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A NEW METHOD OF SOLVING THE BASIC PLANE BOUNDARY VALUE PROBLEMS OF STATICS OF THE ELASTIC MIXTURE THEORY

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Abstract. The basic plane boundary value problems of statics of the elastic mixture theory are considered when on the boundary are given: a displacement vector (the first problem), a stress vector (the second problem); differences of partial displacements and the sum of stress vector components (the third problem). A simple method of deriving Fredholm type integral equations of second order for these problems is given. The properties of the new operators are established. Using these operators and generalized Green formulas we investigate the above-mentioned integral equations and prove the existence and uniqueness of a solution of all the boundary value problems in a finite and an infinite domain.

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1. Some Auxiliary Formulas and Operators

In the two-dimensional case, the basic homogeneous equations of statics of the elastic mixture theory have the form (see [1] and [2]):

$$\begin{aligned} &a_1\Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' = 0,\\ &c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' = 0, \end{aligned} \tag{1.1}$$

where Δ is the two-dimensional Laplacian, grad and div are the principal operators of the field theory, $u' = (u_1', u_2')$ and $u'' = (u_1'', u_2'')$ are partial displacements, a_k, b_k $(k = 1, 2)$, c, d are the known constants characterizing the physical properties of a mixture, and at that

$$\begin{aligned} &a_1 = \mu_1 - \lambda_5, \quad a_2 = \mu_2 - \lambda_5, \quad c = \mu_3 + \lambda_5,\\ &b_1 = \mu_1 + \lambda_1 + \lambda_5 - \rho_2\alpha_2/\rho, \quad b_2 = \mu_2 + \lambda_2 + \lambda_5 + \rho_1\alpha_2/\rho,\\ &d = \mu_3 + \lambda_3 - \lambda_5 - \rho_1\alpha_2/\rho \equiv \mu_3 + \lambda_4 - \lambda_5 + \rho_2\alpha_2/\rho,\\ &\rho = \rho_1 + \rho_2, \quad \alpha_2 = \lambda_3 - \lambda_4, \end{aligned} \tag{1.2}$$

where μ_1, μ_2, μ_3, λ_1, λ_2, λ_3, λ_4, λ_5, ρ_1, ρ_2 are new constants also characterizing the physical properties of the mixture and satisfying the definite conditions (inequalities) [2].

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In the theory of elastic mixtures, the displacement vector is usually denoted by $u=(u',u'')$, while the four-dimensional vector by $u=(u_1,u_2,u_3,u_4)$ or $u_1=u_1'$, $u_2=u_2'$, $u_3=u_1''$, $u_4=u_2''$.

The system of basic equations (1.1) can be rewritten (equivalently) as follows:

$$\begin{aligned}
&a_1\Delta u'+c\Delta u''+b_1\operatorname{grad}\theta'+d\operatorname{grad}\theta''=0,\\
&c\Delta u'+a_2\Delta u''+d\operatorname{grad}\theta'+b_2\operatorname{grad}\theta''=0,
\end{aligned}\tag{1.3}$$

where

$$\theta'=\frac{\partial u_1'}{\partial x_1}+\frac{\partial u_2'}{\partial x_2},\quad \theta''=\frac{\partial u_1''}{\partial x_1}+\frac{\partial u_2''}{\partial x_2}.\tag{1.4}$$

Let us consider the variables $z=x_1+ix_2$, $\bar z=x_1-ix_2$, by which $x_1=\frac{z+\bar z}{2}$, $x_2=\frac{z-\bar z}{2i}$. Then

$$\begin{aligned}
&\frac{\partial}{\partial x_1}=\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar z},\quad \frac{\partial}{\partial x_2}=i\Big(\frac{\partial}{\partial z}-\frac{\partial}{\partial\bar z}\Big),\\
&\frac{\partial}{\partial z}=\frac12\Big(\frac{\partial}{\partial x_1}-i\frac{\partial}{\partial x_2}\Big),\quad \frac{\partial}{\partial\bar z}=\frac12\Big(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\Big).
\end{aligned}\tag{1.5}$$

Elementary calculations give

$$\Delta=4\frac{\partial^2}{\partial z\partial\bar z},\quad \theta'=\frac{\partial U_1}{\partial z}+\frac{\partial\overline{U}_1}{\partial\bar z},\quad \theta''=\frac{\partial U_2}{\partial z}+\frac{\partial\overline{U}_2}{\partial\bar z},\tag{1.6}$$

where

$$U_1=u_1+iu_2,\quad U_2=u_3+iu_4.\tag{1.7}$$

Using formulas (1.5), (1.6) and (1.7), system (1.3) can be written in terms of two complex equations

$$\begin{aligned}
&2a_1\frac{\partial^2U_1}{\partial z\partial\bar z}+2c\frac{\partial^2U_2}{\partial z\partial\bar z}+b_1\frac{\partial\theta'}{\partial\bar z}+d\frac{\partial\theta''}{\partial\bar z}=0,\\
&2c\frac{\partial^2U_1}{\partial z\partial\bar z}+2a_2\frac{\partial^2U_2}{\partial z\partial\bar z}+d\frac{\partial\theta'}{\partial\bar z}+b_2\frac{\partial\theta''}{\partial\bar z}=0.
\end{aligned}$$

After substituting here the values of θ' and θ'' from (1.6) we obtain

$$\begin{aligned}
&(2a_1+b_1)\frac{\partial^2U_1}{\partial z\partial\bar z}+(2c+d)\frac{\partial^2U_2}{\partial z\partial\bar z}+b_1\frac{\partial^2\overline{U}_1}{\partial\bar z^2}+d\frac{\partial^2\overline{U}_2}{\partial\bar z^2}=0,\\
&(2c+d)\frac{\partial^2U_1}{\partial z\partial\bar z}+(2a_2+b_2)\frac{\partial^2U_2}{\partial z\partial\bar z}+d\frac{\partial^2\overline{U}_1}{\partial\bar z^2}+b_2\frac{\partial^2\overline{U}_2}{\partial\bar z^2}=0.
\end{aligned}$$

Hence, by some elementary transformations, we have

$$\frac{\partial^2U}{\partial z\partial\bar z}+\varepsilon^\top\frac{\partial^2\overline{U}}{\partial\bar z^2}=0,\tag{1.8}$$

where

$$\begin{aligned}
&\varepsilon^\top = \begin{bmatrix} \varepsilon_1, & \varepsilon_3 \\ \varepsilon_2, & \varepsilon_4 \end{bmatrix}, \\
&\delta_0\varepsilon_1 = 2(a_2b_1 - cd) + b_1b_2 - d^2, \quad \delta_0\varepsilon_2 = 2(da_1 - cb_1), \\
&\delta_0\varepsilon_3 = 2(da_2 - cb_2), \quad \delta_0\varepsilon_4 = 2(a_1b_2 - cd) + b_1b_2 - d^2, \\
&\delta_0 = (2a_1 + b_1)(2a_2 + b_2) - (2c + d)^2 \equiv 4\Delta_0 d_1 d_2 > 0, \\
&\Delta_0 = m_1m_3 - m_2^2 > 0, \;\; m_1 = l_1 + \frac{l_4}{2}, \;\; m_2 = l_2 + \frac{l_5}{2}, \;\; m_3 = l_3 + \frac{l_6}{2}, \\
&d_1 = (a_1 + b_1)(a_2 + b_2) - (c + d)^2 > 0, \quad d_2 = a_1a_2 - c^2 > 0, \\
&l_1 = \frac{a_2}{d_2}, \;\; l_2 = -\frac{c}{d_2}, \;\; l_3 = \frac{a_1}{d_2}, \\
&l_1 + l_4 = \frac{a_2 + b_2}{d_1}, \;\; l_2 + l_5 = -\frac{c + d}{d_1}, \;\; l_3 + l_6 = \frac{a_1 + b_1}{d_1}.
\end{aligned} \tag{1.9}$$

Equation (1.8) represents basic homogeneous equations of statics of the elastic mixture theory in the complex-vector form. One can likewise esily verify the validity of the identity

$$\varepsilon^\top = -\frac{1}{2}\,\ell m^{-1}, \tag{1.10}$$

where

$$\ell = \begin{bmatrix} \ell_4, & \ell_5 \\ \ell_5, & \ell_6 \end{bmatrix}, \quad m^{-1} = \frac{1}{\Delta_0}\begin{bmatrix} m_3, & -m_2 \\ -m_2, & m_1 \end{bmatrix}, \tag{1.11}$$

ℓ_k $(k = 4, 5, 6)$, Δ_0 and m_k $(k = 1, 2, 3)$ are defined from (1.9).

In addition to the vector

$$U = \begin{pmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{pmatrix}, \tag{1.12}$$

using the formulas

$$\begin{aligned}
iv_1 - v_2 &= m_1\varphi_1(z) + m_2\varphi_2(z) - \frac{z}{2}\Big[\,\ell_4\overline{\varphi_1'(z)} + \ell_5\overline{\varphi_2'(z)}\,\Big] - \overline{\psi_1(z)}, \\
iv_3 - v_4 &= m_2\varphi_1(z) + m_3\varphi_2(z) - \frac{z}{2}\Big[\,\ell_5\overline{\varphi_1'(z)} + \ell_6\overline{\varphi_2'(z)}\,\Big] - \overline{\psi_2(z)},
\end{aligned}$$

(see [3], p.242), we write the vector V as

$$V = \begin{pmatrix} v_1 + iv_2 \\ v_3 + iv_4 \end{pmatrix}, \tag{1.13}$$

where v_1, v_2, v_3, v_4 are the components of the vector v. As is known from [3], U and V are the conjugate vectors, i.e., V, like U, satisfies equation (1.8).

Using analogues of the general Kolosov–Muskhelishvili representations from [3], we can write

$$U = m\varphi(z) + \frac{\ell}{2}\, z\overline{\varphi'(z)} + \overline{\psi(z)}, \quad V = i\Big[-m\varphi(z) + \frac{\ell}{2}\, z\overline{\varphi'(z)} + \overline{\psi(z)}\Big], \tag{1.14}$$

where $\varphi(z)$ and $\psi(z)$ are arbitrary analytic vectors

$$m = \begin{bmatrix} m_1, & m_2 \\ m_2, & m_3 \end{bmatrix}, \tag{1.15}$$

m_1, m_2, m_3 are defined from (1.9).

By (1.14) it is obvious that

$$U + iV = 2m\varphi(z). \tag{1.16}$$

Let us now introduce the vectors

$$\overset{\varkappa}{T}U = \begin{pmatrix} (\overset{\varkappa}{T}u)_2 - i(\overset{\varkappa}{T}u)_1 \\ (\overset{\varkappa}{T}u)_4 - i(\overset{\varkappa}{T}u)_3 \end{pmatrix}, \quad \overset{\varkappa}{T}V = \begin{pmatrix} (\overset{\varkappa}{T}v)_2 - i(\overset{\varkappa}{T}v)_1 \\ (\overset{\varkappa}{T}v)_4 - i(\overset{\varkappa}{T}v)_3 \end{pmatrix}, \tag{1.17}$$

where U and V are defined from (1.12), (1.13), (1.14), $\varkappa$ is an arbitrary constant matrix:

$$\varkappa = \begin{bmatrix} \varkappa_1, & \varkappa_3 \\ \varkappa_3, & \varkappa_2 \end{bmatrix}. \tag{1.18}$$

Using the formula

$$\begin{pmatrix} (\overset{\varkappa}{T}u)_2 - i(\overset{\varkappa}{T}u)_1 \\ (\overset{\varkappa}{T}u)_4 - i(\overset{\varkappa}{T}u)_3 \end{pmatrix} = \frac{\partial}{\partial s(x)}\big[-2\varphi(z) + (2\mu - \varkappa)U\big]$$

(see [3], p. 236) and (1.16), we can rewrite (1.17) as follows:

$$\begin{aligned} \overset{\varkappa}{T}U &= \frac{\partial}{\partial s(x)}\Big[(A - 2E - \varkappa m)\varphi(z) \\ &\quad + (2\mu - \varkappa)\frac{\ell}{2}\, z\overline{\varphi'(z)} + (2\mu - \varkappa)\overline{\psi(z)}\Big], \\ \overset{\varkappa}{T}V &= i\,\frac{\partial}{\partial s(x)}\Big[-(A - 2E - \varkappa m)\varphi(z) \\ &\quad + (2\mu - \varkappa)\frac{\ell}{2}\, z\overline{\varphi'(z)} + (2\mu - \varkappa)\overline{\psi(z)}\Big], \end{aligned} \tag{1.19}$$

where E is the unit matrix and

$$A = 2\mu m, \quad \mu = \begin{bmatrix} \mu_1, & \mu_3 \\ \mu_3, & \mu_2 \end{bmatrix}, \quad \frac{\partial}{\partial s(x)} = n_1\frac{\partial}{\partial x_2} - n_2\frac{\partial}{\partial x_1}, \tag{1.20}$$

$n = (n_1, n_2)$ is an arbitrary unit vector.

If $\varkappa = 0$, then $\overset{\varkappa}{T} \equiv T$, where T is the stress operator. Now (1.19) can be rewritten as

$$\begin{aligned} TU &= \frac{\partial}{\partial s(x)}\Big[(A-2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}\Big], \\ TV &= i\,\frac{\partial}{\partial s(x)}\Big[-(A-2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}\Big], \end{aligned} \tag{1.21}$$

where

$$B = \mu\ell. \tag{1.22}$$

In addition to the operator T, we will need from (1.19) the particular case of the matrix $\varkappa$, where

$$\varkappa = 2\mu - m^{-1}. \tag{1.23}$$

In that case $\overset{\varkappa}{T} \equiv N$, where N is the pseudostress operator which plays an important part in studying the first boundary value problem.

Taking into account (1.14), (1.19) and (1.23), for the operator N we obtain

$$NU = -im^{-1}\,\frac{\partial V}{\partial s(x)}\,, \quad NV = im^{-1}\,\frac{\partial U}{\partial s(x)}\,. \tag{1.24}$$

These relations are important when investigating the basic plane boundary value problems of statics of an elastic mixture.

Put now in (1.19)

$$\begin{aligned} \varphi(z) &= \frac{(A-2E-\varkappa m)^{-1}}{2\pi i}\int_S \ln\sigma g(y)\,dS, \\ \overline{\varphi'(z)} &= -\frac{(A-2E-\varkappa m)^{-1}}{2\pi i}\int_S \frac{\overline{g(y)}}{\overline{\sigma}}\,dS, \\ \overline{\psi(z)} &= -\frac{(2\mu-\varkappa)^{-1}}{2\pi i}\int_S \ln\overline{\sigma}g(y)\,dS + \frac{\ell(A-2E-\varkappa m)^{-1}}{4\pi i}\int_S \frac{\zeta}{\overline{\sigma}}\,\overline{g(y)}\,dS\,, \end{aligned} \tag{1.25}$$

where $\sigma = z-\xi$, $\overline{\sigma} = \overline{z}-\overline{\xi}$, $\xi = y_1 + iy_2$, $g(y)$ is the complex vector we seek for. It is assumed here that

$$\det|A-2E-\varkappa m| \neq 0, \quad \det|2\mu-\varkappa| \neq 0. \tag{1.26}$$

In the sequel we will see that these restrictions are actually fulfilled.

Substituting (1.25) into (1.19) and performing some elementary transformations, we obtain

$$
\begin{aligned}
\overset{\varkappa}{T}U &= \frac{1}{\pi}\int\limits_S \frac{\partial\theta}{\partial s(x)}\, g(y)\, dS \\
&\quad - \frac{(2\mu-\varkappa)\ell(A-2E-\varkappa m)^{-1}}{4\pi i}\int\limits_S \frac{\partial}{\partial s(x)}\,\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\, dS, \\
\overset{\varkappa}{T}V &= -\frac{1}{\pi}\int\limits_S \frac{\partial \ln r}{\partial s(x)}\, g(y)\, dS \\
&\quad - \frac{(2\mu-\varkappa)\ell(A-2E-\varkappa m)^{-1}}{4\pi}\int\limits_S \frac{\partial}{\partial s(x)}\,\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\, dS,
\end{aligned}
\tag{1.27}
$$

where

$$
r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\,, \quad \theta = \operatorname{arctg}\frac{y_2-x_2}{y_2-x_1}\,. \tag{1.28}
$$

By (1.25) and (1.14) we obtain

$$
\begin{aligned}
U &= \frac{m(A-2E-\varkappa m)^{-1}}{2\pi i}\int\limits_S \ln\sigma g(y)\, dS - \frac{(2\mu-\varkappa)^{-1}}{2\pi i}\int\limits_S \ln\overline{\sigma} g(y)\, dS \\
&\quad - \frac{\ell(A-2E-\varkappa m)^{-1}}{4\pi i}\int\limits_S \frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\, dS, \\
V &= -\frac{m(A-2E-\varkappa m)^{-1}}{2\pi}\int\limits_S \ln\sigma g(y)\, dS - \frac{(2\mu-\varkappa)^{-1}}{2\pi}\int\limits_S \ln\overline{\sigma} g(y)\, dS \\
&\quad - \frac{\ell(A-2E-\varkappa m)^{-1}}{4\pi}\int\limits_S \frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\, dS.
\end{aligned}
\tag{1.29}
$$

Let us show that U and V satisfy equation (1.8) for any $\varkappa$. Indeed, by (1.29)

$$
\begin{aligned}
\frac{\partial^2 U}{\partial z\partial\overline{z}} &= \frac{\ell(A-2E-\varkappa m)^{-1}}{4\pi i}\int\limits_S \frac{\overline{g(y)}}{\overline{\sigma}^2}\, dS, \\
\frac{\partial^2 U}{\partial \overline{z}^2} &= \frac{m(A-2E-\varkappa m)^{-1}}{2\pi i}\int\limits_S \frac{\overline{g(y)}}{\overline{\sigma}^2}\, dS.
\end{aligned}
$$

Now by virtue of (1.10) we conclude that U satisfies (1.8). In a similar manner one can show that V, too, satisfies (1.8).

Next, using (1.29), we calculate the operator $\overset{\varkappa_0}{T}$, where $\varkappa_0$ is an arbitrary real matrix (different from $\varkappa$).

By (1.25) we have

$$\overset{\varkappa_0}{T}U=\frac{\partial}{\partial s(x)}\bigg[\frac{(A-2E-\varkappa_0 m)(A-2E-\varkappa m)^{-1}}{2\pi i}\int\limits_S\ln\sigma g(y)\,dS$$
$$-\frac{(2\mu-\varkappa_0)(2\mu-\varkappa)^{-1}}{2\pi i}\int\limits_S\ln\overline{\sigma}g(y)\,dS$$
$$-\frac{(2\mu-\varkappa_0)\ell(A-2E-\varkappa m)^{-1}}{4\pi i}\int\limits_S\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\,dS\bigg],$$
$$\overset{\varkappa_0}{T}V=\frac{\partial}{\partial s(x)}\bigg[\frac{-(A-2E-\varkappa_0 m)(A-2E-\varkappa m)^{-1}}{2\pi}\int\limits_S\ln\sigma g(y)\,dS$$
$$-\frac{(2\mu-\varkappa_0)(2\mu-\varkappa)^{-1}}{2\pi}\int\limits_S\ln\overline{\sigma}g(y)\,dS$$
$$-\frac{(2\mu-\varkappa_0)\ell(A-2E-\varkappa m)^{-1}}{4\pi}\int\limits_S\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\,dS\bigg].$$

Assume that $\varkappa$ and $\varkappa_0$ satisfy the equation

$$(A-2E-\varkappa_0 m)(A-2E-\varkappa m)^{-1}+(2\mu-\varkappa_0)(2\mu-\varkappa_0)^{-1}=0. \quad (1.30)$$

Then the preceding formulas can be rewritten as follows:

$$\begin{aligned}\overset{\varkappa_0}{T}U&=-(2\mu-\varkappa_0)(2\mu-\varkappa)^{-1}\bigg[\frac{1}{\pi i}\int\limits_S\frac{\partial\ln r}{\partial s(x)}\,g(y)\,dS\\&\quad+\frac{(2\mu-\varkappa)\ell(A-2E-\varkappa m)^{-1}}{4\pi i}\int\limits_S\frac{\partial}{\partial s(x)}\,\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\,dS\bigg],\\ \overset{\varkappa_0}{T}V&=(2\mu-\varkappa_0)(2\mu-\varkappa)^{-1}\bigg[\frac{i}{\pi}\int\limits_S\frac{\partial\theta}{\partial s(x)}\,g(y)\,dS\\&\quad-\frac{(2\mu-\varkappa)\ell(A-2E-\varkappa m)^{-1}}{4\pi}\int\limits_S\frac{\partial}{\partial s(x)}\,\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\,dS\bigg].\end{aligned} \quad (1.31)$$

Comparing (1.27) and (1.31), we obtain

$$\overset{\varkappa_0}{T}U=-i(2\mu-\varkappa_0)(2\mu-\varkappa)^{-1}\overset{\varkappa}{T}V,\ \ \overset{\varkappa_0}{T}V=i(2\mu-\varkappa_0)(2\mu-\varkappa)^{-1}\overset{\varkappa}{T}U. \quad (1.32)$$

We introduce the following definition: if $\varkappa$ and $\varkappa_0$ satisfy equation (1.30), then the operators $\overset{\varkappa_0}{T}$ and $\overset{\varkappa}{T}$ are self-conjugate ones, i.e., identities (1.32) are valid. Let us consider some particular cases. Let $\varkappa=0$; then $\overset{\varkappa}{T}\equiv T$ and from (1.30) it follows that

$$\varkappa_0=2\mu-2(A-E)^{-1}\mu. \quad (1.33)$$

Therefore $\overset{\varkappa_0}{T} \equiv L$. Thus the operators T and L are self-conjugate ones. Formulas (1.32) take the form

$$TV = i(A - E)LU, \quad TU = -i(A - E)LV. \tag{1.34}$$

If in (1.30) $\varkappa$ is defined from (1.23), then $\overset{\varkappa}{T} \equiv N$ and to obtain the conjugate operator we have the indefiniteness. But this does not mean that no conjugate operator exists for N. We use here formula (1.24) which implies that for N the conjugate operator is N. Let us rewrite formula (5.15) from [4] as

$$\int\limits_{D^+} \overset{\varkappa}{T}(u,u)\,dy_1\,dy_2 = \int\limits_S u\overset{\varkappa}{T}u\,dS \equiv \operatorname{Im}\int\limits_S U\overset{\varkappa}{T}\overline{U}\,dS, \tag{1.35}$$

where Im is the imaginary part, U and $\overset{\varkappa}{T}\overline{U}$ are defined from (1.12) and (1.17), D^+ is the finite domain bounded by the closed contour S. From the latter formula we obtain two formulas to be used below. For $\varkappa = 0$ and $\varkappa = 2\mu - m^{-1}$, (1.35) respectively yields

$$\int\limits_{D^+} T(u,u)\,dy_1\,dy_2 = \int\limits_S uTu\,dS \equiv \operatorname{Im}\int\limits_S UT\overline{U}\,dS, \tag{1.36}$$

$$\int\limits_{D^+} N(u,u)\,dy_1\,dy_2 = \int\limits_S uNu\,dS \equiv \operatorname{Im}\int\limits_S UN\overline{U}\,dS \equiv \operatorname{Im}\int\limits_S VN\overline{V}\,dS, \tag{1.37}$$

where $T(u,u)$ and $N(u,u)$ are defined in [4], pp. 75–76. Formulas (1.35), (1.36) and (1.37) hold for the infinite domain $D^- = E_2\backslash\overline{D^+}$ as well provided that conditions (5.22) from [4] are fulfilled. In that case we have

$$\int\limits_{D^-} \overset{\varkappa}{T}(u,u)\,dy_1\,dy_2 = -\int\limits_S u\overset{\varkappa}{T}u\,dS \equiv -\operatorname{Im}\int\limits_S U\overset{\varkappa}{T}\overline{U}\,dS, \tag{1.38}$$

$$\int\limits_{D^-} T(u,u)\,dy_1\,dy_2 = -\int\limits_S uTu\,dS \equiv -\operatorname{Im}\int\limits_S UT\overline{U}\,dS, \tag{1.39}$$

$$\int\limits_{D^-} N(u,u)\,dy_1\,dy_2 = -\int\limits_S uNu\,dS \equiv -\operatorname{Im}\int\limits_S UN\overline{U}dS \equiv -\operatorname{Im}\int\limits_S VN\overline{V}dS. \tag{1.40}$$

These formulas play an essential role in investigating the basic plane boundary value problems of statics for an elastic mixture.

2. The First Boundary Value Problem

The first boundary value problem is formulated as follows: Find, in the domain $D^+(D^-)$, a vector U which belongs to the class $C^2(D^+) \cap C^{1,\alpha}(D^+ \cup S)[C^2(D^-) \cap C^{1,\alpha}(D^- \cup S)]$, is a solution of equation (1.8) and satisfies the boundary condition

$$(u)^{\pm} = f(t), \tag{2.1}$$

where f is a given vector on the boundary, the signs $+$ and $--$ denote the limits from inside and from outside. Note that for the infinite domain D^- the vector U additionally satisfies the following conditions at infinity:

$$U = O(1), \quad \frac{\partial U}{\partial x_k} = O(\rho^{-2}), \quad k = 1, 2, \tag{2.2}$$

where $\rho^2 = x_1^2 + x_2^2$.

The direction of the external normal is assumed to be the positive direction of the normal, i.e., the direction from D^+ into D^-.

First we are to write a Fredholm integral equation for the first boundary value problem. Using formulas (1.14) and choosing $\varphi(z)$ and $\overline{\psi(z)}$ in the form

$$\begin{aligned}
\varphi(z) &= \frac{m^{-1}}{2\pi i} \int\limits_S \frac{\partial \ln \sigma}{\partial s(y)}\, g(y)\, dS, \\
\overline{\psi(z)} &= -\frac{1}{2\pi i} \int\limits_S \frac{\partial \ln \overline{\sigma}}{\partial s(y)}\, g(y)\, dS + \frac{\ell m^{-1}}{4\pi i} \int\limits_S \frac{\partial}{\partial s(y)} \frac{\zeta}{\overline{\sigma}}\, \overline{g(y)}\, dS,
\end{aligned}$$

after some simple transformations we have by (1.10) that

$$U = \frac{1}{\pi} \int\limits_S \frac{\partial \theta}{\partial s(y)}\, g(y)\, dS + \frac{\varepsilon^\top}{2\pi i} \int\limits_S \frac{\partial}{\partial s(y)} \frac{\sigma}{\overline{\sigma}}\, \overline{g(y)}\, dS, \tag{2.3}$$

$$V = -\frac{1}{\pi} \int\limits_S \frac{\partial \ln r}{\partial s(y)}\, g(y)\, dS + \frac{\varepsilon^\top}{2\pi} \int\limits_S \frac{\partial}{\partial s(y)} \frac{\sigma}{\overline{\sigma}}\, \overline{g(y)}\, dS, \tag{2.4}$$

where θ and r are defined from (1.28), $g(y)$ is the complex vector we seek for, while the values of σ and $\overline{\sigma}$ are given above (see §1).

Let us first investigate the first internal problem. Passing to the limit in (2.3) as $x \to t \in S$ and using the boundary condition (2.1), to define the vector g we obtain the following Fredholm integral equation of second order:

$$g(t) + \frac{1}{\pi} \int\limits_S \frac{\partial \theta}{\partial s(y)}\, g(y)\, dS + \frac{\varepsilon^\top}{2\pi i} \int\limits_S \frac{\partial}{\partial s(y)} \frac{\sigma}{\overline{\sigma}}\, \overline{g(y)}\, dS = f(t), \tag{2.5}$$

where $f \in C^{1,\beta}(s)$ $(\beta > 0)$ is a given vector on the boundary, and

$$\theta = \operatorname{arctg} \frac{y_2 - x_2}{y_1 - x_1}, \quad \sigma = t - \zeta, \quad \overline{\sigma} = \bar{t} - \bar{\zeta}, \quad t = t_1 + i t_2. \tag{2.6}$$

From (2.5) we have

$$\frac{\partial g}{\partial s(t)} + \frac{1}{\pi} \int\limits_S \frac{\partial^2 \theta}{\partial s(t) \partial s(y)}\, g(y)\, dS + \frac{\varepsilon^\top}{2\pi i} \int\limits_S \frac{\partial^2}{\partial s(t) \partial s(y)} \frac{\sigma}{\overline{\sigma}}\, \overline{g(y)}\, dS = \frac{\partial f}{\partial s(t)}\,.$$

Taking into account that $s \in C^{2,\alpha}$ $(\alpha > 0)$, the latter formula implies

$$\int\limits_S \frac{\partial^2 \theta}{\partial s(t) \partial s(y)}\, g(y)\, dS \in C^{0,\alpha}(S) \quad \text{and} \quad \int\limits_S \frac{\partial^2}{\partial s(t) \partial s(y)} \frac{\sigma}{\overline{\sigma}}\, \overline{g(y)}\, dS \in C^{0,\alpha}(S)$$

and since $f \in C^{1,\beta}$ $(0<\beta<\alpha\le 1)$, from (2.5) we obtain $g \in C^{1,\beta}(s)$.

Let us prove that the homogeneous equation corresponding to (2.5) has only a trivial solution. Assume that it has a nontrivial solution denoted by g_0. By the same reasoning as above we can easily find that $g_0 \in C^{1,\alpha}(s)$ and $NU_0 \in C^{0,\alpha}(s)$. Applying (1.37), we have $u_0(x)=C$, $x\in D^+$, where C is an arbitrary constant vector. But since $(U_0)^+\equiv 0$, we have $U_0(x)=0$. Taking into account that for $g_0\in C^{1,\alpha}(s)$, $(NU_0)^+-(NU_0)^-=0$, we obtain $(NU_0)^-=0$. Using now formula (1.40), we obtain $U_0(x)=C$ $x\in D^-$ for the domain D^-. Since the potential $U_0(x)$ is equal to zero at infinity, we have $U_0(x)=0$ for $x\in D^-$ and, using the formula $(U_0)^+-(U_0)^-=2g_0$, we obtain $g_0=0$. Therefore the homogeneous equation has only a trivial solution. We have proved that, by the first Fredholm theorem, equation (2.5) is solvable for an arbitrary right-hand part $f\in C^{1,\beta}(s)$ $(\beta>0)$.

One can prove that a solution of equation (2.5) exists iff $s\in C^{1,\alpha}$ and $f\in C^{1,\beta}(s)$, $0<\beta<\alpha\le 1$. The proof is the same as in [5].

Let us now consider the first external boundary value problem. Its solution is sought for in the form

$$W=U(x)+U(0), \tag{2.7}$$

where U is defined by formula (2.3), and

$$U(0)=\frac{1}{\pi}\int\limits_S \frac{\partial}{\partial s(y)}\ \operatorname{arctg}\frac{y_2}{y_1}\, g(y)\,dS+\frac{\varepsilon^\top}{2\pi i}\int\limits_S \frac{\partial}{\partial s(y)}\,\frac{\zeta}{\overline{\zeta}}\,\overline{g(y)}\,dS. \tag{2.8}$$

The origin is assumed to lie in the domain D^+.

Taking into account the boundary behavior of the potential $U(x)$ and the boundary condition (2.1), to define the unknown vector g we obtain from (2.7) the Fredholm integral equation of second order

$$-g(t)+\frac{1}{\pi}\int\limits_S \frac{\partial\theta}{\partial s(y)}\,g(y)\,dS+\frac{\varepsilon^\top}{2\pi i}\int\limits_S \frac{\partial}{\partial s(y)}\,\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\,dS+U(0)=f(t). \tag{2.9}$$

where θ, σ, $\overline{\sigma}$ are defined by (2.6). Quite in the same manner as above, one can prove that a solution of equation (2.9), if it exists, belongs to the class $C^{1,\beta}(s)$ for $s\in C^{2,\alpha}$ and $f^{1,\beta}(s)$, $0<\beta<\alpha\le 1$.

Let us show now that equation (2.9) is always solvable. For this it is sufficient that the homogeneous equation corresponding to (2.9) have only a trivial solution. Denote the homogeneous equation (which we do not write) by $(2.9)_0$ and assume that it has a solution different from zero which is denoted by g_0. Denote the corresponding potentials and values by W_0, U_0 and $U_0(0)$. Using (1.40), we obtain $W_0(x)=\alpha$. But $(W_0)^-=0$ and therefore $\alpha=0$ and $W_0(x)=0$, $x\in D^-$. Hence, for $x\to\infty$, we obtain

$$U_0(0)=0. \tag{2.10}$$

In that case, (2.7) implies $W\equiv U_0(x)=0$ and $NU_0(x)=0$, $x\in D^-$. Since under our restrictions $(NU_0)^+-(NU_0)^-=0$, and $(NU_0)^-=0$, we obtain

$(NU_0)^+ = 0$. Hence by (1.37) we have $U_0(x) = \beta$, $x \in D^+$. But by (2.10) $U_0(0) = 0$ and, obviously, $\beta = 0$. Thus $U_0(x) = 0$, $x \in D^+$. Taking into account that $(U_0)^+ - (U_0)^- = 2g_0$ and $(U_0)^+ = 0$, $(U_0)^- = 0$ we obtain $g_0 = 0$. Thus our assumption is not valid. Equation (2.9) has a solution for an arbitrary right-hand part.

As above, here, too, we can note that a solution of equation (2.9) exists iff $s \in C^{1,\alpha}$ and $f \in C^{1,\beta}(s)$, $0 < \beta < \alpha \leq 1$.

So far it has been assumed that the principal vector of external forces, stress components and rotation at infinity are equal to zero. The general case with these values given and different from zero is considered applying a reasoning similar to that used in [6]. When D^+ is a multiply-connected finite domain, the proof of the existence of solutions for this domain is easy and carried out as in [6].

3. The Second Boundary Value Problem

The second boundary value problem is posed as folows: Find, in the domain $D^+(D^-)$, a vector U which belongs to the class $C^2(D^+) \cap C^{1,\alpha}(\overline{D}^+)[C^2(D^-) \cap C^{1,\alpha}(D^- \cup S)]$, is a solution of equation (1.8) and satisfies the boundary condition

$$(TU)^{\pm} = F(t), \tag{3.1}$$

where F is a given vector on the boundary. For an infinite domain we have conditions (2.2).

To derive Fredholm integral equations of second order for the second boundary value problem is not difficult. Indeed, after substituting $\varkappa = 0$ into (1.25) and (1.27), we obtain

$$\begin{aligned} TU &= \frac{1}{\pi} \int\limits_S \frac{\partial \theta}{\partial s(x)}\, g(y)\, dS - \frac{H}{2\pi i} \int\limits_S \frac{\partial}{\partial s(x)}\, \frac{\sigma}{\overline{\sigma}}\, \overline{g(y)}\, dS, \\ TV &= -\frac{1}{\pi} \int\limits_S \frac{\partial \ln r}{\partial s(x)}\, g(y)\, dS - \frac{H}{2\pi} \int\limits_S \frac{\partial}{\partial s(x)}\, \frac{\sigma}{\overline{\sigma}}\, \overline{g(y)}\, dS, \end{aligned} \tag{3.2}$$

where U and V are defined as follows:

$$\begin{aligned} U &= \frac{m(A-2E)^{-1}}{2\pi i} \int\limits_S \ln \sigma g(y)\, dS - \frac{\mu^{-1}}{4\pi i} \int\limits_S \ln \overline{\sigma} g(y)\, dS \\ &\quad - \frac{\ell(A-2E)^{-1}}{4\pi i} \int\limits_S \frac{\sigma}{\overline{\sigma}}\, \overline{g(y)}\, dS, \\ V &= -\frac{m(A-2E)^{-1}}{2\pi} \int\limits_S \ln \sigma g(y)\, dS - \frac{\mu^{-1}}{4\pi} \int\limits_S \ln \overline{\sigma} g(y)\, dS \\ &\quad - \frac{\ell(A-2E)^{-1}}{4\pi} \int\limits_S \frac{\sigma}{\overline{\sigma}}\, \overline{g(y)}\, dS. \end{aligned} \tag{3.3}$$

Here, as above, $g(\overline{g})$ is the complex vector we seek for,

$$H = \begin{bmatrix} H_1, & H_2 \\ H_3, & H_4 \end{bmatrix},$$

where

$$\begin{aligned}
&H_1 = 1 - \frac{2\lambda_5}{\Delta_2 d_2}\Big[(a_1+c)A_3 + (a_2+c)(A_4-2)\Big], \\
&H_2 = \frac{2\lambda_5}{\Delta_2 d_2}\Big[(a_1+c)(A_2-2) + (a_2+c)A_2\Big], \\
&\Delta_2 = (A_1-2)(A_4-2) - A_2A_3 > 0, \quad H_3 = 1-H_1, \quad H_4 = 1-H_2, \\
&(A-2E)^{-1} = \frac{1}{\Delta_2}\begin{bmatrix} A_4-2, & -A_2 \\ -A_3 & A_1-2 \end{bmatrix}, \quad \mu^{-1} = \frac{1}{\Delta_1}\begin{bmatrix} \mu_2, & -\mu_3 \\ -\mu_3, & \mu_1 \end{bmatrix};
\end{aligned} \tag{3.4}$$

the other values contained in (3.2) and (3.3) are defined in the preceding paragraph.

Let us first consider the second boundary value problem for the domain D^+. By (3.1), to define the vector g we find from (3.2) that

$$-g(t) + \frac{1}{\pi}\int_S \frac{\partial\theta}{\partial s(t)}\, g(y)\, dS - \frac{H}{2\pi i}\int_S \frac{\partial}{\partial s(t)}\, \frac{\sigma}{\overline{\sigma}}\, \overline{g(y(}\, dS = F(t), \tag{3.5}$$

where θ, σ and $\overline{\sigma}$ are given from (2.6). We think that it is advisable to modify equation (3.5). To do so, we add to the left-hand side the expression

$$\frac{1}{2\pi}\left[\frac{\partial\theta}{\partial s(t)}\int_S g(y)\, dS - \frac{H}{2\pi i}\, \frac{\partial}{\partial s(t)}\, \frac{\sigma(t)}{\overline{\overline{\sigma(t)}}}\int_S \overline{g}\, dS\right] + \frac{1}{4\pi i}\begin{pmatrix}1\\1\end{pmatrix}\frac{\partial}{\partial s(t)}\, \frac{1}{\overline{t}}\cdot M,$$

where

$$M = \left[-i\,\frac{\partial}{\partial z}(U_1+U_2) + i\,\frac{\partial}{\partial \overline{z}}(\overline{U}_1+\overline{U}_2)\right]_{x_1=x_2=0}, \tag{3.6}$$

U_1 and U_2 and their conjugates are defined in (3.3). As above, it is assumed that the origin lies in the domain D^+.

Thus we obtain the equation

$$\begin{aligned}
-g(t) &+ \frac{1}{\pi}\int_S \frac{\partial\theta}{\partial s(t)}\, g(y)\, dS - \frac{H}{2\pi i}\int_S \frac{\partial}{\partial s(t)}\, \frac{\sigma}{\overline{\sigma}}\, \overline{g(y)}\, dS + \frac{1}{2\pi}\,\frac{\partial\theta(t)}{\partial s(t)}\int_S g(y)\, dS \\
&- \frac{H}{4\pi i}\, \frac{\partial}{\partial s(t)}\, \frac{\sigma(t)}{\overline{\overline{\sigma(t)}}}\int_S \overline{g(y)}\, dS + \frac{1}{4\pi i}\begin{pmatrix}1\\1\end{pmatrix}\frac{\partial}{\partial s(t)}\, \frac{1}{\overline{t}}\, M = F(t).
\end{aligned} \tag{3.7}$$

Let us now show that if equation (3.7) has a solution, then it is necessary that

$$\int_S g\, dS = 0 \tag{3.8}$$

and

$$M = 0, \tag{3.9}$$

provided that the principal vector and the principal moment of external forces are equal to zero.

Indeed, by some simple calculations, from (3.7) we obtain

$$\int_S g\,dS = \int_S F\,dS, \quad M = \int_S \operatorname{Re}\bar{t}(F_1 + F_2)dS. \tag{3.10}$$

If the principal vector $\int_S FdS$ and the principal moment $\int_S \operatorname{Re}\bar{t}(F_1 + F_2)dS$ are equal to zero, then $\int_S gdS = 0$ and $M = 0$, which was required to be shown.

Thus if the requirement that the principal vector and the principal moment of external forces be equal to zero is fulfilled, then any solution g of equation (3.7) is simultaneously a solution of the initial equation (3.5).

Let us now prove that equation (3.7) is always solvable. To this end, consider the homogeneous equation obtained from (3.7) for $F = 0$ and prove that it has no solutions different from zero. Let g_0 be any solution of this homogeneous equation. Since $F = 0$, it is obvious that conditions (3.8) and (3.9) are fulfilled for g_0. In that case the obtained homogeneous equation corresponds to the boundary condition

$$(TU_0(t))^+ = 0, \tag{3.11}$$

where $U_0(x)$ is obtained from (3.3) if g is replaced by g_0. Using (3.11) and (1.36) we obtain

$$U_0(x) = \alpha + i\beta_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} z, \quad x \in D^+, \tag{3.12}$$

where α is an arbitrary constant vector, and β_1 is an arbitrary constant value.

Since $M_0 = 0$, (3.12) implies $\beta_1 = 0$ and we obtain $U_0(x) = \alpha, \quad x \in D^+$.

Hence, in view of (1.24), we have

$$NU_0(x) = -im^{-1}\frac{\partial V_0(x)}{\partial s(x)} = 0$$

and

$$V_0(x) = \beta, \quad x \in D^+, \tag{3.13}$$

where $V_0(x)$ are defined from (3.3) if $g = g_0$ and β is an arbitrary constant vector. By (3.2), from (3.13) we obtain $TV_0(x) = 0$, $x \in D^+$, and since $(TV_0(t))^+ - (TV_0(t))^- = 0$, we have $(TV_0(x))^- = 0$. Applying now formula (1.39) we find

$$V_0(x) = \alpha + i\beta_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} z, \quad x \in D^-,$$

Hence, since $V_0(x)$ is bounded at infinity, we have $\beta_1 = 0$ and $V_0(x) = \alpha$, $x \in D^-$. In that case $U_0(x) = \beta$ and $TU_0(x) = 0$, $x \in D^-$. Taking into account that $(TU_0)^- - (TU_0)^+ = 2g_0$ and $(TU_0)^+ = 0$, $(TU_0)^- = 0$ we obtain $g_0 = 0$.

Thus we proved that the homogeneous equation corresponding to equation (3.7) has no solutions different from zero.

Therefore equation (3.7) has one and only one solution g. On substituting this value g into formula (3.3), we obtain a solution of the second boundary value problem provided that the requirement for the principal vector and the principal moment of external forces to be equal to zero is fulfilled. Displacements U are defined to within rigid disolacement, while stresses are defined precisely.

Let us now consider the second boundary value problem in the domain D^-. Its solution is to be sought for in the form

$$W(x) = U(x) - \frac{\mu^{-1}}{8\pi}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\bar{z}}\,M, \tag{3.14}$$

where U are defined from (3.3), and

$$M = -i\Big[\frac{\partial(V_1+V_2)}{\partial z} - \frac{\partial(\overline{V}_1+\overline{V}_2)}{\partial \bar{z}}\Big]_{x_1=x_2=0}, \tag{3.15}$$

V are given from (3.3).

From (3.14) we readily have $TW = TU(x) - \frac{1}{4\pi}\begin{pmatrix}1\\1\end{pmatrix}\frac{\partial}{\partial s(x)}\frac{1}{\bar{z}}\cdot M$.

Passing here to the limit as $z \to t \in S$ and taking into account (3.1), we obtain

$$\begin{aligned} g(t) + \frac{1}{\pi}\int\limits_S \frac{\partial\theta}{\partial s(t)}\,g(y)\,dS - \frac{H}{2\pi i}\int\limits_S \frac{\partial}{\partial s(t)}\,\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\,dS& \\ - \frac{1}{4\pi}\begin{pmatrix}1\\1\end{pmatrix}\frac{\partial}{\partial s(t)}\,\frac{1}{\bar{t}}\,M = F(t).& \end{aligned} \tag{3.16}$$

Performing integration, from (3.16) we esily have $2\int_S g\,dS = \int_S F\,dS$.

So far it has been assumed that the principal vector of external forces is equal to zero. This means that

$$\int\limits_S g\,dS = 0. \tag{3.17}$$

The latter condition implies that the vector $U(x)$ from (3.3) is unique and bounded.

Now let us show that equation (3.16) is always solvable. To this end, consider the homogeneous equation which is obtained from (3.16) when $F = 0$. We have to prove that this homogeneous equation has no solutions different from zero. Assume the contrary and denote by g_0 some solution of this homogeneous equation. Since $F = 0$, condition (3.17) is fulfilled for g_0.

Note that the homogeneous equation corresponds to the boundary condition $(TW_0(t))^- = 0$. Using formula (1.39), we have $W_0(x) = 0, \quad x \in D^-$, or, by

(3.14), we can write

$$U_0(x) - \frac{\mu^{-1}}{8\pi} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\bar{z}} M_0 = 0, \quad x \in D^-.$$

Obviously, for the conjugate vector we have

$$V_0(x) - \frac{i\mu^{-1}}{8\pi} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\bar{z}} M_0 = c, \quad x \in D^-, \tag{3.18}$$

where c is an arbitrary constant vector. Calculating the stress vector and taking into account (3.2), from (3.18) we obtain

$$-\frac{1}{\pi} \int_S \frac{\partial \ln r}{\partial s(t)} g_0(y)\, dS - \frac{H}{2\pi} \int_S \frac{\partial}{\partial s(t)} \frac{\sigma}{\overline{\sigma}} \overline{g_0(y)}\, dS$$
$$- \frac{i}{4\pi} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{\partial}{\partial s(t)} \frac{1}{\bar{t}} \cdot M_0 = 0, \quad t \in S.$$

Now we have to calculate the principal moment of this vector. After lengthy but obvious trasformations we have $M_0 = 0$.

We have thus proved that $V_0(x) = C$ for $x \in D^-$. But since $(TV_0)^- = (TV_0)^+ = 0$, by (1.36) we obtain

$$V_0(x) = \alpha + i\beta_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} z, \quad x \in D^+, \tag{3.19}$$

where α is an arbitrary constant vector, and β_1 is an arbitrary constant scalar.

Since $M_0 = 0$, (3.19) readily implies $0 = M_0 = 4\beta_1 = 0$.

In that case we have $V_0(x) = \alpha$, $x \in D^+$, and $U_0(x) = \beta$, $x \in D^+$, where β is a constant vector. Thus $(TU_0(t))^- - (TU_0(t))^+ = 2g_0 = 0$.

Therefore the homogeneous equation corresponding to equation (3.16) has no solutions different from zero. This means that equation (3.16) has one and only one solution when the principal vector of external forces is equal to zero.

Like in §2, here too we can note that a solution of the second boundary value exists iff the principal vector of external forces, and stress and rotation components are given values. It is understtod that in that case the displacement vector at infinity is unbounded.

The existence of solutions of the second boundary value problem can also be proved when D^+ is a finite multiply-connected domain.

4. The Third Boundary Value Problem

In the case of the third boundary value problem the following values are given at the boundary:

$$\frac{\partial}{\partial s(x)} \Big[u_3 - u_1 + i(u_4 - u_2) \Big], \quad (Tu)_2 + (Tu)_1 - i\Big[(Tu)_4 + (Tu)_3 \Big],$$

where u_1, u_2, u_3, u_4 are the components of the four-dimensional vector U, and $(TU)_1$, $(TU)_2$, $(TU)_3$, $(TU)_4$ are the components of the stress vector TU.

By virtue of (1.12), (1.14) and (1.21), these value can be rewritten as

$$\frac{\partial}{\partial s(x)}\left\{\begin{bmatrix} m_2-m_1, & m_3-m_2 \\ B_1+B_3, & B_2+B_4\end{bmatrix}\varphi(z)+\begin{bmatrix}\frac{\ell_5-\ell_4}{2}, & \frac{\ell_6-\ell_5}{2} \\ B_1+B_3, & B_2+B_4\end{bmatrix} z\overline{\varphi'(z)}\right.$$
$$\left.+\begin{bmatrix} -1, & 1 \\ 2(\mu_1+\mu_3), & 2(\mu_2+\mu_3)\end{bmatrix}\overline{\psi(z)}\right\}=F(x), \tag{4.1}$$

where

$$F(x)=\begin{pmatrix}\frac{\partial}{\partial s(x)}[u_3-u_1+i(u_4-u_2)] \\ (Tu)_2+(Tu)_4-i[(Tu)_1+(Tu)_3]\end{pmatrix}\equiv\begin{pmatrix}U_2-U_1 \\ (TU)_1+(TU)_2\end{pmatrix}, \tag{4.2}$$

U_1, U_2, $(TU)_1$ and $(TU)_2$ are defined from (1.12) and (1.21).

Now we can formulate the thirs boundary value problem: Find, in the domain $D^+(D^-)$, a vector U which belongs to the class $C^2(D^+)\cap C^{1,\alpha}(\overline{D^+})[C^2(D^-)\cap C^{1,\alpha}(D^-\cup S)]$, is a solution of equations (1.8), and satisfies the boundary condition

$$(F(t))^{\pm}=F(t), \tag{4.3}$$

where $F(t)$ is a given vector on the boundary. Conditions (2.2) are fulfilled at infinity.

For this problem we need to derive Fredholm integral equations of second order.

Let

$$\varphi(z)=\frac{1}{2\pi i\Delta_3}\begin{bmatrix} B_2+B_4, & m_2-m_3 \\ -(B_1+B_3), & m_2-m_1\end{bmatrix}\int_S \ln\sigma g(y)\,dS, \tag{4.4}$$

where g is the complex vector we seek for, and

$$\Delta_3=2(m_1+m_3-2m_2-\Delta_0 a_0)>0, \tag{4.5}$$

where $a_0=\mu_1+\mu_2+2\mu_3\equiv a_1+a_2+2c$. Note that the proof for $\Delta_3>0$ is given in [7].

After lengthy but elementary calculations we obtain

$$\frac{1}{\Delta_3}\begin{bmatrix}\frac{\ell_5-\ell_4}{2}, & \frac{\ell_6-\ell_5}{2} \\ B_1+B_3, & B_2+B_4\end{bmatrix}\begin{bmatrix} B_2+B_4, & m_2-m_3 \\ -(B_1+B_3), & m_2-m_1\end{bmatrix}=\begin{bmatrix}-K_0, & -\alpha_0 \\ 0, & 1\end{bmatrix}, \tag{4.6}$$

where

$$K_0=\frac{a_0(b_1b_2-d^2)}{2\Delta_3 d_1 d_2},\quad \alpha_0=\frac{\Delta_0}{\Delta_3}(\varepsilon_1+\varepsilon_3-\varepsilon_2-\varepsilon_4), \tag{4.7}$$

while ε_1, ε_2, ε_3, ε_4 are defined from (1.9).

In view of (4.4) and (4.6), expression (4.1) takes the form

$$\frac{\partial}{\partial s(x)}\Bigg\{\frac{1}{2\pi i}\int\limits_S \ln\sigma g(y)\,dS+\frac{K}{2\pi i}\int\limits_S \frac{z}{\overline{\sigma}}\,\overline{g(y)}\,dS + \begin{bmatrix}\dfrac{-1}{2(\mu_1+\mu_3)}, & \dfrac{1}{2(\mu_2+\mu_3)}\end{bmatrix}\overline{\psi(z)}\Bigg\}=F(x), \tag{4.8}$$

where

$$K=\begin{bmatrix}K_0, & \alpha_0\\ 0, & -1\end{bmatrix} \tag{4.9}$$

Let us now choose $\overline{\psi(z)}$ in (4.8) as follows:

$$\begin{bmatrix}\dfrac{-1}{2(\mu_1+\mu_3)}, & \dfrac{1}{2(\mu_2+\mu_3)}\end{bmatrix}\overline{\psi(z)}=-\frac{1}{2\pi i}\int\limits_S \ln\overline{\sigma} g(y)\,dS-\frac{K}{2\pi i}\int\limits_S \frac{\zeta}{\overline{\sigma}}\,g(y)\,dS,$$

Then (4.8) takes the final form

$$\frac{1}{\pi}\int\limits_S \frac{\partial\theta}{\partial s(x)}\,g(y)\,dS+\frac{K}{2\pi i}\int\limits_S \frac{\partial}{\partial s(x)}\,\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\,dS=F(x). \tag{4.10}$$

First we consider the third boundary value problem in the domain D^+. By the boundary condition, to define the unknown vector g we find from (4.10) the Fredholm integral equation of second order

$$-g(t)+\frac{1}{\pi}\int\limits_S \frac{\partial\theta}{\partial s(t)}\,g(y)\,dS+\frac{K}{2\pi i}\int\limits_S \frac{\partial}{\partial s(t)}\,\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\,dS=F(t), \tag{4.11}$$

where θ, σ and $\overline{\sigma}$ are defined from (2.6).

To investigate (4.11), its advisable to consider, instead of (4.11), the equation

$$-g(t)+\frac{1}{\pi}\int\limits_S \frac{\partial\theta}{\partial s(x)}\,g(y)\,dS+\frac{K}{2\pi i}\int\limits_S \frac{\partial}{\partial s(y)}\,\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\,dS-\frac{i}{2\pi}\begin{pmatrix}0\\1\end{pmatrix}\frac{\partial}{\partial s(t)}\,\frac{1}{\overline{t}}\,M + \frac{1}{2\pi}\left[\frac{\partial\theta(t)}{\partial s(t)}\int\limits_S g\,dS-\frac{H}{2i}\,\frac{\partial}{\partial s(t)}\,\frac{t}{\overline{t}}\int\limits_S \overline{g}\,dS\right]=F(t), \tag{4.12}$$

where

$$M=(-i)\Big(\frac{\partial U_2}{\partial z}-\frac{\partial\overline{U}_2}{\partial\bar{z}}\Big)_{x_1=x_2=0}, \tag{4.13}$$

and $\theta(t)$ are given from (2.6) at $y_1=y_2=0$.

From (4.12) we readily obtain

$$\int\limits_S g\,dS=\int\limits_S F\,dS,\quad M=\mathrm{Re}\int\limits_S \bar{t}F_2(t)\,dS. \tag{4.14}$$

Let us prove that equation (4.12) is always solvable. Assume the contrary and denote any solution of the homogeneous equation corresponding to (4.12) by g_0. On account of (4.14) we have

$$\int_S g_0\, dS = 0, \quad M_0 = 0, \tag{4.15}$$

where M_0 is obtained from (4.13) after replacing g by g_0.

In that case the homogeneous equations for (4.12) and (4.11) coincide, but to the homogeneous equation for (4.11) there corresponds the boundary condition $(F(t))^+ = 0$.

Using (1.16) and taking into account that $(U_{01})^+ - (U_{02})^+ = C$ and $\int_S (T\overline{U_0})^+ dS = 0$, we obtain

$$U_0(x) = \alpha + i\beta_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} z, \quad x \in D^+,$$

where α and β_1 are arbitrary constants. By (4.15) we have $U_0(x) = \alpha, \quad x \in D^+$.

The conjugate vector $V_0(x)$ defined from (1.14) has the form $V_0(x) = \beta, \quad x \in D^+$, where β is another constant value. By repeating the arguments used above for the vector U_0 we get

$$\begin{aligned} &(V_{01} - V_{02})^- = (V_{01} - V_{02})^+ = \alpha_1, \\ &\Big[(TV_0)_1 + (TV_0)_2\Big]^- = \Big[(TV_0)_1 + (TV_0)_2\Big]^+ = 0. \end{aligned}$$

Using now (1.39) for the vector V_0, we have

$$V_0(x) = \gamma + i\delta_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} z, \quad z \in D^-,$$

where γ is an arbitrary constant vector, and δ_0 is an arbitrary constant scalar. Since the vector $V_0(x)$ is equal to zero at infinity, we have $\delta_0 = 0$, $\gamma = 0$ and $V_0(x) = 0$, $x \in D^-$. In that case, the conjugate vector $U_0(x)$ is $U_0(x) = C$, $x \in D^-$. Applying (4.15), we have

$$U_0(x) = 0, \quad x \in D^-. \tag{4.18}$$

Taking into account (4.17) and (4.18) it is easy to obtain

$$\begin{aligned} &\frac{\partial}{\partial s(t)}(U_{01} - U_{02})^- - \frac{\partial}{\partial s(t)}(U_{01} - U_{02})^+ = 2g_{01} = 0, \\ &\Big[(TU_0)_1 + (TU_0)_2\Big]^- - \Big[(TU_0)_1 + (TU_0)_2\Big]^+ = 2g_{02} = 0. \end{aligned}$$

This means that $g_0 = 0$ and our assumption is not correct.

Thus we have proved that equation (4.12) is solvable for any right-hand part. By substitution this value of g into (4.11) we obtain a solution of the third boundary value problem when $\int_S F_1 dS = 0$ and the principal moment of external forces is equal to zero.

The investigation of the third internal value is finished.

Finally, let investigate the third boundary value problem in the domain D^-. Its solution will be sought for in the form

$$\frac{1}{\pi}\int_S \frac{\partial\theta}{\partial s(x)}\, g(y)\, dS + \frac{K}{2\pi i}\int_S \frac{\partial}{\partial s(x)}\,\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\, dS \\ - \frac{1}{2\pi}\begin{pmatrix}0\\1\end{pmatrix}\frac{\partial}{\partial s(x)}\,\frac{1}{\overline{z}}\, M = F(x), \tag{4.19}$$

where

$$M = (-i)\Big(\frac{\partial V_2}{\partial z} - \frac{\partial \overline{V}_2}{\partial \overline{z}}\Big)_{x_1=x_2=0}. \tag{4.20}$$

To define the vector g, from (4.19) we obtain the Fredholm integral equation of second order

$$g(t) + \frac{1}{\pi}\int_S \frac{\partial\theta}{\partial s(t)}\, g(y)\, dS + \frac{K}{2\pi i}\int_S \frac{\partial}{\partial s(t)}\,\frac{\sigma}{\overline{\sigma}}\,\overline{g(y)}\, dS \\ - \frac{1}{2\pi}\begin{pmatrix}0\\1\end{pmatrix}\frac{\partial}{\partial s(t)}\,\frac{1}{\overline{t}}\, M = F(t). \tag{4.21}$$

After obvious transformations, the latter equation gives

$$2\int_S g\, dS = \int_S F\, dS. \tag{4.22}$$

Let us prove that equation (4.21) is always solvable. Assume that the homogeneous equation corresponding to (4.21) has a nonzero solution denoted by g_0. Since $F = 0$, (4.22) implies $\int_S g_0\, dS = 0$.

This means that both U_0 and the conjugate vector V_0 are equal to zero at infinity. Applying (1.39), we obtain $U_0(x) = 0, \quad V_0(x) = 0$.

On account of (4.21), for $F = 0$ we have

$$-\frac{1}{\pi}\int_S \frac{\partial \ln r}{\partial s(t)}\, g_0\, dS - \frac{K}{2\pi}\int_S \frac{\partial}{\partial s(t)}\,\frac{\sigma}{\overline{\sigma}}\,\overline{g}_0\, dS - \frac{i}{2\pi}\begin{pmatrix}0\\1\end{pmatrix}\frac{\partial}{\partial s(t)}\,\frac{1}{\overline{t}}\, M_0 = 0. \tag{4.23}$$

The latter equation implies $M_0 = 0$. Now (4.23) takes the form $(TV_0(t))^+ = (TV_0(t))^- = 0$. Applying (1.36) for the vector V_0 we obtain

$$V_0(x) = \alpha + i\beta_1\begin{pmatrix}1\\1\end{pmatrix} z, \quad x \in D^+.$$

Since $M_0 = 0$, we have $\beta_1 = 0$ and $V_0(x) = \alpha$, from which it follows that $U_0(x) = \beta$, $x \in D^+$, where β is an arbitrary constant vector. Now by the boundary condition we have

$$\frac{\partial}{\partial s(t)}(U_{01}-U_{02})^{-}-\frac{\partial}{\partial s(t)}(U_{01}-U_{02})^{+}=2g_{01}=0,$$
$$\Big[(TU_0)_1+(TU_0)_2\Big]^{-}-\Big[(TU_0)_1+(TU_0)_2\Big]^{+}=2g_{02}=0.$$

Thus we have proved that the homogeneous equation corresponding to (4.21) has only a trivial solution. By choosing among solutions of (4.21) those satisfying the condition $\int_S F_1 dS=0$ we obtain a solution of the third boundary value problem in the domain D^-.

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