p)$, $c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$

In this section, we characterize the matrix mappings from the sequence spaces $c_0^\lambda(u,p), c^\lambda(u,p)$ and $\ell_\infty^\lambda(u,p)$ into any given sequence space. We shall write throughout for brevity that

$$\begin{aligned}\tilde{a}_{nk} &= \Delta\left[\frac{a_{nk}}{(\lambda_k-\lambda_{k-1})u_k}\right]\lambda_k\\ &= \left[\frac{a_{nk}}{(\lambda_k-\lambda_{k-1})u_k}-\frac{a_{n,k+1}}{(\lambda_{k+1}-\lambda_k)u_{k+1}}\right]\lambda_k\end{aligned}$$

for all $n,k\in\mathbb{N}$. We will also use the similar notation with other letters and use the convention that any term with negative subscript is equal to naught.

Suppose throughout that the entries of the infinite matrices $A=(a_{nk})$ and $C=(c_{nk})$ are connected with the relation

$$c_{nk}=\tilde{a}_{nk}\quad\left(\text{or equivalently } a_{nk}=\sum_{j=k}^{\infty}\frac{(\lambda_k-\lambda_{k-1})}{\lambda_k}u_kc_{nj}\right)\quad(n,k\in\mathbb{N}). \tag{24}$$

Now, we may give our basic theorem.

Theorem 5.1 *Let μ be any given sequence space. Then, $A \in (c_0^{\lambda}(u,p) : \mu)$ if and only if $C \in (c_0(p) : \mu)$ and*

$$\left\{ \frac{\lambda_k}{(\lambda_k - \lambda_{k-1})u_k} a_{nk} B^{-1/p_k} \right\} \in c_0, \quad (\forall n \in \mathbb{N}, \exists B \in \mathbb{N}_2). \tag{25}$$

Proof. Suppose that (24) holds and μ be any given sequence space. Let $A \in (c_0^{\lambda}(u,p) : \mu)$ and take any $y \in c_0(p)$. Then, $(a_{nk})_{k\in\mathbb{N}} \in [c_0^{\lambda}(u,p)]^{\beta}$ which yields that (25) is necessary and $(c_{nk})_{k\in\mathbb{N}} \in \ell_1$ for each $n \in \mathbb{N}$. Hence, Cy exists and thus letting $m \to \infty$ in the equality

$$\sum_{k=0}^{m} c_{nk} y_k = \sum_{k=0}^{m} \sum_{j=k}^{m} \frac{(\lambda_k - \lambda_{k-1})}{\lambda_k} u_k c_{nj} x_k, \quad (n, m \in \mathbb{N})$$

we have that $Cy = Ax$ which leads us to the consequence $C \in (c_0(p) : \mu)$.

Conversely, let $C \in (c_0(p) : \mu)$ and (25) holds, and take any $x \in c_0^{\lambda}(u,p)$. Then, we have $(a_{nk})_{k\in\mathbb{N}} \in [c_0^{\lambda}(u,p)]^{\beta}$ for each $n \in \mathbb{N}$. Hence, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m-1} c_{nk} y_k + \frac{\lambda_m}{(\lambda_m - \lambda_{m-1})u_m} a_{nm} y_m; \quad (n, m \in \mathbb{N})$$

as $m \to \infty$ that $Ax = Cy$ and this shows that $A \in (c_0^{\lambda}(u,p) : \mu)$. This completes the proof.

Theorem 5.2 *Let μ be any given sequence space. Then,*
(i) $A \in (c^{\lambda}(u,p) : \mu)$ if and only if $C \in (c(p) : \mu)$ and (25) holds.
(ii)$A \in (\ell_{\infty}^{\lambda}(u,p) : \mu)$ if and only if $C \in (\ell_{\infty}(p) : \mu)$ and (25) holds.

Proof. This may be obtained by proceedings as in Theorem 5.1, above. So, we omit the details.

Now, we may quote our corollaries on the characterization of some matrix classes concerning with the sequence spaces $c_0^{\lambda}(u,p), c^{\lambda}(u,p)$ and $\ell_{\infty}^{\lambda}(u,p)$. Before giving the corollaries, let us consider the following conditions:

$$\sup_{n\in\mathbb{N}} \left[\sum_k \left| \Delta\left(\frac{a_{nk}}{(\lambda_k - \lambda_{k-1})u_k} \right) \right| \lambda_k B^{1/p_k} \right]^{q_n} < \infty, \quad (\forall B \in \mathbb{N}), \tag{26}$$

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} \Delta\left[\frac{a_{nk}}{(\lambda_k - \lambda_{k-1})u_k} \right] \lambda_k B^{1/p_k} \right|^{q_n} < \infty, \quad (\forall B \in \mathbb{N}), \tag{27}$$

$$\sup_{n\in\mathbb{N}} \sum_k \left| \Delta\left[\frac{a_{nk}}{(\lambda_k - \lambda_{k-1})u_k} \right] \lambda_k \right| B^{1/p_k} < \infty, \quad (\forall B \in \mathbb{N}), \tag{28}$$

$$\exists(\alpha_k) \subset \mathbb{R} \ni \lim_{n\to\infty} \left[\sum_k \left| \Delta \left(\frac{a_{nk}}{(\lambda_k - \lambda_{k-1})u_k} \right) \lambda_k - \alpha_k \right| B^{1/p_k} \right]^{q_n} = 0, \quad (\forall B \in \mathbb{N}), \tag{29}$$

$$\sup_{n\in\mathbb{N}} \left[\sum_k \left| \Delta \left(\frac{a_{nk}}{(\lambda_k - \lambda_{k-1})u_k} \right) \lambda_k \right| B^{-1/p_k} \right]^{q_n} < \infty, \quad (\exists B \in \mathbb{N}), \tag{30}$$

$$\sup_{n\in\mathbb{N}} \left| \sum_k \Delta \left[\frac{a_{nk}}{(\lambda_k - \lambda_{k-1})u_k} \right] \lambda_k \right|^{q_n} < \infty, \tag{31}$$

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} \Delta \left[\frac{a_{nk}}{(\lambda_k - \lambda_{k-1})u_k} \right] \lambda_k B^{-1/p_k} \right|^{q_n} < \infty, \quad (\exists B \in \mathbb{N}), \tag{32}$$

$$\sum_n \left| \sum_k \Delta \left[\frac{a_{nk}}{(\lambda_k - \lambda_{k-1})u_k} \right] \lambda_k \right|^{q_n} < \infty, \tag{33}$$

$$\exists \alpha \in \mathbb{R} \ni \lim_{n\to\infty} \left| \sum_k \Delta \left[\frac{a_{nk}}{(\lambda_k - \lambda_{k-1})u_k} \right] \lambda_k - \alpha \right|^{q_n} = 0, \tag{34}$$

$$\exists(\alpha_k) \subset \mathbb{R} \ni \lim_{n\to\infty} \left| \Delta \left[\frac{a_{nk}}{(\lambda_k - \lambda_{k-1})u_k} \right] \lambda_k - \alpha_k \right|^{q_n} = 0, \quad (\forall k \in \mathbb{N}), \tag{35}$$

$$\exists(\alpha_k) \subset \mathbb{R} \ni \sup_{n\in\mathbb{N}} N^{1/q_n} \sum_k \left| \Delta \left[\frac{a_{nk}}{(\lambda_k - \lambda_{k-1})u_k} \right] \lambda_k - \alpha_k \right| B^{-1/p_k} < \infty, \quad (\forall N, \exists B \in \mathbb{N}). \tag{36}$$

Corollary 5.3 *(i) $A \in (\ell_\infty^\lambda(u,p) : \ell_\infty(q))$ if and only if (25) and (26) hold.*
(ii) $A \in (\ell_\infty^\lambda(u,p) : \ell(q))$ if and only if (25) and (27) hold.
(iii) $A \in (\ell_\infty^\lambda(u,p) : c(q))$ if and only if (25), (28) and (29) hold.
(iv)$A \in (\ell_\infty^\lambda(u,p) : c_0(q))$ if and only if (25) holds and (29) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.

Corollary 5.4 *(i) $A \in (c^\lambda(u,p) : \ell_\infty(q))$ if and only if (25),(30) and (31) hold.*
(ii) $A \in (c^\lambda(u,p) : \ell(q))$ if and only if (25),(32) and (33) hold.
(iii) $A \in (c^\lambda(u,p) : c(q))$ if and only if (25),(34),(35) and (36) hold, and (30) also holds with $q_n = 1$ for all $n \in \mathbb{N}$.
(iv) $A \in (c^\lambda(u,p) : c_0(q))$ if and only if (25) holds, and (34),(35) and (36) also hold with $\alpha = 0$, $\alpha_k = 0$ for all $k \in \mathbb{N}$, respectively.

Corollary 5.5 *(i) $A \in (c_0^\lambda(u,p) : \ell_\infty(q))$ if and only if (25) and (30) hold.*
(ii) $A \in (c_0^\lambda(u,p) : \ell(q))$ if and only if (25) and (32) hold.
(iii) $A \in (c_0^\lambda(u,p) : c(q))$ if and only if (25), (35) and (36) hold and (30) also holds with $q_n = 1$ for all $n \in \mathbb{N}$.

(iv) $A \in (c_0^\lambda(u,p) : c_0(q))$ *if and only if (25) holds and (29) also holds with* $\alpha_k = 0$ *for all* $k \in \mathbb{N}$.

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