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ON THE EQUIVALENCE OF QUILLEN'S AND SWAN'S K-THEORIES

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Abstract. The K-theory of rings can be defined in terms of nonabelian derived functors as described in [9]; see also the books [7] and [8] of Inassaridze for a similar approach. In fact both Swan's theory and Quillen's theory can be described this way. The equivalence of both K-theories is proved by Gersten [5]. In this paper we give a proof using these descriptions that involve nonabelian derived functors.

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Introduction

Quillen's higher algebraic K-groups of a unital ring R are defined as the homotopy groups of the space that is obtained from the classifying space of the elementary group of the ring R:

$$K_n(R) = \pi_n(BE(R)^+) \qquad (\text{for } n \geqslant 2).$$

In Section 4 we replace this space by a simplicial set $\mathbb{Z}_\infty \bar{W}E(R)$ also depending functorially on R, having a geometric realization which is homotopy equivalent to $BE(R)^+$. This description depends on the notion of integral completion in the sense of Bousfield and Kan [3].

Swan's K-groups of a (nonunital) ring R are defined by means of a free simplicial resolution of R. In Section 8 we consider a simplicial group $H(R)$ depending functorially on R, having Swan's K-groups of R as homotopy groups. Writing $K'_n(R)$ for these groups the formula becomes

$$K'_n(R) = \pi_{n-2}(H(R)).$$

The main result is the theorem in Section 8, which says that Swan's K-groups coincide with Quillen's when extended to the category of nonunital rings in the standard way: $K'_n(R) \cong K_n(R^+, R)$. Here R^+ stands for the ring obtained from R by formally adjoining a unity element. The proof uses Gersten's result [5]: free associative nonunital rings have trivial K-theory which historically was the missing part of the earliest proof of the equivalence of these K-theories by Anderson [1].

It should be noted that the proof in Section 8 is a corrected and improved version of the proof in the unpublished paper [11].

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1. The Plus Construction

The so-called plus construction is used in Quillen's definition of the higher K-groups of a unital ring. Its defining properties are described in the theorem below; see also Property P1. Good references for the plus construction are Loday's thesis [13] and the book by Berrick [2].

Theorem and definition ([13], Théorème 1.1.1)**.** *Let X be a connected CW-complex with basepoint $*$, and let N be a perfect normal subgroup of π $(=\pi_1(X,*))$. Then there exists a connected CW-complex X^+ and a map $j\colon X \to X^+$ such that*

(i) $\pi_1(j)\colon \pi_1(X,*) \to \pi_1(X^+, j(*))$ *identifies with the canonical projection* $\pi \to \pi/N$.

(ii) *The map j induces an isomorphism on integral homology.*

In the sequel the following properties will be used.

Property P1 ([13], Proposition 1.1.2)**.** *The pair (X^+, j) is universal (in the homotopy category) among pairs (Y, f), where Y is a connected space and $f\colon X \to Y$ a continuous map satisfying $\pi_1(f)(N) = 0$.*

A consequence is that the plus construction is functorial up to homotopy ([13], Corollaire 1.1.3).

Property P2 ([13], Proposition 1.1.7)**.** *Let G be a group with a perfect commutator subgroup $[G,G]$. Then $B[G,G]^+$ is up to homotopy the universal covering of BG^+, where in both cases the plus construction is relative to the subgroup $[G,G]$.* (As usual BG is the classifying space of the group G.)

The plus construction is used in the definition of higher K-groups as follows. Let R be a unital ring. The general linear group $GL(R)$ of R has the elementary subgroup $E(R)$ as the commutator subgroup (the 'Whitehead Lemma'). Apply the plus construction to the classifying space $BGL(R)$ of $GL(R)$ relative to the subgroup $E(R)$ (which is perfect), and finally take homotopy groups.

Definition. $K_n(R) = \pi_n(BGL(R)^+)$ for $n \geqslant 1$.

From Property P2 one deduces

$$K_n(R) = \pi_n(BE(R)^+) \quad \text{for } n \geqslant 2$$

(and of course $\pi_1(BE(R)^+) = 0$). In this paper this identity is taken as definition of K_n for $n \geqslant 2$. An important property of the space $BE(R)^+$ is the following.

Property P3. *$BE(R)^+$ is an H-space.* ([13], §1.3.4.)

2. The Integral Completion of a Group

In Section 4 the space $BE(R)^+$ will be replaced by the integral completion (in the sense of Bousfield and Kan [3]) of $BE(R)$. The definition and properties of this completion are given in the next section. We will need the notion of integral completion of a group.

Let G be a group. The subgroups $\Gamma_i G$ (for $i \geqslant 1$) are defined inductively by

$$\Gamma_1 G = G,$$
$$\Gamma_{i+1} G = [\Gamma_i G, G].$$

As usual $[H_1, H_2]$ denotes the subgroup generated by the commutators $[h_1, h_2] = h_1 h_2 h_1^{-1} h_2^{-1}$ with $h_1 \in H_1$ and $h_2 \in H_2$. Clearly, the subgroups $\Gamma_i G$ are normal subgroups of G. The series of subgroups

$$G \trianglerighteq \Gamma_2 G \trianglerighteq \Gamma_3 G \cdots$$

is known as the *lower central series* of the group G. It induces a tower of groups

$$\cdots \to G/\Gamma_3 G \to G/\Gamma_2 G \to 1$$

in which every group homomorphism $G/\Gamma_{i+1} G \to G/\Gamma_i G$ is a central extension.

The *integral completion* CG ($= G^{\wedge}_{\mathbb{Z}}$ in the terminology of [3]) of G is defined as the inverse limit of the tower of groups:

$$CG = \varprojlim_i G/\Gamma_i G.$$

It was pointed out to the authors that this construction also appears in the literature under the name "pronilpotent completion".

In an obvious way C is a functor $\mathbf{Gr} \to \mathbf{Gr}$, where $\mathbf{Gr}$ denotes the category of groups. This functor can be viewed as the inverse limit of the functor which assigns to every homomorphism $G \to N$, with N a nilpotent group, the group N, and to every commutative triangle

$$\begin{tikzcd} & N \arrow[dd] \\ G \arrow[ur] \arrow[dr] & \\ & N' \end{tikzcd}$$

with N and N' nilpotent, the map $N \to N'$. The existence of this inverse limit follows from the existence of small cofinal diagrams, e.g. given by the tower of groups above.

3. The Integral Completion of a Space

We will use here the simplicial terminology. For a category $\mathbf{C}$ the category of simplicial $\mathbf{C}$-objects is denoted by s$\mathbf{C}$. The category of reduced simplicial sets is denoted by rs$\mathbf{Set}$. It is the full subcategory of s$\mathbf{Set}$, the category of simplicial sets, consisting of those $X \in$ s$\mathbf{Set}$ which have only one vertex, i.e. X_0 is a one-element set.

The functor $G\colon$ rs$\mathbf{Set} \to$ s$\mathbf{Gr}$ assigns to a reduced simplicial set its loop group GX, which is a simplicial group satisfying $\pi_i(GX) \cong \pi_{i+1}(X)$ for all $i \geqslant 0$. This functor G has a right adjoint $\bar{W}\colon$ s$\mathbf{Gr} \to$ rs$\mathbf{Set}$, which is the simplicial analogue of the classifying space functor.

The reduced simplicial set $\bar{W}H$ is called the *classifying complex* of the simplicial group H. The adjunction of G and $\bar{W}$ induces a natural simplicial map

$X \to \bar{W}GX$, which induces isomorphisms on the homotopy groups when X is a Kan complex. A good reference for this is May [14].

A functor $T\colon \mathbf{Gr} \to \mathbf{Gr}$ determines a functor $\tilde{T}\colon \mathrm{rs}\mathbf{Set} \to \mathrm{rs}\mathbf{Set}$ in the following way: let X be a reduced simplicial set; first form its loop group GX, next apply T dimension-wise to obtain a simplicial group TGX, and finally take the classifying complex. In a formula: $\tilde{T} = \bar{W}TG$, where T stands for T applied dimension-wise. In particular the integral completion $C\colon \mathbf{Gr} \to \mathbf{Gr}$ as defined in Section 2 determines a functor

$$\mathbb{Z}_\infty = \tilde{C}\colon \mathrm{rs}\mathbf{Set} \to \mathrm{rs}\mathbf{Set}.$$

This functor assigns to a reduced simplicial set its so-called *integral completion*. This integral completion functor was introduced by Bousfield and Kan [3]. The definition given here is in fact one of various possible definitions: it is the definition they give in Chapter III of [3]. Some of the main properties of the integral completion functor are:

Property I1 ([3], Ch. I, Lemma 5.5, p. 25)**.** *Let $f\colon X \to Y$ be a map in* rs**Set**. *Then $\mathbb{Z}_\infty f\colon \mathbb{Z}_\infty X \to \mathbb{Z}_\infty Y$ is a homotopy equivalence if and only if f induces an isomorphism on integral homology.*

Property I2 ([3], Ch. V, Proposition 3.4, p. 134)**.** *For $X \in$* rs**Set** *there is a natural map $i\colon X \to \mathbb{Z}_\infty X$ which is a weak homotopy equivalence if X is nilpotent.*

Property I3 ([3], Ch. II, Lemma 5.4, p. 63)**.** *Let $p\colon E \to B$ (in* rs**Set***) be a fibration with connected fibre F such that the Serre action of $\pi_1(B)$ on $H_i(F;\mathbb{Z})$ is nilpotent for all $i \geqslant 0$. Then $\mathbb{Z}_\infty(p)\colon \mathbb{Z}_\infty E \to \mathbb{Z}_\infty B$ is a fibration and the inclusion $\mathbb{Z}_\infty F \to \mathbb{Z}_\infty(p)^{-1}(*)$ is a homotopy equivalence* ([3],Ch. II, Lemma 5.1, p. 62)*. The action of $\pi_1(B)$ on $H_*(F;\mathbb{Z})$ is in particular nilpotent if $\pi_1(E)$ acts nilpotently on $\pi_i(F)$ for all $i \geqslant 1$.*

4. The Integral Completion of $\bar{W}E(R)$

For any group H, there is a simplicial group which is H in every dimension, having the identity as degeneracy and boundary maps. This object is a *constant simplicial group* and we denote it by H again.

Proposition 1. *Let R be a unital ring. Then the geometric realization $|\mathbb{Z}_\infty\bar{W}E(R)|$ of $\mathbb{Z}_\infty\bar{W}E(R)$ is homotopy equivalent to $BE(R)^+$, the equivalence being functorial in R.*

Proof. The plus construction has its simplicial analogue in rs**Set**, the category of reduced simplicial sets. There exists a map $j\colon \bar{W}E(R) \to \bar{W}E(R)^+$ in rs**Set** such that its geometric realization $|j|\colon BE(R) \to |\bar{W}E(R)^+|$ is the map $j\colon BE(R) \to BE(R)^+$ of Section 1. Consider the commutative square (which is functorial in R)

$$\begin{tikzcd} \bar{W}E(R) \arrow[r, "j"] \arrow[d, "i"'] & \bar{W}E(R)^+ \arrow[d, "i"'] \\ \mathbb{Z}_\infty\bar{W}E(R) \arrow[r, "\mathbb{Z}_\infty(j)"] & \mathbb{Z}_\infty(\bar{W}E(R)^+) \end{tikzcd}$$

It suffices to prove that $i\colon \bar{W}E(R)^+ \to \mathbb{Z}_\infty(\bar{W}E(R)^+)$ and $\mathbb{Z}_\infty(j)$ are homotopy equivalences. The first map is a homotopy equivalence because of Property I2 and Property P3: the space $BE(R)^+$ is an H-space, so it is nilpotent. The map $\mathbb{Z}_\infty(j)$ is a homotopy equivalence because of Property I1. □

The proof above can also be found in [5], where it is attributed to E. Dror. In exactly the same way one proves the homotopy equivalence of the spaces $|\mathbb{Z}_\infty\bar{W}GL(R)|$ and $BGL(R)^+$.

Corollary. *For $n \geqslant 2$ we have $K_n \cong \pi_n\mathbb{Z}_\infty\bar{W}E$.*

5. Derived Functors

Let **Gr** be the category of groups. In this section we will review the theory of (left) derived functors of a given functor $T\colon \mathbf{Gr} \to \mathbf{Gr}$ as introduced in [9]. The situation is analogous to the Abelian case where projective resolutions are used.

Let G be a simplicial group. For each $n \geqslant 1$ we define

$$Z_n(G) = \{\, (x_0, \dots, x_{n+1}) \in G_n^{n+2} \mid d_i x_j = d_{j-1} x_i \text{ for all } 0 \leqslant i < j \leqslant n+1 \,\},$$

a subgroup of G_n^{n+2} ($= G_n \times \cdots \times G_n$, $n+2$ times). The elements of $Z_n(G)$ are those $n+2$ -tuples of elements of G_n that fit together in exactly the same way as the $n+2$ faces of an $n+1$ -simplex do. The group $Z_n(G)$ will be called the *group of n-spheres* in the simplicial group G. There is an obvious homomorphism $d\colon G_{n+1} \to Z_n(G)$ which assigns to an $n+1$ -simplex x the n-sphere $(d_0x, \dots, d_{n+1}x)$ of its faces. A simplicial group is called *aspherical* if for each $n \geqslant 1$ the map $d\colon G_{n+1} \to Z_n(G)$ is surjective, that is if every n-sphere is the boundary of an $n+1$ -simplex.

For $n > 0$ there is an isomorphism

$$\bar{\alpha}\colon \pi_n(G) \to Z_n(G)/dG_{n+1},$$

which is induced by the homomorphism

$$\alpha\colon \tilde{G}_n \to Z_n(G), \quad g \mapsto (1, \dots, 1, g),$$

where $\tilde{G}_n = \bigcap_i \operatorname{Ker} d_i$.

The isomorphism $\bar{\alpha}$ has the useful property that it respects the natural action of G_0 on G, given by conjugating G dimension-wise by the images of elements of G_0 under the degeneracy maps. The action obviously induces actions on $\tilde{G}_n$ and $Z_n(G)$ by restricting the actions on G_n and the $n+2$ -fold product $G_n \times \cdots \times G_n$ respectively.

For any set X let FX be the free group on the elements of X. A simplicial group G is called *free* if there is a subset X_n of G_n for each $n \geqslant 0$ such that $G \cong FX_n$ and moreover the degeneracy maps $s_i\colon G_n \to G_{n+1}$ $(i = 0, \ldots, n)$ map X_n into X_{n+1} for each $n \geqslant 0$. An example of a free simplicial group is the loop group GX of a reduced simplicial set X.

Let H be a group. A *free resolution* (G, ε), or simply G, of H consists of:

(1) a free aspherical simplicial group G;
(2) a group homomorphism $\varepsilon\colon G_0 \to H$, which induces an isomorphism $\pi_0(G) \to H$.

Free resolutions do exist. What is more, there are functorial free resolutions $G(H)$. One example is the cotriple resolution $G_n(H) = F^{n+1}(H)$. Another example is $G\bar{W}(H)$, the loop group on the classifying complex of H.

Let G be a free resolution of a group H and G' a free resolution of a group H'. In [9] it is proved that a homomorphism $h\colon H \to H'$ can be covered by a simplicial homomorphism $g\colon G \to G'$, i.e. $\pi_0(g)\colon \pi_0(G) \to \pi_0(G')$ induces f via the isomorphisms $\pi_0(G) \to H$ and $\pi_0(G') \to H'$. Moreover, two such simplicial homomorphisms are **Gr**-homotopic. As a consequence one can define derived functors of a functor $T\colon \mathbf{Gr} \to \mathbf{Gr}$. On objects the n-th derived functor $L_nT\colon \mathbf{Gr} \to \mathbf{Gr}$ is defined as follows. Let $H \in \mathbf{Gr}$; take a free resolution G of H; then put

$$L_nT = \pi_n(TG),$$

where TG means: T applied dimension-wise to G. On morphisms L_nT is defined by

$$(L_nT)(h) = \pi_n(Tg),$$

where $g\colon G \to G'$ covers $h\colon H \to H'$, G and G' being free resolutions of H and H' respectively.

Example. Let $T\colon \mathbf{Gr} \to \mathbf{Gr}$ be the Abelianization funtor. What are its derived functors? $G\bar{W}H$ is a free resolution of $H \in \mathbf{Gr}$. Therefore

$$\begin{aligned}(L_nT)(H) &= \pi_n(TG\bar{W}H) = \pi_n(G\bar{W}H/[G\bar{W}H, G\bar{W}H]) \\ &= H_{n+1}(\bar{W}H; \mathbb{Z}) = H_{n+1}(H; \mathbb{Z})\end{aligned}$$

(cf. [14], p.121). So L_nT is the homology functor $H_{n+1}(-; \mathbb{Z})$.

One can also consider functors $T\colon \mathbf{Rg} \to \mathbf{Gr}$ on the category $\mathbf{Rg}$ of nonunital rings instead of $\mathbf{Gr}$. The role of the free group is then taken over by the free ring: for X a set FX is the ring of polynomials without constant term in the non-commuting variables $x \in X$, with coefficients in $\mathbb{Z}$. Analogously one then considers: free simplicial rings, rings of n-spheres in a simplicial ring, etc. Then too one has a theory of derived functors for functors from $\mathbf{Rg}$ to $\mathbf{Gr}$. More generally, the procedure is applicable to functors $T\colon \mathbf{A} \to \mathbf{Set}_*$, where $\mathbf{A}$ is a category of triple algebras and $\mathbf{Set}_*$ the category of pointed sets. In fact this is how the theory is presented in [9].

6. Derived Functors of the Integral Completion

We will consider the derived functor L_nC of $C\colon \mathbf{Gr} \to \mathbf{Gr}$, the integral completion functor as described in Section 2. Let H be a group. Note that for any simplicial set X the simplicial group GX is free. It follows that the simplicial group $G\bar{W}H$ is a free resolution of H. The groups $(L_nC)(H)$ are therefore the homotopy groups of $CG\bar{W}H$, a simplicial group which has as classifying complex the integral completion of $\bar{W}H$ (cf. Section 3):

$$\mathbb{Z}_\infty \bar{W}H = \bar{W}CG\bar{W}H.$$

Hence

$$(L_nC)(H) = \pi_n(CG\bar{W}H) \cong \pi_{n+1}(\bar{W}CG\bar{W}H) = \pi_{n+1}(\mathbb{Z}_\infty \bar{W}H).$$

In the special case $H = GL(R)$ with R a unital ring, we obtain

Lemma. *For each $n \geqslant 0$ there is a canonical isomorphism*

$$(L_nC)(GL(R)) \cong \pi_{n+1}(\mathbb{Z}_\infty \bar{W}GL(R)) \cong K_{n+1}(R).$$

7. Derived Functors of GL

In [9] higher K-functors were defined as derived functors of $GL\colon \mathbf{Rg} \to \mathbf{Gr}$ by the formula

$$K'_n(R) = L_{n-2}GL \qquad (n \geqslant 3)$$

and K'_1 and K'_2 are then defined by the exactness of

$$1 \to K'_2 \to L_0GL \to GL \to K'_1 \to 1.$$

Since $\mathrm{St}(FX) \cong GL(FX)$ for free rings FX, we have $L_0GL = L_0\,\mathrm{St}$, where St denotes the Steinberg group. It is easily seen that $L_0\,\mathrm{St} = \mathrm{St}$, i.e. St is a right exact functor, see [10] for details. Hence the exact sequence above becomes

$$1 \to K'_2 \to \mathrm{St} \to GL \to K'_1 \to 1,$$

which shows that the functors K'_1 and K'_2 coincide with the classical ones. It will be proven in Section 8 that the functors as defined above are isomorphic to the functors K_n as defined in Section 1.

Remark. The groups $K'_n(R)$ defined in this section coincide with the groups $K_n(R)$ as defined by Gersten [4], since Gersten uses the cotriple resolution of R for their definition, which is simply one of possible resolutions of R. In [17] Swan proved that his functor K_n, which he defined in [16], coincides with Gersten's.

8. Comparison of Both K-theories

In this section we prove the main theorem.

Theorem. *Let $R \in \mathbf{Rg}$. Then for all $n \geqslant 2$,*

$$K'_n(R) \cong K_n(R^+, R).$$

Let R be a nonunital ring. Form the simplicial ring FR by applying the free resolution functor. Adjoining a unit in every dimension of FR we get a split homomorphism of simplicial unital rings,

$$\begin{tikzcd} (FR)^+ \arrow[r, shift left] & \mathbb{Z} \arrow[l, shift left, bend left] \end{tikzcd},$$

where the right-hand side is interpreted as a constant simplicial ring.

The adjunction of G and $\bar{W}$ induces a cotriple on s**Gr**. By first applying this cotriple resolution to the diagram $E((FR)^+) \to E(\mathbb{Z})$, and then the integral completion functor C, we define the two following bisimplicial groups together with a split homomorphism.

$$Q^+_{pq} = (C(G\bar{W})^{q+1}E((FR)^+))_p \longrightarrow Z_{pq} = (C(G\bar{W})^{q+1}E(\mathbb{Z}))_p$$

Let Q denote the fibre (or kernel) of this map. Taking homotopy in each row of these bisimplicial groups, we can consider the usual long exact sequence of a fibration. Since the homomorphism splits, this sequence of simplicial groups degenerates into split short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \pi^h_q Q \arrow[r] & \pi^h_q Q^+ \arrow[r, shift left] & \pi^h_q Z \arrow[l, shift left, bend left] \arrow[r] & 1. \end{tikzcd}$$

Again computing homotopy, each of these fibrations induces a long exact sequence, which too is split. Hence for every p and q we have a split short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \pi^v_p\pi^h_q Q \arrow[r] & \pi^v_p\pi^h_q Q^+ \arrow[r, shift left] & \pi^v_p\pi^h_q Z \arrow[l, shift left, bend left] \arrow[r] & 1. \end{tikzcd}$$

Repeating this process, but now taking vertical homotopy groups first, we have

$$\begin{tikzcd} 1 \arrow[r] & \pi^h_q\pi^v_p Q \arrow[r] & \pi^h_q\pi^v_p Q^+ \arrow[r, shift left] & \pi^h_q\pi^v_p Z \arrow[l, shift left, bend left] \arrow[r] & 1. \end{tikzcd}$$

Note that for each p, the simplicial group Q^+_{p*} is the result of an application of the functor C to a free **Gr**-resolution of $E((F_pR)^+)$. Hence,

$$\pi^h_q\pi^v_p Q^+ = (L_pC)(E((F_qR)^+)) = \pi_{p+1}\mathbb{Z}_\infty\bar{W}E((F_qR)^+).$$

Using the description of the Quillen K-groups from Section 6, it follows that for all p and q we have a short exact sequence

$$0 \longrightarrow \pi^h_q\pi^v_p Q \longrightarrow K_{p+1}((F_qR)^+) \longrightarrow K_{p+1}(\mathbb{Z}) \longrightarrow 0.$$

Using Gersten's theorem on the K-theory of free rings [5], we find that for all p and q

$$\pi^h_q\pi^v_p Q = 0,$$

and hence, e.g. by Quillen's spectral sequence [15] we also have

$$\pi^v_p\pi^h_q Q = 0.$$

By letting $p = 0$ and applying this formula to the relevant split exact sequence above, we obtain an isomorphism

$$\pi_0^v \pi_q^h Q^+ \xrightarrow{\sim} \pi_0^v \pi_q^h Z.$$

We will now determine these homotopy groups. Note that the functorial homomorphisms $G\bar{W}(H) \to H$ are homotopy equivalences for any simplicial group H. From this it follows that all the maps $d_i^v\colon Q^+_{p,q+1} \to Q^+_{p,q}$ are homotopy equivalences too, since the functor C is applied dimension-wise. Hence,

$$\pi_p^v \pi_q^h Q^+ = 0 \qquad \text{for } p > 0$$

and

$$\pi_0^v \pi_q^h Q^+ = \pi_q C G \bar{W} E((FR)^+) = \pi_{q+1} \mathbb{Z}_\infty \bar{W} E((FR)^+).$$

Performing the same calculation for Z and substituting this into the isomorphism above, we find the following proposition, which can be seen as a generalization of Gersten's theorem to include some types of free simplicial rings:

Proposition 2. *For each $q \geqslant 0$ we have*

$$\pi_q \mathbb{Z}_\infty \bar{W} E((FR)^+) \cong \pi_q \mathbb{Z}_\infty \bar{W} E(\mathbb{Z}).$$

Let FR once again be the cotriple resolution of R in **Rg**. Then $F_0 R \to R$ induces a surjective homomorphism $E(FR) \to E(R)$. Its kernel is denoted by HR. From the long exact sequence of the fibration $HR \to E(FR) \to E(R)$ it follows that $\pi_n HR = K'_{n+2}(R)$ for all $n > 0$ and $\pi_0 HR = \mathrm{Ker}(\mathrm{St}(R) \to E(R))$. Hence we have

$$\pi_n HR = K'_{n+2}(R) \quad \text{for all } n \geqslant 0.$$

Note that the simplicial group HR is also the kernel of the homomorphism $E((FR)^+) \to E(R^+)$. To see this, apply the snake lemma to the following diagram having split exact rows:

$$\begin{tikzcd}
 & HR \arrow[d] & & & \\
1 \arrow[r] & E(FR) \arrow[r] \arrow[d] & E(((FR)^+)) \arrow[r] \arrow[d] & E(\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\
1 \arrow[r] & E(R) \arrow[r] & E(R^+) \arrow[r] & E(\mathbb{Z}) \arrow[r] & 1
\end{tikzcd}$$

This gives the desired identification of the simplicial group HR with the kernel of $E((FR)^+) \to E(R^+)$.

The point of introducing this simplicial group HR is that the fibration

$$1 \to HR \to E((FR)^+) \to E(R^+) \to 1$$

has good behavior under application of the composite functor $\mathbb{Z}_\infty \bar{W}$. We need to verify the requirements of Property I3. Taking classifying complexes we obtain the following fibration in s**Set**:

$$\bar{W} HR \to \bar{W} E((FR)^+) \to \bar{W} E(R^+).$$

Recall that for any simplicial group G the usual action of $\pi_1(\bar{W}G) = \pi_0(G)$ on $\pi_i(\bar{W}G)$ is the action induced by dimension-wise conjugation of G by degenerate elements originating from G_0.

Lemma. *There is a natural isomorphism $GL(Z_i(FR)) \to Z_i(GL(FR))$ which is induced by the projection maps $Z_i(FR) \to F_iR$.*

Proof. The simplicial kernel $Z_i(FR)$ is an inverse limit of a suitable system of rings and the functor GL preserves inverse limits. By inspecting the pullback diagrams, it is clear that $Z_i(FR)$ is completely determined by the projection maps $p_*\colon Z_i(FR) \to F_iR$. We have a homomorphism $(GL(p_1), \dots, GL(p_{i+2}))$: $GL(Z_i(FR)) \to (GL(F_iR))^{i+2}$. Its image is precisely $Z_i(GL(FR))$. □

Proposition 3. *The usual action of $\pi_1(\bar{W}E((FR)^+))$ on $\pi_i(\bar{W}E((FR)^+))$ is trivial for $i > 1$.*

Proof. For $i > 0$ we have the isomorphism

$$\pi_i(E(FR)^+) \to Z_i(E((FR)^+))/dE((FR)^+)_{i+1}$$

which commutes with the action of $E((FR)^+)$ (see Section 5.) Using this we have that

$$\begin{aligned} Z_i(E((FR)^+)) &\cong Z_i(E(FR) \rtimes E(\mathbb{Z})) = Z_i(GL(FR) \rtimes E(\mathbb{Z})) \\ &\cong Z_i(GL(FR)) \rtimes E(\mathbb{Z}) \cong GL(Z_i(FR) \rtimes E(\mathbb{Z})). \end{aligned}$$

The image of the group $dE((FR)^+)_{i+1}$ under this composition is the group $E(Z_i(FR)) \rtimes E(\mathbb{Z})$. To see this, note that $d\colon (FR)^+_{i+1} \to Z_i((FR)^+)$ is onto and that the functor E preserves such maps.

The images of the elements of $E((F_0R)^+) \subseteq Z_i(E((FR)^+))$ are contained in $E(Z_i(FR)) \rtimes E(\mathbb{Z})$. Hence the action on $Z_i(E((FR)^+))$ corresponds to an action which becomes trivial when passing to quotients. It follows that the action of $E((FR)^+)$ on $\pi_{i+1}(\bar{W}E((FR)^+))$ is trivial. □

Corollary. *The action of $\pi_1(\bar{W}E((FR)^+))$ on $\pi_i(\bar{W}HR)$ is trivial for all i.*

Proof. For $i > 1$ this is a consequence of the previous lemma. For $i = 1$ the action is also trivial because $\pi_1(\bar{W}HR)$ maps isomorphically onto the kernel of $\pi_1(\bar{W}E((FR)^+)) \to E(R^+)$, which identifies with the central extension $\mathrm{St}(R) \rtimes E(\mathbb{Z}) \to E(R) \rtimes E(\mathbb{Z})$. □

Corollary. *The action of $\pi_1(\bar{W}HR)$ on $\pi_i(\bar{W}HR)$ is trivial.*

Proof. The homomorphism $\pi_1(\bar{W}HR) \to \pi_1(\bar{W}E((FR)^+))$ is injective by the long exact sequence of a fibration. Now use the previous corollary. □

Proposition 4. *The induced map $\mathbb{Z}_\infty\bar{W}E((FR)^+) \to \mathbb{Z}_\infty\bar{W}E(R^+)$ is a fibration and its fibre is homotopy equivalent to $\bar{W}HR$.*

Proof. From Property I3 and the above corollary it follows that the map $\mathbb{Z}_\infty\bar{W}E((FR)^+) \to \mathbb{Z}_\infty\bar{W}E(R^+)$ is a fibration and that the canonical map from $\mathbb{Z}_\infty\bar{W}HR$ to the fibre is a homotopy equivalence. From Property I2 and this corollary it follows that the natural map $i\colon \bar{W}HR \to \mathbb{Z}_\infty \bar{W}HR$ is a weak homotopy equivalence. Since all simplicial sets involved are Kan complexes, this map is a fortiori a homotopy equivalence. $\square$

Now we can finish the proof of the theorem.

Proof. Let X be the fibre of the map $\mathbb{Z}_\infty \bar{W}E(R^+) \to \mathbb{Z}_\infty \bar{W}E(\mathbb{Z})$. We have the following diagram which consists horizontally and vertically of long exact sequences of suitable fibrations:

$$\begin{tikzcd}
\pi_{p+1}(X) \arrow[r, "\sim"] \arrow[d] & \pi_p(\cdots) \arrow[r] \arrow[d, "\wr"] & 0 \arrow[r] \arrow[d] & \pi_p(X) \arrow[d] \\
\pi_{p+1}\mathbb{Z}_\infty \bar{W}E(R^+) \arrow[r] \arrow[d] & \pi_p \bar{W}HR \arrow[r] \arrow[d] & \pi_p \mathbb{Z}_\infty \bar{W}E((FR)^+) \arrow[r] \arrow[d, "\wr"] & \pi_p \mathbb{Z}_\infty \bar{W}E(R^+) \arrow[d] \\
\pi_{p+1}\mathbb{Z}_\infty \bar{W}E(\mathbb{Z}) \arrow[r] & 0 \arrow[r] & \pi_p \mathbb{Z}_\infty \bar{W}E(\mathbb{Z}) \arrow[r, "="] & \pi_p \mathbb{Z}_\infty \bar{W}E(\mathbb{Z})
\end{tikzcd}$$

The map in the third column is an isomorphism by Proposition 2. The other relations are evident from the diagram. The group $K_{p+1}(R^+, R)$ equals $\pi_{p+1}(X)$ for $p \geqslant 1$, which is in turn isomorphic to $\pi_p(\bar{W}H(R))$ by the above diagram. The latter group equals $K'_{p+1}(R)$ for $p \geqslant 1$. $\square$

Final remark. An alternative way to prove the equivalence of both algebraic K-theories is by showing that the Quillen K-theory satisfies the axioms for multirelative K-theory given in [12]. To do so, the Quillen K-theory has to be extended to include multirelative groups. The main concern is then to extend long exact sequences in such a way that they include K_0-groups as well.

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