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Analytic Functions in Their Debye's Form

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Abstract

We show that any analytic function in complex variable admits a splitting in the same sense as Debye potentials in electromagnetic theory.

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For the electromagnetic field, the solution of Maxwell equations without sources can be written [2,5,11-13,15] in terms of scalar generators (Debye potentials), ψ_E and ψ_M, which satisfy the wave equation:

$$\Box\psi_E = \Box\psi_M = 0, \qquad \Box = \frac{\partial^2}{c^2\partial t^2} - \nabla^2 , \tag{1.a}$$

in according to:

$$\phi = -c\frac{\vec{r}}{r}\cdot\vec{\nabla}\left(r\psi_E\right), \qquad \vec{A} = -\vec{r}\times\vec{\nabla}\psi_M + \vec{r}\,\frac{\partial\psi_E}{c\partial t}, \tag{1.b}$$

up to gauge transformations, where:

$$\vec{r} = x\hat{i} + y\hat{j}, \qquad r = \sqrt{x^2 + y^2}, \qquad \vec{\nabla} = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y}. \tag{1.c}$$

The existence of ψ_E and ψ_M follows from results of several authors [6,8-10,14].

If f is an analytic function of the complex variable $z = x + iy$, then it has the form $f(z) = u(x, y) + iv(x, y)$, and besides are valid the Cauchy-Riemann relations [1]:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{2.a}$$

that is [7]:

$$\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)(u + iv) = 0 . \tag{2.b}$$

We note that (2.b) is very interesting because it permits to generalize easily the conditions (2.a) to four dimensions [3,7] via quaternions. The functions u and v are harmonic because they verify the Laplace equation:

$$\nabla^2 u = \nabla^2 v = 0, \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} . \tag{3}$$

The conditions (2.a,3) allow to deduce a splitting of $f(z)$ which has strong similarity with the Debye expressions (1.b) for the electromagnetic potentials. In fact, we introduce the notation:

$$\left[\vec{r} \times \vec{\nabla} g\right]_3 \equiv x\frac{\partial g}{\partial y} - y\frac{\partial g}{\partial x} , \tag{4.a}$$

then:

$$\frac{\vec{r}}{r} \cdot \vec{\nabla}(ru) = u + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}, \qquad \left[\vec{r} \times \vec{\nabla} v\right]_3 \equiv x\frac{\partial v}{\partial y} - y\frac{\partial v}{\partial x},$$

therefore:

$$\frac{\vec{r}}{r} \cdot \vec{\nabla}(ru) - \left[\vec{r} \times \vec{\nabla} v\right]_3 = u + x\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \overset{(2.a)}{=} u ; \tag{4.b}$$

besides:

$$\frac{\vec{r}}{r}\cdot\vec{\nabla}(rv)-\left[\vec{r}\times\vec{\nabla}u\right]_3=v+x\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\overset{(2.a)}{=}v. \qquad (4.c)$$

We see that (4.b,c) have a similar structure to (1.b), and they imply the following non-trivial splitting for the analytic function $f(z)=u+iv$:

$$f(z)=\frac{\vec{r}}{r}\cdot\vec{\nabla}\left(rf(z)\right)+i\left[\vec{r}\times\vec{\nabla}f(z)\right]_3, \qquad (5.a)$$

which means that $f(z)$ is a Debye potential for itself, and where each term is a harmonic function:

$$\nabla^2\left[\frac{\vec{r}}{r}\cdot\vec{\nabla}(rf)\right]=\nabla^2\left[\vec{r}\times\vec{\nabla}f\right]_3=0. \qquad (5.b)$$

The expression (5.a) is a reformulation [4] of the Cauchy-Riemann conditions (2.a), and it is a strong motivation for the existence of Debye generators in electromagnetic theory.

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