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Relatively Prime Uniform Partitions

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Abstract

A partition of a positive integer n is a sequence of non increasing positive integers say $\lambda = (\lambda_1, \cdots, \lambda_1 (f_1 \text{ times}), \cdots, \lambda_k, \cdots, \lambda_k\ (f_k \text{ times}))$, with $\lambda_i > \lambda_{i+1}$, whose sum equals n. If $\gcd(\lambda_1, \cdots, \lambda_k) = 1$, we say that λ is a relatively prime partition. If $f_i = 1\ \forall i = 1, 2, \cdots, k$, we say that λ is a distinct partition. If $f_i = f_j\ \forall i \neq j$, then we say that λ is an uniform partition. In this note, the class of partitions which are both uniform and relatively prime are analyzed. They are found to have connections to distinct partitions. Furthermore, we term a partition of a certain kind as Conjugate Closed (abbreviated CC) if its conjugate is also of the same kind. Enumeration of CC-uniform partitions, CC-relatively prime partitions, and CC-uniform relatively prime partitions are studied; these enumeration formulas involves functions from multiplicative number theory such as divisors counting function and Liouville's function.

Keywords: *Relatively prime partitions, uniform partitions, conjugate, conjugate closed partitions.*

1 Introduction and Definitions

The study of partition identities was initiated by the mathematician L. Euler [2]. Much of the work done in partition theory since its beginnings has centered on the problem of finding asymptotic estimate, congruence identities, reccurence relations, parity results and equinumerous results for the counting function of a defined class of partitions. In this sequence, we introduce a new kind of partitions namely uniform partitions, which are partitions with identical number of occurences of each part, and we establish its connection to earlier defined partitions. The enumeration of uniform partitions in finite sets was studied by Karen Meagher and Lucia Moura [4]. Present definition of uniform integer partition is quite an analogue to the definition of uniform set partition.

The motivation of the present work stems from the work of Mohamed El Bachraoui. In [6], Mohamed El Bachraoui discuss a kind of partitions called relatively prime partitions, which are partitions whose parts form a relatively prime set. The idea of relatively prime set was found by M. B. Nathanson [7]. We intertwine the definition of relatively prime partition with uniform partition, and find new results. We study the enumeration of relatively prime partitions that are uniform and we find that: number of relatively prime partitions that are uniform is equal to the number of distinct partitions. Moreover, we term a partition of a certain kind as *conjugate closed* if its conjugate is also of the same kind. In the latter part of this article, we are concerned with the enumeration of conjugate closed uniform partitions, conjugate closed uniform partitions that are relatively prime, conjugate closed uniform partitions that are self conjugate, etc.,

Now, we state some basic definitions that are required for our study.

Let n and k be integers with $k \leq n$. By a partition of n, we mean a sequence of non increasing positive integers say $\lambda = (\lambda_1, \cdots, \lambda_1 (f_1\ times), \cdots, \lambda_k, \cdots, \lambda_k\ (f_k\ times))$ whose sum equals n, where λ_is are distinct integers. We sometimes write $\lambda = (\lambda_1^{f_1} \lambda_2^{f_2} \cdots \lambda_k^{f_k})$. Each λ_i is called part of the partition λ. If $\lambda = \left(\lambda_1^{f_1} \cdots \lambda_k^{f_k}\right)$, then we say that λ has precisely k diferent sizes namely $\lambda_1, \cdots, \lambda_k$. The value of f_i is termed as frequency of the part λ_i. We define $p(n)$ to be the number of partitions of n.

Definition 1. *Let $\lambda = ({\lambda_1}^{f_1} {\lambda_2}^{f_2} \cdots {\lambda_k}^{f_k})$ be a partition of n. We say that λ is uniform if all f_is are identical. We define $u(n)$ to be the number of uniform partitions of n.*

Example 2. Among the partitions of 4, the partitions (1^4), (2^2), $(1^1 3^1)$, and (4^1) are uniform partitions, whereas the partition $(1^2 2^1)$ is not an uniform partition of 4.

Definition 3. *Let λ be a partition of a positive integer n. We say that λ is relatively prime if its parts form a relatively prime set. We define $p_\psi(n)$ to be the number of relatively prime partitions of n.*

Example 4. Among the partitions of 4, the partitions (1^4), $(1^1 3^1)$, and $(1^2 2^1)$ are relatively prime partitions, whereas the partitions(4^1), and (2^2) are not relatively prime partitions of 4.

Definition 5. *A partition say λ of n is said to be relatively prime uniform partition if it is both uniform and relatively prime. We define $u_\psi(n)$ to be the number of relatively prime uniform partitions of n.*

Example 6. Among the partitions of 4, the partitions (1^4), and $(1^1 3^1)$ are relatively prime uniform partitions, whereas the partition $(1^2 2^1)$ is relatively prime but not uniform, and the partitions (2^2) and (4^1) are uniform but not relatively prime partitions.

Definition 7. *Let $\lambda = \left(\lambda_1^{f_1} \cdots \lambda_k^{f_k}\right)$ be a partition of n. If $\gcd(f_1, \cdots, f_k) = 1$, then we say that λ has relatively prime frequencies. We define $f_\psi(n)$ to be the number of partitions of n with relatively prime frequencies.*

Example 8. Among the partitions of 4, the partitions $(1^1 3^1)$, $(1^2 2^1)$, and (4^1) have relatively prime frequencies, whereas the partitions (1^4), and (2^2) are partitions with non relatively prime frequencies.

Given a partition $\lambda = \left(\lambda_1^{f_1} \cdots \lambda_k^{f_k}\right)$ of n. There is a simple way of representing λ geometrically by using display of n lattice points called a *graph*. The construction of the graph of λ goes as follows: first f_1 rows should have λ_1 points; next f_2 rows should have λ_2 points, and so on.

For example, the partition of 15 given by $(6^1 3^2 2^1 1^1)$ can be represented by using 15 lattice points arranged in five rows as follows:

· · · · · · 6
· · · 3
· · · 3
· · 2
· 1

If we read this graph vertically, we get

· · · · · 5
· · · · 4
· · · 3
· 1
· 1
· 1

The corresponding partition to this graph is $\lambda^* = (5^1 4^1 3^1 1^3)$. The partition λ^* is said to be conjugate of λ.

Definition 9. *Let A be a set of partitions of n. We say that $\lambda \in A$ is conjugate closed in A (abbreviated CC) if $\lambda^* \in A$. We say that λ is self conjugate if, $\lambda = \lambda^*$.*

Based on the above mentioned definitions, we derive several convolution identities, parity results, and equinumerous results. This is the core of section 2. In section 3, we confine to the enumeration of CC-partitions of the above mentioned kinds of partitions.

Throughout this paper, we denote the number of distinct partitions of n (partitions with distinct parts) by $q(n)$; number of distinct partitions of n with k parts by $q(n,k)$; number of distinct relatively prime partitions of n by $q_\psi(n)$; number of relatively prime distinct partitions of n with exactly k parts by $q_\psi(n,k)$, and the number of positive divisors of n by $\tau(n)$.

2 Main Results

2.1 Convolution Identities

In this section, we derive several convolution identites relating the functions $u(n)$; $q(n)$; $u_\psi(n)$, and $q_\psi(n)$.

Theorem 10. *Let n be a positive integer. Then we have:*

(i)

$$u(n) = \sum_{d|n} q\left(\frac{n}{d}\right) \tag{1}$$

(ii)

$$q(n) = \sum_{d|n} \mu(d) u\left(\frac{n}{d}\right) \tag{2}$$

(iii)

$$u_\psi(n) = \sum_{d|n} q_\psi\left(\frac{n}{d}\right) \tag{3}$$

(iv)

$$q_\psi(n) = \sum_{d|n} \mu(d) u_\psi\left(\frac{n}{d}\right) \tag{4}$$

(v)

$$u(n) = \sum_{d|n} \tau(d) q_\psi\left(\frac{n}{d}\right) \tag{5}$$

where μ is the möbius function.

Proof. Let U_n be the set of all uniform partitions of n. We define $F(\lambda) = f$ when $\lambda = \left(\lambda_1^f \lambda_2^f \cdots \lambda_k^f\right)$. In U_n, define a relation say R in the following way: $\lambda_1 R \lambda_2$ if, and only if, $F(\lambda_1) = F(\lambda_2)$. Clearly, R is an equivalence relation. Further, we observe that, each equivalence class of R equals the set $\{\lambda \in U_n : F(\lambda) = f\}$ for some $f|n$; we denote this set by F_f. We notice that, the mapping

$$\left(\lambda_1^f \lambda_2^f \cdots \lambda_k^f\right) \to (\lambda_1 \lambda_2 \cdots \lambda_k)$$

from the set F_f to the set of all distinct partitions of $\frac{n}{f}$ is a bijection. Consequently, we have $|F_f| = q\left(\frac{n}{f}\right)$. Furthermore, $u(n) = |U_n| = \sum_{f|n} |F_f|$; hence, (i) follows.

As an application of möbius inversion formula, one can get (ii) from (i), and (iv) from (iii). Similar proof goes well to (iii). To prove (v), following identity is required:

$$q(n) = \sum_{d|n} q_\psi\left(\frac{n}{d}\right) \tag{6}$$

Now, we establish the identity 6; to that end, we denote the set of all distinct partitions of n by Q_n, and let $\lambda = (\lambda_1, \cdots, \lambda_k) \in Q_n$. Define $\gcd(\lambda) = \gcd(\lambda_1, \cdots, \lambda_k)$. Further, we define a relation say R_p in Q_n in the following way: $\lambda_1 R_p \lambda_2$ if, and only if, $\gcd(\lambda_1) = \gcd(\lambda_2)$. Clearly, R_p is an equivalence relation, and each equivalence class of R_p equals the set $\{\lambda \in Q_n : \gcd(\lambda) = d\}$ for some $d|n$; we denote this set by R_{pd}. Then, from the equation: $\sum d\lambda_i = d\sum \lambda_i = n$, where λ_i s are distinct and $\gcd(\lambda_1, \cdots) = 1$, it follows that $|R_{pd}| = q_\psi\left(\frac{n}{d}\right)$. Since $\sum_{d|n} |R_{pd}| = |Q_n| = q(n)$; (6) follows. As an application of the following well known identity:

$$\sum_{d|n} \sum_{k|d} f(k) = \sum_{d|n} \tau(d) f\left(\frac{n}{d}\right) \tag{7}$$

one can get (v) from (6) and (i). □

P. A. MacMahon [5] studied the partition function $t(n, k, s)$ which counts the number of partitions of n into k parts of precisely s different sizes. Recently, Nesrine Benyahia Tani and Sadek Bouroubi [8] bring out an exciting expression for $t(n, k, 2)$. Here, similar enumeration theorems were proved. Following result is concerned with uniform partitions of n into parts of precisely s different sizes. Total number of such partitions is denoted by $u(n, s)$.

Theorem 11. *Let n and s be two positive integers with $n \geq \frac{s(s+1)}{2}$. Then we have:*

(i)

$$u(n,s)=\sum_{d|n} q\left(\frac{n}{d},s\right) \tag{8}$$

(ii)

$$q(n,s)=\sum_{d|n} \mu(d)u\left(\frac{n}{d},s\right) \tag{9}$$

(iii)

$$u(n,s)=\sum_{d|n} \tau(d)q_{\psi}\left(\frac{n}{d},s\right) \tag{10}$$

(iv)

$$u(n,2)=\sum_{d|n} \lfloor\frac{\frac{n}{d}-1}{2}\rfloor \tag{11}$$

Proof. Define the relation R as it is in the proof of theorem 10 on the set of all uniform partitions of n with precisely s different sizes. Then, it is clear that, each equivalence class of R contains precisely $q\left(\frac{n}{k},s\right)$ elements for some $k|n$, provided $\frac{n}{k}\geq\frac{s(s+1)}{2}$. Thus, (i) follows. As an application of möbius inversion formula, one can get (ii) from (i). Since $\sum_{d|n}\sum_{k|d}f(k)=\sum_{d|n}\tau(d)f\left(\frac{n}{d}\right)$ for any arithmetical function f, and $q(n,s)=\sum_{d|n}q_{\psi}\left(\frac{n}{d},s\right)$, one can get (iii) from (i). Since $q(n,2)=\lfloor\frac{n-1}{2}\rfloor$, one can get (iv) from (i). □

An additional constraint on uniform partitions with exactly k parts yields new results.

Definition 12. Let n,k and d be three positive integers such that $d\leq\frac{n}{k}$. We define the function, $u(n,k,\geq d)$, to be the number of uniform partitions of n with exactly k parts and least part greater than or equal to d.

Theorem 13. *Let n be a positive integer. Then we have:*

(i) $u(n,k,\geq 1)=\sum_{f|n,f|k} q\left(\frac{n}{f},\frac{k}{f}\right)$

(ii) $u(n,k,\geq d)=u(n-(d-1)k,k,\geq 1)$ *for* $d>1$

(iii) $u(n,k,\geq d)=\sum_{f|n-(d-1)k,f|k} q\left(\frac{n-(d-1)k}{f},\frac{k}{f}\right)$

Proof. Let $U_{n,k,\geq 1}$ be the set of all uniform partitions of n with exactly k parts, and let $\lambda=\left(\lambda_1^f\lambda_2^f\cdots\lambda_r^f\right)\in U_{n,k,\geq 1}$. Then evidently, $r\cdot f=k$ therefore, $f|k$. Since $\sum f\lambda_i=n$, we have: $f|n$. Define relation R as it is in the proof of theorem 10 on the set $U_{n,k,\geq 1}$. Then evidently, R is an equivalence relation, and each equivalence class of R contains $q\left(\frac{n}{f}, \frac{k}{f}\right)$ elements where f is some divisor of both n and k. This enumeration can be had from the equation $f \cdot \left(\lambda_1 + \lambda_2 + \cdots + \lambda_{\frac{k}{f}}\right) = n$; λ_is being distinct positive integers. Thus, (i) follows.

Let $U_{n,k,\geq d}$ be the set of all uniform partitions of n with exactly k parts and least part greater than or equal to d, where $d > 1$ is a positive integer. If $(\lambda_1, \lambda_2, \cdots, \lambda_k) \in U_{n,k,\geq d}$, then $(\lambda_1 - 1, \cdots, \lambda_k - 1) \in U_{n-k,k,\geq d-1}$. Furthermore , if $(\lambda_1, \cdots, \lambda_k) \in U_{n-k,k,\geq d-1}$, then $(\lambda_1 + 1, \cdots, \lambda_k + 1) \in U_{n,k,\geq d}$.Thus, the mapping $f : U_{n,k,\geq d} \to U_{n-k,k,\geq d-1}$ defined by

$$f((\lambda_1, \cdots, \lambda_k)) = (\lambda_1 - 1, \cdots, \lambda_k - 1)$$

has inverse function. Consequently, $u(n, k, \geq d) = |U_{n,k,\geq d}| = |U_{n-k,k,\geq d-1}| = u(n-k, k, \geq d-1)$. Repeated application of this equality for $d-1$ times gives (ii), and an application of (i) in (ii) gives (iii). $\square$

Corollary 14. *Let n and k be two coprime integers with $k \leq n$. Then $u(n, k, \geq 1) = q(n, k)$ and $u(n, k, \geq d) = q(n - (d-1)k, k)$*

2.2 Parity Results

Theorem 15. *Let n be a positive integer. Then we have:*

$$u(n) \equiv \begin{cases} 1(\mod 2) & if \ \sum_{d|n, d=\frac{3k^2 \pm k}{2}} I(d) \ is \ odd \\ 0(\mod 2) & if \ \sum_{d|n, d=\frac{3k^2 \pm k}{2}} I(d) \ is \ even \end{cases} \tag{12}$$

where unit function I is defined by $I(n) = 1$ for every positive integer n.

Proof. Partition theoretic version of Euler's pentagonal number theorem [2] (for proof see [3]) states that

$$d_e(n) - d_o(n) = \begin{cases} (-1)^k \ \ if \ n = \frac{3k^2 \pm k}{2}, \\ 0 \ otherwise \end{cases} \tag{13}$$

where $d_e(n)$ $(resp.\ d_o(n))$ denotes the number of distinct partitions of n with even (resp. odd) number of parts.

Consequently, we have

$$q(n) \equiv \begin{cases} 1(\mod 2) \ \ if \ n = \frac{3k^2 \pm k}{2}, \\ 0(\mod 2) \ otherwise \end{cases} \tag{14}$$

Since $u(n) = \sum_{d|n} q(d)$; the result follows. $\square$

Theorem 16. *Let p be a prime number. Then we have*

$$q_\psi(p^m) \equiv 0(\mod 2)\ \forall m \geq 3 \tag{15}$$

Proof. From Euler's pentagonal number theorem it follows that

$$q(n) \equiv \begin{cases} 1(\mod 2) & if\ n = \frac{3k^2 \pm k}{2}, \\ 0(\mod 2) & otherwise \end{cases} \tag{16}$$

We claim that, $q(p^m) \equiv 0(\mod 2)$ for all positive integer $m \geq 2$. For otherwise, we have: $2p^m = k(3k \pm 1)$ for some positive integer k.

When $p = 2$, we have $k = 2^\alpha$ for some positive integer α with $\alpha \leq m$; this gives $3k \pm 1 = 3.2^\alpha \pm 1$. Consequently, $3k \pm 1$ does not divide 2^{m+1}; a contradiction.

If p is an odd prime number, then $k = 1$ is not an actual possibility. Therefore, 2 divides either k or $3k \pm 1$. Also, since $m \geq 2$, the case $k = 2$ will be ruled out. Therefore, if 2 divides k, then we have $k = 2 \cdot p^\alpha$ for some positive integer α with $\alpha < m$, which in turn implies that, $3k \pm 1 = 3 \cdot 2 \cdot p^\alpha \pm 1$. Consequently, $3k \pm 1$ does not divide p^β for any positive integer $\beta > 1$; a contradiction. In case if, 2 divides $3k \pm 1$, then we have $3k \pm 1 = 2p^\alpha$ for some positive integer α with $\alpha < m$, and $k = p^\beta$ for some positive integer $\beta > 1$; this is absurd. Thus, claim follows. Since $q_\psi(p^m) = q(p^m) - q(p^{m-1})$; the result follows. □

2.3 Equinumerous Theorems

In [2], L. Euler proved that: number of partitions of n with distinct parts is equal to the number of partitions of n with odd parts. In this section, we give two such equinumerous results; these results are the direct consequence of previously proved convolution identities.

Theorem 17. *Let n be a positive integer. Then we have:*

(i)

$$p_\psi(n) = f_\psi(n) \tag{17}$$

(ii)

$$q(n) = u_\psi(n) \tag{18}$$

Proof. In [6], Mohamed El Bachraoui proved that:

$$p(n) = \sum_{d|n} p_\psi\left(\frac{n}{d}\right) \tag{19}$$

Then from möbius inversion formula it follows that:

$$p_\psi(n) = \sum_{d|n} \mu(d) p\left(\frac{n}{d}\right) \tag{20}$$

We define $f_{d\psi}(n)$ to be the number of partitions of n with greatest common divisor of its frequencies equals d. Then, we have $f_{d\psi}(n) = f_\psi\left(\frac{n}{d}\right)$. The validity of this equality follows from the equation: $\sum(df_i)\lambda_i = d\left(\sum f_i\lambda_i\right) = n$ where $\gcd(f_1, f_2, \cdots) = 1$. Moreover, we have: $\sum_{d|n} f_{d\psi}(n) = p(n)$. Thus, $p(n) = \sum_{d|n} f_\psi\left(\frac{n}{d}\right)$. By möbius inversion formula, we get that

$$f_\psi(n) = \sum_{d|n} \mu(d) p\left(\frac{n}{d}\right) \tag{21}$$

Then from (20) and (21), we get (i).

To prove (18), we need the following identity

$$u(n) = \sum_{d|n} u_\psi(d) \tag{22}$$

Proof of the above identity is similar to the proof of the identity (6). Application of möbius inversion formula in (22) gives

$$u_\psi(n) = \sum_{d|n} \mu(d) u\left(\frac{n}{d}\right) \tag{23}$$

Then comparison of (23) with (ii) of theorem 10 gives (ii). □

3 Enumeration of Conjugate Closed Partitions

In the following result, we give a beautiful expression for enumeration of CC uniform partitions of n. Further, we show that relatively prime uniform partition of a positive integer n is conjugate closed only when n is a triangular number.

Theorem 18. *Let A be the class of uniform partitions of a positive integer n. Then the number of CC-uniform partitions in A is equal to*

$$\sum_{T_k|n} \tau\left(\frac{n}{T_k}\right)$$

where T_k denote the k^{th} triangular number.

Proof. We consider a uniform partition λ of n. If frequency of each part of λ is k, that is, $\lambda = \left(a_1^k a_2^k \cdots a_r^k\right)$, then the graph of λ got the following form: first k rows has a_1 points; next k rows has a_2 points; $\cdots$, and last k rows has a_r points. And conjugate of λ got the following form: first a_r rows has rk points; next $a_{r-1}-a_r$ rows has $(r-1)k$ points; next $a_{r-2}-a_{r-1}$ rows has $(r-2)k$ points, and so on, finally, $a_1 - a_2$ rows has k points. If conjugate of λ is uniform, then we have: $a_1 - a_2 = a_2 - a_3 = a_3 - a_4 = \cdots = a_{r-1} - a_r = a_r = s$ (say). This implies: $a_1 = rs,\ a_2 = (r-1)s, \cdots, a_{r-1} = 2s,\ a_r = s$. Thus, $n = k \cdot s \cdot T_r$, where T_r is the r^{th} triangular number.

Conversely, if n is of the form $n = k \cdot s \cdot T_r$, then we can find a uniform partition of n namely $\lambda = \left((sr)^k (s(r-1))^k \cdots s^k\right)$. Clearly λ is conjugate closed. Thus enumeration of uniform partitions of n that are conjugate closed equals the enumeration of (k, s, r) with $n = k{\cdot}s{\cdot}T_r$. We notice that enumeration of the triplets (k, s, r) with $n = k \cdot s \cdot T_r$ equals $\sum_{T_r|n} \tau\left(\frac{n}{T_r}\right)$. Thus, the result follows. □

Corollary 19. *Let n be a positive integer. If no triangular number divides n other than 1, then number of uniform partitions of n which are conjugate closed is equal to $\tau(n)$*

Theorem 20. *If n is a positive integer, then relatively prime uniform partition of n will be conjugate closed only when n is a triangular number. In particular, if n is a triangular number, then there exist only one relatively prime uniform partition which is conjugate closed, and if n is not a triangular number, then there will be no conjugate closed relatively prime uniform partition of n.*

Proof. Let $\lambda = \left(a_1^k a_2^k \cdots a_r^k\right)$ be a conjugate closed relatively prime uniform partition. Adopt the notation as it is in the first paragraph of the proof of theorem 18. Since λ is relatively prime; we get $s = 1$. And, since λ is conjugate closed; we get $k = 1$. Thus, the result follows. □

To present the next theorem we need Liouville's function $\lambda(n)$ which is defined as follow:

$$\lambda(n) = (-1)^{g(n)}$$

where $g(n)$ is the number of prime factors of n counted with multiplicity.

Theorem 21. *Let A be the set of all uniform partitions of n. Then the number of CC- partitions in A that are self conjugate is equal to*

$$\sum_{T_r|n} \sum_{d|\frac{n}{T_r}} \tau(d)\lambda\left(\frac{n}{T_r \cdot d}\right)$$

where T_r denote the r^{th} triangular number.

Proof. Let $\lambda = \left(a_1^k a_2^k \cdots a_r^k\right)$ be a uniform partition which is self conjugate. Adopt the notations as it is in the first paragraph of the proof of theorem 18. Since λ is self conjugate; we get $s = k$. Thus, enumeration of self conjugate uniform partitions of n equals the enumeration of (k, r) with $n = k^2 \cdot T_r$. We notice that, enumeration of (k, r) with $n = k^2 \cdot T_r$ equals $\sum_{T_r|n} \tau^*\left(\frac{n}{T_r}\right)$, where $\tau^*(m)$ counts the number of square divisors of m. Also, it is well known that, $\tau^*(m) = \sum_{d|m} \sum_{k|d} \lambda(k) = \sum_{d|m} \tau(d)\lambda\left(\frac{m}{d}\right)$. Thus, the result follows. □

Theorem 22. . *Let A be the set of all partitions of n with relatively prime frequencies. Then the number of CC-partitions in A is equal to*

$$\sum_{d|n} \mu(d) p_\psi\left(\frac{n}{d}\right)$$

Proof. Let $\lambda = \left(\lambda_1^{f_1} \lambda_2^{f_2} \cdots \lambda_r^{f_r}\right)$ be a partition of n with relatively prime frequencies. Then the graph of λ has f_1 rows with λ_1 points; f_2 rows with λ_2 points, and so on. Further, we observe that, conjugate of λ has λ_r rows with $(f_1 + f_2 + \cdots + f_r)$ points; $\lambda_{r-1} - \lambda_r$ rows with $(f_1 + f_2 + \cdots + f_{r-1})$ points, and so on, finally $\lambda_1 - \lambda_2$ rows have f_1 points. That is, frequencies of conjugate of λ are λ_r, $\lambda_{r-1} - \lambda_r$, $\cdots$, $\lambda_1 - \lambda_2$. It is not hard to see that, $\gcd(\lambda_1, \lambda_2, \cdots, \lambda_r) = 1$ if, and only if, $\gcd(\lambda_r, \lambda_{r-1} - \lambda_r, \cdots, \lambda_1 - \lambda_2) = 1$. Accordingly, we have the following equivalence: conjugate of λ will have relatively prime frequencies if, and only if λ is a relatively prime partition of n. Thus, enumeration of conjugate closed partitions of n with relatively prime frequencies equals enumeration of relatively prime partitions with relatively prime frequencies. Now, we count the number of relatively prime partitions with relatively prime frequencies; towards that end, we define $rf_d(n)$ to be the number of relatively prime partitions of n with greatest common divisor of its frequencies equals d. This gives $p_\psi(n) = \sum_{d|n} rf_d(n) = \sum_{d|n} rf_1\left(\frac{n}{d}\right)$. The validity of this equality can be seen from the following equation: $\sum(df_i)\lambda_i = d\left(\sum f_i\lambda_i\right) = n$, where $\gcd(f_1, f_2, \cdots) = 1$ Then from möbius inversion formula, it follows that, $rf_1(n) = \sum_{d|n} \mu(d) p_\psi\left(\frac{n}{d}\right)$. Since $rf_1(n)$ counts the relatively prime partitions of n with relatively prime frequencies; the result follows. □

Similar arguments gives the following result.

Theorem 23. *Let A be the set of all relatively prime partitions of n. Then the number of CC- partitions in A is equal to $\sum_{d|n} \mu(d) p_\psi\left(\frac{n}{d}\right)$*

4 Further Work

we note that, one may get new enumeration formula for CC partitions in various kinds of partitions other than what is mentioned in this note.

The other matters of question which has to be taken into account are:

(i) Can a bijection be defined between set of all distinct partitions of n and set of all relatively prime uniform partitions of n?

(ii) Does Andrew's partition ideal [1] or similar tool be wielded to answer the equinumerous results found in this article?

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