

Gen. Math. Notes, Vol. 30, No. 1, September 2015, pp.1-11 ISSN 2219-7184; Copyright ©ICSRS Publication, 2015 www.i-csrs.org Available free online at http://www.geman.in

Some Stronger Forms of g^* Pre

Continuous Functions in Topology

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(Received: 10-6-15 / Accepted: 16-8-15)

Abstract

In this paper, we introduce and study some stronger forms of g^*p -continuous functions namely, strongly g^*p -continuous, perfectly g^*p -continuous and completely g^*p -continuous functions in topological spaces. Further we introduce the concepts of strongly g^*p -closed and strongly g^*p -open maps and obtain some of their properties.

Keywords: g^*p -closed sets, strongly g^*p -continuous functions, perfectly g^*p -continuous functions, completely g^*p -continuous functions, strongly g^*p -closed maps and strongly g^*p -open maps.

1 Introduction

In 1982, Mashhour et. al. [12] introduced preopen sets and pre-continuity in topology. Levine [8] introduced the class of generalized closed (g-closed) sets in topological spaces. The generalized continuity was studied in recent years by Balachandran et.al. Devi et.al, Maki et.al, [3, 5, 9]. Levine [6], Noiri [16]

and Arya and Gupta [2] introduced and investigated the concept of strongly continuous , perfectly continuous and completely continuous functions respectively which are stronger than continuous functions. Later, Sundaram [18] defined and studied strongly g-continuous functions and perfectly g-continuous functions in topological spaces. Generalized closed (g-closed) maps were introduced by Malghan[11]. In 2013, P.G.Patil et al.[17] studied the concept of stronger forms of $\omega \alpha$ - continuous functions in topological spaces. Veerakumar [19] introduced and studied the concept of g^*p -closed sets, g^*p -continuity, g^*p -irresolute functions, g^*p -closed maps, g^*p -open maps and T_p^* -spaces for general topology.

In this paper, we introduce and study some stronger forms of g^*p -continuous functions namely, strongly g^*p -continuous, perfectly g^*p -continuous and completely g^*p -continuous functions in topological spaces. Further, we introduce the concepts of strongly g^*p -closed and strongly g^*p -open maps and obtain some of their properties.

2 Preliminaries

Throughout this paper (X, τ) , (Y, σ) and (Z, η) represent non-empty topological spaces on which no separation axioms are assumed unless explicitly stated and they are simply written X, Y and Z respectively. For a subset A of a topological space (X, τ) , the closure of A, the interior of A with respect to τ are denoted by cl(A) and int(A) respectively. The complement of A is denoted by A^c . The α -closure (resp. pre-closure) of A is the smallest α -closed (resp. preclosed) set containing A and is denoted by $\alpha cl(A)$ (resp. pcl(A)).

Before entering into our work we recall the following definitions from various authors.

Definition 2.1 A subset A of a topological space (X, τ) is called a

- 1. pre-open set [12] if $A \subseteq int(cl(A))$ and pre-closed set if $cl(int(A)) \subseteq A$.
- 2. α -open set [15] if $A \subseteq int(cl(int(A)))$ and α -closed set if $cl(int(cl(A))) \subseteq A$.

Definition 2.2 A subset A of a topological space (X, τ) is called g-closed [8] (resp. g^*p -closed[19]) set if $cl(A) \subseteq G$ (resp. $pcl(A) \subseteq G$) whenever $A \subseteq G$ and G is open (resp.g -open) in (X, τ) .

Definition 2.3 A of a topological space (X, τ) is called a T_p^* -space [19] if every g^*p -closed set is closed.

Some Stronger Forms of g^* Pre...

Definition 2.4 A map $f : X \to Y$ is called pre-continuous [12] (resp. gcontinuous [3] and g^*p -continuous [19]) if $f^{-1}(V)$ is pre-closed (resp. g-closed and g^*p -closed)in X for every closed set V in Y.

Definition 2.5 A map $f : X \to Y$ is called a irresolute [4] (resp. gcirresolute [5] and g^*p -irresolute [19]) if $f^{-1}(V)$ is semi-closed (resp. g-closed and g^*p -closed) in X for every semi-closed (resp. g-closed and g^*p -closed) set V of Y.

Definition 2.6 A map $f: X \to Y$ is called a

- 1. strongly continuous [6] if $f^{-1}(V)$ is both open and closed in X for each subset V in Y.
- 2. perfectly continuous [16] if $f^{-1}(V)$ is both open and closed in X for each open set V in Y.
- 3. completely continuous [2] if $f^{-1}(V)$ is regular-open in X for each open set V in Y.
- 4. strongly g-continuous [18] if $f^{-1}(V)$ is g-open in X for each open set V in Y.
- 5. perfectly g-continuous [18] if $f^{-1}(V)$ is both open and closed in X for each g-open set V in Y.

Definition 2.7 A map $f : X \to Y$ is called a

- 1. M-preopen (resp. M-preclosed) [14] if f(V) is preopen (resp. preclosed) set in Y for every preopen (resp. preclosed) set V of X.
- 2. g^*p -open (resp. g^*p -closed) [19] if f(V) is g^*p -open (resp. g^*p -closed) in Y for each open (resp. closed) set V in Y.

3 Stronger Forms of Continuous Functions

In this section, we introduce strongly g^*p -continuous functions, perfectly g^*p -continuous functions and completely g^*p -continuous functions and study some of their properties.

Definition 3.1 A function $f : X \to Y$ is said to be strongly g^* -precontinuous (briefly strongly g^*p -continuous) if the inverse image of every g^*p -closed set in Y is closed in X.

Theorem 3.2 A function $f : X \to Y$ is strongly g^*p -continuous if and only if the inverse image of every g^*p -open set in Y is open in X.

Theorem 3.3 Every strongly g^*p -continuous function is continuous and thus pre-continuous and g^*p -continuous.

Proof: The proof follows from the definitions.

The converse of the above theorem need not be true as seen from the following examples.

Example 3.4 Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{a, c\}\}$ and $\sigma = \{X, \phi, \{a, b\}\}$. Define a function $f : X \to Y$ by f(a) = a, f(b) = c and f(c) = b. Then f is continuous but not strongly g^*p -continuous function, since for the g^*p -closed set $\{a, b\}$ in $Y, f^{-1}(\{a, b\}) = \{a, c\}$ is not closed in X.

Example 3.5 Let X, Y, τ and σ be as in Example 3.4, define a function $f: X \to Y$ by f(a) = b, f(b) = a and f(c) = c. Then f is pre-continuous and g^*p -continuous but not strongly g^*p -continuous function, since for the g^*p -closed set $\{c\}$ in Y, $f^{-1}(\{c\}) = \{c\}$ is not closed in X.

Theorem 3.6 Every strongly continuous function is strongly g^*p -continuous but not conversely.

Proof: The proof follows from the definitions.

Example 3.7 Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{X, \phi, \{a\}\}$. Then the identity function $f : X \to Y$ is strongly g^*p continuous but not strongly continuous function, since for the subset $\{b, c\}$ in $Y, f^{-1}(\{b, c\}) = \{b, c\}$ is closed but not open in X.

Theorem 3.8 Every strongly g^*p -continuous is g^*p -irresolute and thus every strongly continuous map is g^*p -irresolute.

Proof: Let $f : X \to Y$ be a strongly g^*p -continuous function and V be a g^*p -closed set in Y. Then $f^{-1}(V)$ is closed and hence g^*p -closed set in X from [19]. Hence f is g^*p -irresolute. By Theorem 3.6, f is g^*p -irresolute function.

The converse of the above theorem need not be true as seen from the following example.

Example 3.9 Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Define a function $f : X \to Y$ by f(a) = b, f(b) = a and f(c) = c. Then f is g^*p -irresolute but not strongly g^*p -continuous function, since for the g^*p -closed set $\{a, c\}$ in Y, $f^{-1}(\{a, c\}) = \{b, c\}$ is not closed in X. Some Stronger Forms of g^* Pre...

Theorem 3.10 Let (X, τ) be any topological space, (Y, σ) be a T_p^* -space and $f: X \to Y$ be a function. Then the following are equivalent: (1) f is strongly g^*p -continuous. (2) f is continuous.

Proof: $(1) \Rightarrow (2)$: Follows from the Theorem 3.3.

 $(2) \Rightarrow (1)$: Let U be any g^*p -open set in Y. Since (Y, σ) is T_p^* -space, U is open in (Y, σ) . Again since f is continuous, we have $f^{-1}(U)$ is open in X. Therefore f is strongly g^*p -continuous.

Definition 3.11 A topological space (X, τ) is called a g^*p -space if every subset in it is g^*p -closed. i.e. $g^*p(X, \tau) = P(X)$.

Example 3.12 Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Then the space (X, τ) is g^*p -space, because $g^*p(X, \tau) = P(X)$.

Theorem 3.13 Let X be discrete topological space and Y be a g^*p -space and $f: X \to Y$ be a function. Then the following statements are equivalent: (1) f is strongly continuous. (2) f is strongly g^*p -continuous.

Proof: (1) \Rightarrow (2): Follows from the Theorem 3.6.

 $(2) \Rightarrow (1)$: Let U be any g^*p -open set in Y. Since Y is g^*p -space, U is a g^*p -open subset of Y and by hypothesis, $f^{-1}(U)$ is open in X. But X is a discrete topological space and so $f^{-1}(U)$ is also closed in X. That is $f^{-1}(U)$ is both open and closed in X and hence f is strongly continuous.

Remark 3.14 Strongly g^*p -continuity and strongly g-continuity are independent of each other as shown in the following examples.

Example 3.15 Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. Define a function $f : X \to Y$ by f(a) = b, f(b) = a and f(c) = c. Then f is strongly g-continuous but not strongly g^*p -continuous since for the g^*p -closed set $\{b\}$ in Y, $f^{-1}(\{b\}) = \{a\}$ is not closed in X.

Example 3.16 Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Define a map $f : X \to Y$ by f(a) = b, f(b) = a and f(c) = c. Then f is strongly g^*p -continuous but not strongly g-continuous since for the g-closed set $\{a, b\}$ in Y, $f^{-1}(\{a, b\}) = \{a, b\}$ is not closed in X.

Theorem 3.17 The composition of two strongly g^*p -continuous functions is strongly g^*p -continuous function.

Proof: Let $f: X \to Y$ and $g: Y \to Z$ be two strongly g^*p -continuous maps. Let V be a g^*p -closed set in Z. Since g is strongly g^*p -continuous, $g^{-1}(V)$ is closed in Y. Then $g^{-1}(V)$ is g^*p -closed in Y. Again since f is strongly g^*p -continuous, $f^{-1}(g^{-1}(V))$ is closed in X. That is $gof^{-1}(V)$ is closed in X. Hence gof is strongly g^*p -continuous.

Theorem 3.18 Let $f : X \to Y$ and $g : Y \to Z$ be any two functions such that $gof : X \to Z$. Then

- 1. gof is strongly g^*p -continuous if g is strongly g^*p -continuous and f is continuous.
- 2. gof is g^*p -irresolute if g is strongly g^*p -continuous and f is g^*p -continuous (or f is g^*p -irresolute).
- 3. gof is continuous if g is continuous and f is strongly g^*p -continuous.

Proof: The proof follows from the definitions.

Definition 3.19 A function $f : X \to Y$ is called perfectly g^* -pre continuous (briefly perfectly g^*p -continuous) if the inverse image of every g^*p -closed set in Y is both open and closed in X.

Theorem 3.20 Every perfectly g^*p -continuous function is strongly g^*p -continuous but not conversely.

Example 3.21 In Example 3.7, the function f is strongly g^*p -continuous but not perfectly g^*p -continuous function, since for the g^*p - closed set $\{c\}$ in Y, $f^{-1}(\{c\}) = \{c\}$ is closed but not open in X.

Theorem 3.22 Every strongly continuous function is perfectly g^*p -continuous but not conversely.

Example 3.23 Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. Define a function $f : X \to Y$ by f(a) = b, f(b) = c and f(c) = a. Then f is perfectly g^*p -continuous but not strongly continuous, since for the subset $\{b\}$ in Y, $f^{-1}(\{b\}) = \{a\}$ is open but not closed in X.

Theorem 3.24 Let $f : X \to Y$ and $g : Y \to Z$ be any two functions such that $gof : X \to Z$. Then

1. gof is perfectly g^*p -continuous if f and g are perfectly g^*p -continuous functions.

Some Stronger Forms of g^* Pre...

2. gof is perfectly g^*p -continuous if f is perfectly g^*p -continuous and g is g^*p -irresolute.

Proof: Straight forward.

Definition 3.25 A function $f : X \to Y$ is called completely g^* -precontinuous (briefly completely g^*p -continuous) if the inverse image of every g^*p -closed set in Y is regular-closed in X.

Theorem 3.26 A function $f : X \to Y$ is completely g^*p -continuous if and only if the inverse image of every g^*p -open set in Y is regular-open in X.

Theorem 3.27 If a function $f : X \to Y$ is completely g^*p -continuous then f is continuous.

The converse of the above theorem need not be true as seen from the following example.

Example 3.28 Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. Then the identity function $f : X \to Y$ is continuous but not completely g^*p -continuous.

Theorem 3.29 Every completely g^*p -continuous function is completely continuous but not conversely.

Example 3.30 Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Then the identity function $f : X \to Y$ is completely continuous but not completely g^*p -continuous since for the g^*p -open set $\{a, b\}$ in Y, $f^{-1}(\{a, b\}) = \{a, b\}$ is not regular-open in X.

Theorem 3.31 Every completely g^*p -continuous function is strongly g^*p -continuous but not conversely.

Example 3.32 In Example 3.7, the map f is strongly g^*p -continuous but not completely g^*p -continuous since for the g^*p -closed set $\{c\}$ in Y, $f^{-1}(\{c\}) = \{c\}$ is not regular-closed in X.

Theorem 3.33 If a map $f : X \to Y$ is completely continuous and Y is T_n^* -space, then f is completely g^*p -continuous.

Theorem 3.34 Let $f : X \to Y$ and $g : Y \to Z$ be any two functions such that $gof : X \to Z$. Then

1. gof is completely g^*p -continuous if f is completely continuous and g is completely g^*p -continuous.

- 2. gof is completely g^*p -continuous if f is completely g^*p -continuous and g is g^*p -irresolute.
- 3. gof is completely g*p-continuous if f is completely g*p-continuous and g is g*p-continuous.

Proof: Follows from the definitions.

4 Strongly g^*p -Closed Maps and Strongly g^*p -Open Maps

In this section, we introduce strongly g^*p -closed and strongly g^*p -open maps in topological spaces and investigate some of their properties.

Definition 4.1 A map $f : X \to Y$ is called strongly g^*p -closed (resp. strongly g^*p -open) map if the image of every g^*p -closed (resp. g^*p -open) set in X is g^*p -closed (resp. g^*p -open) set in Y.

Theorem 4.2 If a map $f : X \to Y$ is strongly g^*p -closed then it is g^*p -closed.

The converse of the above theorem need not be true as seen from the following example.

Example 4.3 Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then the identity map $f : X \to Y$ is g^*p -closed but not strongly g^*p -closed, since for the g^*p -closed set $\{b\}$ in X, $f(\{b\}) = \{b\}$ is not g^*p -closed in Y.

Theorem 4.4 The composition of two strongly g^*p -closed maps is again a strongly g^*p -closed map.

Remark 4.5 Strongly g^*p -closed maps and g^*p -irresolute maps are independent of each other as seen from the following examples.

Example 4.6 Let $X = Y = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, c\}\}$. Define a map $f : X \to Y$ by f(a) = a, f(b) = c and f(c) = b. Then f is strongly g^*p -closed but not g^*p -irresolute since for the g^*p -closed set $\{c\}$ in Y, $f(\{c\}) = \{b\}$ is not g^*p -closed in X.

Example 4.7 In Example 4.3, the map f is g^*p -irresolute but not strongly g^*p -closed map.

Some Stronger Forms of g^* Pre...

Theorem 4.8 A map $f : X \to Y$ is strongly g^*p -closed if and only if for each subset B of Y and for each g^*p -open set U of X containing $f^{-1}(B)$, there exists a g^*p -open set V of Y such that $B \subset V$ and $f^{-1}(V) \subset U$.

Proof: Necessity: Suppose that f is a strongly g^*p -closed map. Let B be any subset of Y and U be a g^*p -open set of X containing $f^{-1}(B)$. Put V = Y - f(X-U). Then V is a g^*p -open set in Y containing B such that $f^{-1}(V) \subset U$.

Sufficiency: Let F be any g^*p -closed subset of X. Then $f^{-1}(Y - f(F)) \subset X - F$. Put B = Y - f(F). We have $f^{-1}(B) \subset X - F$. Also X - F is g^*p -open in X and $f^{-1}(V) \subset X - F$. Therefore we have f(F) = Y - V and hence f(F) is g^*p -closed in Y. Therefore f is strongly g^*p -closed map.

Theorem 4.9 If $f : X \to Y$ is g-irresolute and M-preclosed, then f is strongly g^*p -closed map.

Proof: Let A be a g^*p -closed set in X. Let V be any g-open set in Y containing f(A). Then $A \subset f^{-1}(V)$. Since f is g-irresolute, $f^{-1}(V)$ is g-open set in X. Again since A is g^*p -closed in X, $pcl(A) \subset f^{-1}(V)$ and hence $f(A) \subset f(pcl(A)) \subset V$. As f is M-preclosed and pcl(A) is preclosed in X, f(pcl(A)) is preclosed in Y and hence $pcl(f(A)) \subset pcl(f(pcl(A))) \subset V$. This shows that f(A) is g^*p -closed set in Y. Hence f is strongly g^*p -closed.

Theorem 4.10 Let $f : X \to Y$ and $g : Y \to Z$ be two maps such that $gof : X \to Z$. Then

(1) gof is g^*p -closed if f is closed and g is strongly g^*p -closed.

(2) gof is g^*p -closed if f is closed and g is g-irresolute and M-preclosed.

(3) g is g^*p -closed if f is continuous surjections and gof is strongly g^*p -closed.

Proof: (1) By Theorem 4.2, g is g^*p -closed map. Let K be a closed set in X. Since f is closed, then f(K) is closed in Y. Therefore g(f(K)) is g^*p -closed in Z as g is g^*p -closed map. That is gof(K) is g^*p -closed set in Z. Hence gofis g^*p -closed map.

(2). By Theorem 4.9, g is strongly g^*p -closed map. Hence by (1), gof is g^*p -closed.

(3). Let F be a closed set of Y. Since f is continuous, $f^{-1}(F)$ is closed in X and hence $f^{-1}(F)$ is g^*p -closed in X. Since gof is strongly g^*p -closed, $(gof)(f^{-1}(F))$ is g^*p -closed in Z. Again since f is surjective, g(F) is g^*p -closed in Z. Hence g is g^*p -closed.

Acknowledgements: The authors are greatful to the University Grants Commission, New Delhi,India for its financel support under UGC-SAP-DRS to the Department of Mathematics, Karnatak University, Dharwad, India.

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