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ON ESTIMATING INTEGRAL MODULI OF CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES IN TERMS OF FOURIER COEFFICIENTS

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ABSTRACT. Inequalities are derived which enable one to estimate integral moduli of continuity of functions of several variables in terms of Fourier coefficients.

In the paper inequalities are derived which give estimates of integral moduli of continuity in terms of Fourier coefficients. These inequalities generalize S. A. Telyakovskiĭ's results [1] to the multi-dimansional case. For simplicity and brevity we shall limit our discussion to the two-dimensional case.

First we give some one-dimensional theorems for the series

$$\frac{a_0}{2}+\sum_{i=1}^{\infty} a_i \cos ix, \tag{1}$$

$$\sum_{i=1}^{\infty} a_i \sin ix. \tag{2}$$

Let

$$\begin{aligned} &\Delta a_i = a_i - a_{i+1}, \\ &\delta a_i = \sum_{\nu=1}^{\left[\frac{i}{2}\right]} \frac{\Delta a_{i-\nu}-\Delta a_{i+\nu}}{\nu}, \quad i \geq 2; \quad \delta a_i = 0, \quad i<2, \end{aligned} \tag{3}$$

$$\alpha_{i,m}(p) = \begin{cases} (im^{-1})^p, & 1 \leq i \leq m, \\ 1, & i > m, \\ 0, & i < 1. \end{cases} \tag{4}$$

In [1] and [2] the following theorems are proved.

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Theorem A$_1$. *Let*

$$\lim_{i\to\infty} a_i = 0,$$

$$\sum_{i=0}^{\infty} \big(|\Delta a_i| + |\delta a_i|\big) < \infty.$$

Then series (1) *converges for all* $x \neq 0$ *and for its sum* $f(x)$ *we have*

$$\int_{-\pi}^{\pi} |f(x)|dx = O\bigg(\sum_{i=0}^{\infty} \big(|\Delta a_i| + |\delta a_i|\big)\bigg),$$

$$\omega_p\Big(f, \frac{1}{m}\Big)_L = O\bigg(\sum_{i=0}^{\infty} \alpha_{i,m}(p)\big(|\Delta a_i| + |\delta a_i|\big)\bigg), \tag{5}$$

where $\omega_p\big(f, \frac{1}{m}\big)_L$ *is an integral modulus of continuity of order* p.

Theorem A$_2$. *Let the conditions of Theorem* A_1 *be fulfilled and*

$$\sum_{i=1}^{\infty} \frac{|a_i|}{i} < \infty.$$

Then series (2) *converges for all* $x \in (-\pi, \pi]$ *and for its sum* $g(x)$ *we have the estimates*

$$\int_{\frac{\pi}{2\mu+1}}^{\pi} |g(x)|dx = \sum_{i=1}^{\mu} \frac{|a_i|}{i} + O\bigg(\sum_{i=0}^{\infty} \big(|\Delta a_i| + |\delta a_i|\big)\bigg),$$

$$\omega_p\Big(g, \frac{1}{m}\Big) = \frac{2^p}{\pi} \sum_{i=m}^{\infty} \frac{|a_i|}{i} + O\bigg(\sum_{i=0}^{\infty} \alpha_{i,m}(p)\big(|\Delta a_i| + |\delta a_i|\big)\bigg). \tag{6}$$

Note that in formulas (5) and (6) the coefficients in the O-terms depend only on p.

Now let a double sequence of numbers $a_{i,j}$, $i, j = 0, 1, \dots$, be given and set

$$\Delta_1 a_{i,j} = a_{i,j} - a_{i+1,j}, \quad \Delta_2 a_{i,j} = a_{i,j} - a_{i,j+1},$$

$$\Delta_1 \Delta_2 a_{i,j} = \Delta_1(\Delta_2 a_{i,j}) = \Delta_2(\Delta_1 a_{i,j}),$$

$$\Delta_1^2 a_{i,j} = \Delta_1(\Delta_1 a_{i,j}), \quad \Delta_2^2 a_{i,j} = \Delta_2(\Delta_2 a_{i,j}),$$

$$\delta_1 a_{i,j} = \sum_{\nu=1}^{[\frac{i}{2}]} \frac{\Delta_1 a_{i-\nu,j} - \Delta_1 a_{i+\nu,j}}{\nu}, \quad i \geq 2;\ \delta_1 a_{i,j} = 0, \quad i < 2,$$

$$\delta_2 a_{i,j} = \sum_{\mu=1}^{[\frac{j}{2}]} \frac{\Delta_2 a_{i,j-\mu} - \Delta_2 a_{i,j+\mu}}{\mu}, \;\; j \geq 2; \; \delta_2 a_{i,j} = 0, \;\; j < 2,$$
$$\delta_1 \Delta_1 a_{i,j} = \delta_1(\Delta_1 a_{i,j}) = \Delta_1(\delta_1 a_{i,j}) = \Delta_1 \delta_1 a_{i,j},$$
$$\delta_1 \Delta_2 a_{i,j} = \delta_1(\Delta_2 a_{i,j}) = \Delta_2(\delta_1 a_{i,j}) = \Delta_2 \delta_1 a_{i,j},$$
$$\delta_2 \Delta_1 a_{i,j} = \delta_2(\Delta_1 a_{i,j}) = \Delta_1(\delta_2 a_{i,j}) = \Delta_1 \delta_2 a_{i,j},$$
$$\delta_2 \delta_1 a_{i,j} = \delta_2(\delta_1 a_{i,j}) = \delta_1(\delta_2 a_{i,j}) = \delta_1 \delta_2 a_{i,j}.$$

Given a function $f \in L(T^2)$, $T = (-\pi, \pi]$, consider the expressions

$$\Delta_t^{(p,0)} f(x,y) = \sum_{\mu=0}^{p} (-1)^\mu \binom{p}{\mu} f\big(x + (p-2\mu)t, y\big),$$
$$\Delta_s^{(0,q)} f(x,y) = \sum_{\nu=0}^{q} (-1)^\nu \binom{q}{\nu} f\big(x, y + (p-2\nu)s\big),$$
$$\Delta_{t,s}^{(p,q)} f(x,y) = \Delta_t^{(p,0)} \big(\Delta_s^{(0,q)} f(x,y)\big) = \Delta_s^{(0,q)} \big(\Delta_t^{(p,0)} f(x,y)\big),$$
$$\omega^{(p,0)}(h,f) = \sup_{|t| \leq h} \int_{T^2} \big|\Delta_t^{(p,0)} f(x,y)\big| dx\, dy,$$
$$\omega^{(0,q)}(\eta,f) = \sup_{|s| \leq \eta} \int_{T^2} \big|\Delta_s^{(0,q)} f(x,y)\big| dx\, dy,$$
$$\omega^{(p,q)}(h,\eta,f) = \sup_{\substack{|t| \leq h \\ |s| \leq \eta}} \int_{T^2} \big|\Delta_{t,s}^{(p,q)} f(x,y)\big| dx\, dy.$$

Consider the series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \lambda_{i,j} a_{i,j} \cos ix \cos jy, \tag{7}$$

where $\lambda_{0,0} = \frac{1}{4}$, $\lambda_{0,j} = \lambda_{i,0} = \frac{1}{2}$.

Theorem 1. *Let*

$$\lim_{i+j \to \infty} a_{i,j} = 0, \tag{8}$$
$$A_{i,j} = |\Delta_1 \Delta_2 a_{i,j}| + |\delta_1 \Delta_2 a_{i,j}| + |\Delta_1 \delta_2 a_{i,j}| + |\delta_1 \delta_2 a_{i,j}|, \tag{9}$$
$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_{i,j} < \infty. \tag{9'}$$

Then series (7) *converges for all* $x \neq 0$, $y \neq 0$ *and for its sum* $f(x,y)$ *we have*

$$\int\limits_{T^2} |f(x,y)|dx\,dy = O\biggl(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{i,j}\biggr), \tag{10}$$

$$\int\limits_{T}\biggl(\sup_{|s|\le\frac{1}{n}}\int\limits_{T}|\Delta_s^{(0,q)}f(x,y)|dy\biggr)dx = O\biggl(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{j,n}(q)A_{i,j}\biggr) = O(B_n), \tag{11}$$

$$\int\limits_{T}\biggl(\sup_{|t|\le\frac{1}{m}}\int\limits_{T}|\Delta_t^{(p,0)}f(x,y)|dx\biggr)dy = O\biggl(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{i,m}(p)A_{i,j}\biggr) = O(C_m), \tag{12}$$

$$\sup_{|t|\le\frac{1}{m}}\int\limits_{T}\biggl(\sup_{|s|\le\frac{1}{n}}\int\limits_{T}|\Delta_{t,s}^{(p,q)}f(x,y)|dy\biggr)dx =$$

$$= O\biggl(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{i,m}(p)\alpha_{j,n}(q)A_{i,j}\biggr) = O(D_{m,n}), \tag{13}$$

$$\sup_{|s|\le\frac{1}{n}}\int\limits_{T}\biggl(\sup_{|t|\le\frac{1}{m}}\int\limits_{T}|\Delta_{t,s}^{(p,q)}f(x,y)|dx\biggr)dy = O(D_{m,n}). \tag{14}$$

Note that the coefficient in the O-term depends only on q in (11), only on p in (12), and only on p and q in (13) and (14).

Proof. (9) and (9′) imply

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_1\Delta_2 a_{i,j}| < \infty. \tag{15}$$

This and (8) give for each $j \ge 0$

$$\sum_{i=0}^{\infty}|\Delta_1 a_{i,j}| = \sum_{i=0}^{\infty}\Bigl|\sum_{\nu=j}^{\infty}\Delta_2\Delta_1 a_{i,\nu}\Bigr| \le \sum_{i=0}^{\infty}\sum_{\nu=j}^{\infty}|\Delta_1\Delta_2 a_{i,\nu}| < \infty \tag{16}$$

and

$$\lim_{j\to\infty}\sum_{i=0}^{\infty}|\Delta_1 a_{i,j}| = 0. \tag{16′}$$

In a similar manner, for any $i \ge 0$ we obtain

$$\sum_{j=0}^{\infty}|\Delta_2 a_{i,j}| < \infty$$

and

$$\lim_{i\to\infty}\sum_{j=0}^{\infty}|\Delta_2 a_{i,j}|=0. \tag{17}$$

Further, applying the Hardy transformation, we have

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\lambda_{i,j}a_{i,j}\cos ix\cos jy=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}D_i(x)D_j(y)\Delta_1\Delta_2 a_{i,j}+$$

$$+\sum_{i=0}^{m-1}D_i(x)\cos ny\Delta_1 a_{i,n}+\sum_{j=0}^{n-1}D_j(y)\cos mx\Delta_2 a_{m,j}+D_m(x)D_n(y)a_{m,n}.$$

Now, using the property of Dirichlet kernels and (8), (15), (16′), (17), one can easily verify that series (7) converges for all $x\neq 0$, $y\neq 0$. Moreover, (8) and (16) imply that the series

$$\frac{a_{0,j}}{2}+\sum_{i=1}^{\infty}a_{i,j}\cos ix=f_j(x) \tag{18}$$

converges for each $j\geq 0$ and all $x\neq 0$, $y\neq 0$.

As it is known (see, e.g., [3], p. 14) if a double series $\sum\limits_{i,j=0}^{\infty}u_{ij}$ is convergent to S and the series $\sum\limits_{i=0}^{\infty}u_{ij}$ is convergent for each j, then the double series is repeatedly convergent to S, i.e.,

$$\sum_{i=0}^{\infty}\left(\sum_{i=0}^{\infty}u_{ij}\right)=S.$$

Therefore, series (7) will converge repeatedly and for its sum $f(x,y)$ the equality

$$f(x,y)=\frac{f_0(x)}{2}+\sum_{j=1}^{\infty}f_j(x)\cos jy \tag{19}$$

will be fulfilled. Hence, in turn, it follows that for each $x\neq 0$

$$\lim_{j\to\infty}f_j(x)=0 \tag{20}$$

and for each fixed $t\in\left[-\frac{1}{n},\frac{1}{n}\right]$ we have

$$\Delta_t^{(p,0)}f(x,y)=\frac{\Delta_t^{(p,0)}f_0(x)}{2}+\sum_{j=1}^{\infty}\Delta_t^{(p,0)}f_j(x)\cos jy \tag{21}$$

for all (except the finite number of values of) x.

From (18) we obtain

$$\Delta f_j(x) = \frac{\Delta_2 a_{0,j}}{2} + \sum_{i=1}^{\infty} \Delta_2 a_{i,j} \cos ix, \tag{22}$$

$$\delta f_j(x) = \frac{\delta_2 a_{0,j}}{2} + \sum_{i=1}^{\infty} \delta_2 a_{i,j} \cos ix, \tag{23}$$

while (9) and (9') imply

$$\sum_{i=0}^{\infty} \big(|\Delta_1 \Delta_2 a_{i,j}| + |\delta_1 \Delta_2 a_{i,j}| + |\Delta_1 \delta_2 a_{i,j}| + |\delta_1 \delta_2 a_{i,j}| \big) < \infty \tag{24}$$

for each $j \geq 0$.

Moreover, for each $j \geq 0$ (8) yields

$$\lim_{i \to \infty} \Delta_2 a_{i,j} = 0, \quad \lim_{i \to \infty} \delta_2 a_{i,j} = 0. \tag{25}$$

In view of the latter two relations Theorem A_1 being applied to (22) and (23) results

$$\int_T |\Delta f_j(x)| dx = O\bigg(\sum_{i=0}^{\infty} \big(|\Delta_1 \Delta_2 a_{i,j}| + |\delta_1 \Delta_2 a_{i,j}| \big) \bigg), \tag{26}$$

$$\int_T |\delta f_j(x)| dx = O\bigg(\sum_{i=0}^{\infty} \big(|\Delta_1 \delta_2 a_{i,j}| + |\delta_1 \delta_2 a_{i,j}| \big) \bigg), \tag{27}$$

$$\begin{aligned} \int_T \big|\Delta_t^{(p,0)}(\Delta f_j(x))\big| dx = O\bigg(\sum_{i=0}^{\infty} \alpha_{i,m}(p) \big(|\Delta_1 \Delta_2 a_{i,j}| + \\ + |\delta_1 \Delta_2 a_{i,j}| \big) \bigg), \quad |t| \leq \frac{1}{m}, \end{aligned} \tag{28}$$

$$\begin{aligned} \int_T \big|\Delta_t^{(p,0)}(\delta f_j(x))\big| dx = O\bigg(\sum_{i=0}^{\infty} \alpha_{i,m}(p) \big(|\Delta_1 \delta_2 a_{i,j}| + \\ + |\delta_1 \delta_2 a_{i,j}| \big) \bigg), \quad |t| \leq \frac{1}{m}. \end{aligned} \tag{29}$$

By summing (26) and (27) we obtain

$$\int_T \sum_{j=0}^{\infty} \big(|\Delta f_j(x) + |\delta f_j(x)| \big) dx = O\bigg(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \big(|\Delta_1 \Delta_2 a_{i,j}| +$$

$$+|\delta_1\Delta_2 a_{i,j}| + |\Delta_1\delta_2 a_{i,j}| + |\delta_1\delta_2 a_{i,j}|\big)\Big).$$

Since the right-hand side of this inequality is finite by virtue of (9) and (9′), we have

$$\sum_{j=0}^{\infty}\big(|\Delta f_j(x)| + |\delta f_j(x)|\big) < \infty$$

for almost all $x \in T$. Thus for each fixed $|t| \le \frac{1}{m}$ we obtain

$$\sum_{j=0}^{\infty}\Big(\big|\Delta(\Delta_t^{(p,0)} f_j(x))\big| + \big|\delta(\Delta_t^{(p,0)} f_j(x))\big|\Big) < \infty$$

for almost all x.

Using the latter two relations and (20), by virtue of Theorem A_1 we have for series (19), (21)

$$\int_T |f(x,y)|dy = O\bigg(\sum_{j=0}^{\infty}\big(|\Delta f_j(x)| + |\delta f_j(x)|\big)\bigg),$$

$$\sup_{|s|\le\frac{1}{n}}\int_T \big|\Delta_s^{(0,q)} f(x,y)\big|dy = O\bigg(\sum_{j=0}^{\infty}\alpha_{j,n}(q)\big(|\Delta f_j(x)| + |\delta f_j|\big)\bigg),$$

$$\sup_{|s|\le\frac{1}{n}}\int_T \big|\Delta_s^{(0,q)}(\Delta_t^{(p,0)} f(x,y))\big|dy = \sup_{|s|\le\frac{1}{n}}\int_T \big|\Delta_{t,s}^{(p,q)} f(x,y))\big|dx\,dy =$$

$$= O\bigg(\sum_{j=0}^{\infty}\alpha_{j,n}(q)\Big(\big|\Delta(\Delta_t^{(p,0)} f_j(x))\big| + \big|\delta(\Delta_t^{(p,0)} f_j(x))\big|\Big)\bigg).$$

Integrating these inequalities with respect to x and using (26)–(29), we obtain the validity of (10), (11), and (13).

If instead of series (18) we consider the series

$$\frac{a_{i,0}}{2} + \sum_{j=1}^{\infty} a_{i,j}\cos jy,$$

then by applying arguments similar to those above we shall obtain the validity of (12), (14). □

Theorem 2. *Let the conditions of Theorem* 1 *be fulfilled and*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{|a_{i,j}|}{ij} + \frac{|\Delta_1 a_{i,j}| + |\delta_1 a_{ij}|}{j} + \frac{|\Delta_2 a_{i,j}| + |\delta_2 a_{i,j}|}{i}\right) < \infty. \quad (30)$$

Then the series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j}\sin ix\sin jy \tag{31}$$

converges for all $x\in T^2$ *and for its sum* $g(x,y)$ *the following estimates are valid:*

$$\begin{aligned}\int\limits_{\frac{\pi}{2\mu+1}}^{\pi}\int\limits_{\frac{\pi}{2\nu+1}}^{\pi}|g(x,y)|dx\,dy &= \sum_{i=1}^{\mu}\sum_{j=1}^{\nu}\frac{|a_{i,j}|}{ij}+O\bigg(\sum_{i=1}^{\infty}\sum_{j=1}^{\nu}\frac{|\Delta_1 a_{i,j}|+|\delta_1 a_{i,j}|}{j}\bigg)+\\ &+O\bigg(\sum_{i=1}^{\mu}\sum_{j=1}^{\infty}\frac{|\Delta_2 a_{i,j}|+|\delta_2 a_{i,j}|}{i}\bigg)+O\bigg(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}A_{i,j}\bigg),\end{aligned} \tag{32}$$

$$\begin{aligned}\int\limits_{\frac{\pi}{2\mu+1}}^{\pi}\bigg(\sup_{|s|\le\frac1n}\int\limits_T \Delta_s^{(0,q)}g(x,y)|dy\bigg)dx &= \frac{2^q}{\pi}\sum_{i=1}^{\mu}\sum_{j=n}^{\infty}\frac{|a_{i,j}|}{ij}+\\ &+O\bigg(\sum_{i=1}^{\infty}\sum_{j=n}^{\infty}\frac{|\Delta_1 a_{i,j}|+|\delta_1 a_{i,j}|}{j}\bigg)+\\ &+O\bigg(\sum_{i=1}^{\mu}\sum_{j=1}^{\infty}\alpha_{j,n}(q)\frac{|\Delta_2 a_{i,j}|+|\delta_2 a_{i,j}|}{i}\bigg)+O(B_n)=O(E_{n,\mu}),\end{aligned} \tag{33}$$

$$\begin{aligned}\int\limits_{\frac{1}{2\nu+1}}^{\pi}\bigg(\sup_{|t|\le\frac1m}\int\limits_T |\Delta_t^{(p,0)}g(x,y)|dx\bigg)dy &= \frac{2^p}{\pi}\sum_{i=m}^{\infty}\sum_{j=1}^{\nu}\frac{|a_{i,j}|}{ij}+\\ &+O\bigg(\sum_{i=m}^{\infty}\sum_{j=1}^{\infty}\frac{|\Delta_2 a_{i,j}|+|\delta_2 a_{i,j}|}{i}\bigg)+\\ &+O\bigg(\sum_{i=1}^{\infty}\sum_{j=1}^{\nu}\alpha_{i,m}(p)\frac{|\Delta_1 a_{i,j}|+|\delta_1 a_{i,j}|}{j}\bigg)+O(C_m)=O(F_{m,\nu}),\end{aligned} \tag{34}$$

$$\begin{aligned}\sup_{|t|\le\frac1m}\int\limits_T\bigg(\sup_{|s|\le\frac1n}\int\limits_T |\Delta_{t,s}^{(p,q)}g(x,y)|dy\bigg)dx &= \frac{2^{p+q}}{\pi^2}\sum_{i=m}^{\infty}\sum_{j=n}^{\infty}\frac{|a_{i,j}|}{ij}+\\ &+O\bigg(\sum_{i=1}^{\infty}\sum_{j=n}^{\infty}\alpha_{i,m}(p)\frac{|\Delta_1 a_{i,j}|+|\delta_1 a_{i,j}|}{j}\bigg)+\\ &+O\bigg(\sum_{i=m}^{\infty}\sum_{j=1}^{\infty}\alpha_{j,n}(q)\frac{|\Delta_2 a_{i,j}|+|\delta_2 a_{i,j}|}{i}\bigg)+O(D_{m,n})=O(G_{m,n}),\end{aligned} \tag{35}$$

$$\sup_{|t|\le\frac{1}{n}} \int_T \left(\sup_{|t|\le\frac{1}{m}} \int_T |\Delta_{t,s}^{(p,q)} g(x,y)|dx \right) dy = O(C_{m,n}). \tag{36}$$

Note that the coefficient in the O-term depends only on q in (33), only on p in (34) and only on p and q in (35) and (36).

Proof. The convergence of the series

$$\sum_{i=1}^{\infty} a_{i,j} \sin ix$$

and series (31) is proved in the same manner as that of series (18) and (7). Denote their sums by $g_j(x)$ and $g(x,y)$, respectively. The proof of the equalities

$$g_j(x) = \sum_{i=1}^{\infty} a_{i,j} \sin ix, \tag{18$'$}$$

$$g(x,y) = \sum_{j=1}^{\infty} g_j(x) \sin jy \tag{19$'$}$$

repeats that of equalities (18) and (19).

The validity of the equalities

$$\lim_{j\to\infty} g_j(x) = 0, \tag{20$'$}$$

$$\Delta_t^{(p,0)} g(x,y) = \sum_{j=1}^{\infty} \Delta_t^{(p,0)} g_j(x) \sin jy, \quad |t| \le \frac{1}{m}, \tag{21$'$}$$

$$\Delta g_j(x) = \sum_{i=1}^{\infty} \Delta_2 a_{i,j} \sin ix, \tag{22$'$}$$

$$\delta g_j(x) = \sum_{i=1}^{\infty} \delta_2 a_{i,j} \sin ix, \tag{23$'$}$$

is obtained by using the same arguments as for equalities (20)–(23).

Inequality (30) implies that for each $j \ge 1$

$$\sum_{i=1}^{\infty} \frac{|\Delta_2 a_{i,j}|}{i} < \infty, \quad \sum_{i=1}^{\infty} \frac{|\delta_2 a_{i,j}|}{i} < \infty, \tag{37}$$

$$\sum_{i=1}^{\infty} \frac{|a_{i,j}|}{i} < \infty, \quad \sum_{i=1}^{\infty} |\Delta_1 a_{i,j}| < \infty, \quad \sum_{i=1}^{\infty} |\delta_1 a_{i,j}| < \infty. \tag{38}$$

Using (8), (24), (25), (37), and (38), by virtue of Theorem A_2 we obtain for series (18′), (22′), (23′)

$$\int\limits_{\frac{\pi}{2\mu+1}}^{\pi} |g_j(x)|dx = \sum_{i=1}^{\mu} \frac{|a_{i,j}|}{i} + O\Big(\sum_{i=1}^{\infty} \big(|\Delta_1 a_{i,j}| + |\delta_1 a_{i,j}|\big)\Big), \tag{39}$$

$$\int\limits_{\frac{\pi}{2\mu+1}}^{\pi} |\Delta g_j(x)|dx = \sum_{i=1}^{\mu} \frac{|\Delta_2 a_{i,j}|}{i} + O\Big(\sum_{i=1}^{\infty} \big(|\Delta_1 \Delta_2 a_{i,j}| + |\delta_1 \Delta_2 a_{i,j}|\big)\Big), \tag{40}$$

$$\int\limits_{\frac{\pi}{2\mu+1}}^{\pi} |\delta g_j(x)|dx = \sum_{i=1}^{\mu} \frac{|\delta_2 a_{i,j}|}{i} + O\Big(\sum_{i=1}^{\infty} \big(|\Delta_1 \delta_2 a_{i,j}| + |\delta_1 \delta_2 a_{i,j}|\big)\Big), \tag{41}$$

$$\int\limits_{T} \big|\Delta_t^{(p,0)} g_j(x)\big|dx = \frac{2^p}{\pi} \sum_{i=m}^{\infty} \frac{|a_{i,j}|}{i} + O\Big(\sum_{i=1}^{\infty} \alpha_{i,m}(p)\big(|\Delta_1 a_{i,j}| + {}$$
$$+ |\delta_1 a_{i,j}|\big)\Big), \quad |t| \le \frac{1}{m}, \tag{42}$$

$$\int\limits_{T} \big|\Delta_t^{(p,0)} (\Delta g_j(x))\big|dx = \frac{2^p}{\pi} \sum_{i=m}^{\infty} \frac{|\Delta_2 a_{i,j}|}{i} + O\big(\sum_{i=1}^{\infty} \alpha_{i,m}(p)\big(|\Delta_1 \Delta_2 a_{i,j}| + {}$$
$$+ |\delta_1 \Delta_2 a_{i,j}|\big)\Big), \quad |t| \le \frac{1}{m}, \tag{43}$$

$$\int\limits_{T} \big|\Delta_t^{(p,0)} (\delta g_j(x))\big|dx = \frac{2^p}{\pi} \sum_{i=m}^{\infty} \frac{|\delta_2 a_{i,j}|}{i} + O\Big(\sum_{i=1}^{\infty} \alpha_{i,m}(p)\big(|\Delta_1 \delta_2 a_{i,j}| + {}$$
$$+ |\delta_1 \delta_2 a_{i,j}|\big)\Big), \quad |t| \le \frac{1}{m}. \tag{44}$$

After multiplying (39) by $1/j$, passing to the limit in (39)–(41) as $\mu \to \infty$, and then summing them with respect to the index j, we obtain

$$\int\limits_{T} \Big(\sum_{j=1}^{\infty} \Big(\frac{|g_j|}{j} + |\Delta g_j(x)| + |\delta g_j(x)|\Big)\Big)dx = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{|a_{i,j}|}{ij} + {}$$
$$+ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{|\Delta_2 a_{i,j}| + |\delta_2 a_{i,j}|}{i} + O\Big(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{|\Delta_1 a_{i,j}| + |\delta_1 a_{i,j}|}{j}\Big) + {}$$
$$+ O\Big(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \big(|\Delta_1 \Delta_2 a_{i,j}| + |\delta_1 \delta_2 a_{i,j}| + |\Delta_1 \delta_2 a_{i,j}| + |\delta_1 \delta_2 a_{i,j}|\big)\Big).$$

Since by virtue of (9), (9′), and (30) the right-hand side of this inequality is finite, we have

$$\sum_{j=1}^{\infty}\Big(\frac{|g_j|}{j}+|\Delta g_j(x)|+|\delta g_j(x)|\Big)<\infty$$

for almost all $x\in T$. Hence for each fixed $|t|\leq\frac{1}{m}$ we have

$$\sum_{j=1}^{\infty}\Big(\frac{|\Delta_t^{(p,0)}g_j|}{j}+\big|\Delta(\Delta_t^{(p,0)}g_j(x))\big|+\big|\delta(\delta_t^{(p,0)}g_j(x))\big|\Big)<\infty$$

for almost all x.

By virtue of Theorem A_2 the latter two relations and (20′) imply for series (19′) and (21′) that

$$\int\limits_{\frac{\pi}{2\mu+1}}^{\pi}|g(x,y)|dy=\sum_{j=1}^{\nu}\frac{|g_j|}{j}+O\bigg(\sum_{j=1}^{\infty}\big(|\Delta g_j(x)|+|\delta g_j|\big)\bigg),$$

$$\sup_{|s|\leq\frac{1}{n}}\int\limits_T\big|\Delta_s^{(0,q)}g(x,y)\big|dy=\frac{2^q}{\pi}\sum_{j=n}^{\infty}\frac{|g_j|}{j}+O\bigg(\sum_{j=1}^{\infty}\alpha_{j,n}(q)\big(|\Delta g_j|+|\delta g_j|\big)\bigg),$$

$$\sup_{|s|\leq\frac{1}{n}}\int\limits_T\big|\Delta_{t,s}^{(p,q)}g(x,y)\big|dy=\sup_{|s|\leq\frac{1}{n}}\int\limits_T\big|\Delta_s^{(0,q)}(\Delta_t^{(p,0)}g(x,y))\big|dy=$$

$$=\frac{2^q}{\pi}\sum_{j=n}^{\infty}\frac{|\Delta_t^{(p,0)}g_j(x)|}{j}+O\bigg(\sum_{j=1}^{\infty}\alpha_{j,n}(q)\big(|\Delta(\Delta_t^{(p,0)}g_j(x))|+$$

$$+|\delta(\Delta_t^{(p,0)}g_j(x)|\big)\bigg).$$

After integrating these inequalities with respect to x and using (39)–(44), we obtain the validity of (32), (33), (35).

If instead of series (18′) we consider the series

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{i,j}\sin jy,$$

then by a similar reasoning we shall obtain the validity of (34) and (36). □

Corollary 1. *Let the conditions of Theorem* 1 *be fulfilled. Then*

$$\omega^{(p,0)}\Big(\frac{1}{m},f\Big)=O(C_m),\quad \omega^{(0,q)}\Big(\frac{1}{n},f\Big)=O(B_n),$$
$$\omega^{(p,q)}\Big(\frac{1}{m},\frac{1}{n},f\Big)=O(D_{m,n}).$$

Corollary 2. *Let the conditions of Theorem* 2 *be fulfilled. Then*

$$\omega^{(p,0)}\Big(\frac{1}{m},g\Big)=O(F_m),\quad \omega^{(0,q)}\Big(\frac{1}{n},f\Big)=O(E_n),$$
$$\omega^{(p,q)}\Big(\frac{1}{m},\frac{1}{n},f\Big)=O(G_{m,n}).$$

where

$$E_n=\lim_{\mu\to\infty}E_{n,\mu},\quad F_m=\lim_{\nu\to\infty}F_{m,\nu}.$$

It is easy to verify that for each function $\varphi\in L(T^2)$

$$\omega^{(p,0)}\Big(\frac{1}{m},\varphi\Big)=O\bigg(\int\limits_T\Big(\sup_{|t|\le\frac{1}{m}}\int\limits_T\big|\Delta_t^{(p,0)}\varphi(x,y)\big|dx\Big)dy\bigg),\tag{45}$$

$$\omega^{(0,q)}\Big(\frac{1}{n},\varphi\Big)=O\bigg(\int\limits_T\Big(\sup_{|s|\le\frac{1}{n}}\int\limits_T\big|\Delta_s^{(0,q)}\varphi(x,y)\big|dy\Big)dx\bigg),\tag{46}$$

$$\omega^{(p,q)}\Big(\frac{1}{m},\frac{1}{n},\varphi\Big)=O\bigg(\sup_{|t|\le\frac{1}{m}}\int\limits_T\Big(\sup_{|s|\le\frac{1}{n}}\int\limits_T\big|\Delta_{t,s}^{(p,q)}\varphi(x,y)\big|dy\Big)dx\bigg).\tag{47}$$

These equalities and (11)–(13) imply the validity of Corollary 1. Corollary 2 is proved by a similar reasoning, but first we must pass to the limit in (33) as $\mu\to\infty$ and in (34) as $\nu\to\infty$.

Corollary 3. *Let equality* (8) *be fulfilled and*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)(j+1)|\Delta_1^2\Delta_2^2a_{i,j}|<\infty.$$

Then

$$\int\limits_{T^2}|f(x,y)|dx\,dy=O\bigg(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)(j+1)|\Delta_1^2\Delta_2^2a_{i,j}|\bigg),\tag{48}$$

$$\int\limits_T\Big(\sup_{|s|\le\frac{1}{n}}\int\limits_T|\Delta_s^{(0,q)}f(x,y)|dy\Big)dx=$$
$$=O\bigg(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)(j+1)\alpha_{j,n}(q)|\Delta_1^2\Delta_2^2a_{i,j}|\bigg)=O(H_n),\tag{49}$$

$$\int\limits_T\Big(\sup_{|s|\le\frac{1}{m}}\int\limits_T|\Delta_t^{(p,0)}f(x,y)|dx\Big)dy=$$
$$=O\bigg(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)(j+1)\alpha_{i,m}(p)|\Delta_1^2\Delta_2^2a_{i,j}|\bigg)=O(K_n),\tag{50}$$

$$\sup_{|t|\le\frac{1}{m}}\int_T\Big(\sup_{|s|\le\frac{1}{n}}\int_T|\Delta_{t,s}^{(p,q)}f(x,y)|dy\Big)dx=$$
$$=O\Big(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)(j+1)\alpha_{i,m}(p)\alpha_{j,n}(q)|\Delta_1^2\Delta_2^2a_{i,j}|\Big)=O(L_{m,n}), \quad (51)$$
$$\sup_{|s|\le\frac{1}{n}}\int_T\Big(\sup_{|t|\le\frac{1}{m}}\int_T|\Delta_{s,t}^{(p,q)}f(x,y)|dx\Big)dy=O(L_{m,n}). \quad (52)$$

As one can easily verify, to prove Corollary 3 it suffices to estimate the right-hand sides of relations (10)–(14) by the right-hand sides of (48)–(52), respectively.

In [1] it is shown that

$$\sum_{i=0}^{\infty}\alpha_{i,m}(p)|\Delta a_i|=O\Big(\sum_{i=0}^{\infty}(i+1)\alpha_{i,m}(p)|\Delta^2a_i|\Big), \quad (53)$$
$$\sum_{i=0}^{\infty}\alpha_{i,m}(p)|\delta a_i|=O\Big(\sum_{i=0}^{\infty}(i+1)\alpha_{i,m}(p)|\Delta^2a_i|\Big) \quad (54)$$

are valid. In a similar manner one can prove

$$\sum_{i=0}^{\infty}|\Delta a_i|=O\Big(\sum_{i=0}^{\infty}(i+1)|\Delta^2a_i|\Big), \quad (55)$$
$$\sum_{i=0}^{\infty}|\delta a_i|=O\Big(\sum_{i=0}^{\infty}(i+1)|\Delta^2a_i|\Big). \quad (56)$$

Applying (53) and (55) successively, we obtain

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{j,n}(q)|\Delta_1\Delta_2a_{i,j}|=O\Big(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(j+1)\alpha_{j,n}(q)|\Delta_1\Delta_2^2a_{i,j}|\Big)=$$
$$=O\Big(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)(j+1)\alpha_{j,n}(q)|\Delta_1^2\Delta_2^2a_{i,j}|\Big).$$

By a similar technique we derive the estimates

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{j,n}(q)|\delta_1\Delta_2a_{i,j}|=O\Big(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)(j+1)\alpha_{j,n}(q)|\Delta_1^2\Delta_2^2a_{i,j}|\Big),$$
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{j,n}(q)|\Delta_1\delta_2a_{i,j}|=O\Big(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)(j+1)\alpha_{j,n}(q)|\Delta_1^2\Delta_2^2a_{i,j}|\Big),$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{j,n}(q)|\delta_1\delta_2 a_{i,j}| = O\Big(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(i+1)(j+1)\alpha_{j,n}(q)|\Delta_1^2\Delta_2^2 a_{i,j}|\Big).$$

The latter four relations enable us to estimate (see (9)) the right-hand side of (11) by the right-hand side of (49). In the same manner one can prove the validity of relations (48), (50)–(52).

Let

$$\beta_{i,m}(p) = \begin{cases} (im^{-1})^p, & 1 \le i \le m, \\ 1 + \ln \frac{i}{m}, & i > m, \\ 0, & i < 1. \end{cases}$$

Corollary 4. *Let* (8) *be fulfilled and*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ij\ln(i+1)\ln(j+1)|\Delta_1^2\Delta_2^2 a_{i,j}| < \infty.$$

Then

$$\int_{T^2} |g(x,y)|dx\,dy = O\Big(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ij\ln(i+1)\ln(j+1)|\Delta_1^2\Delta_2^2 a_{i,j}|\Big),$$

$$\int_T \Big(\sup_{|s|\le\frac{1}{n}}\int_T |\Delta_s^{(0,q)}g(x,y)|dy\Big)dx =$$

$$= O\Big(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ij\ln(i+1)\beta_{j,n}(q)|\Delta_1^2\Delta_2^2 a_{i,j}|\Big) = O(M_n),$$

$$\int_T \Big(\sup_{|t|\le\frac{1}{m}}\int_T |\Delta_t^{(p,0)}g(x,y)|dx\Big)dy =$$

$$= O\Big(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ij\ln(i+1)\beta_{i,m}(p)|\Delta_1^2\Delta_2^2 a_{i,j}|\Big) = O(N_m),$$

$$\sup_{|t|\le\frac{1}{m}}\int_T \Big(\sup_{|s|\le\frac{1}{n}}\int_T |\Delta_{t,s}^{(p,q)}g(x,y)|dy\Big)dx =$$

$$= O\Big(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ij\beta_{i,m}(p)\beta_{j,n}(q)|\Delta_1^2\Delta_2^2 a_{i,j}|\Big) = O(R_{m,n}),$$

$$\sup_{|s|\le\frac{1}{n}}\int_T \Big(\sup_{|t|\le\frac{1}{m}}\int_T |\Delta_{t,s}^{(p,q)}g(x,y)|dx\Big)dy = O(R_{m,n}).$$

We have

$$\sum_{i=m}^{\infty}\frac{|a_i|}{i}=\sum_{i=m}^{\infty}\frac{1}{i}\Big|\sum_{\mu=i}^{\infty}\sum_{\nu=\mu}^{\infty}\Delta^2 a_\nu\Big|\le\sum_{i=m}^{\infty}\frac{1}{i}\sum_{\nu=i}^{\infty}\sum_{\mu=i}^{\nu}|\Delta^2 a_\nu|\le$$

$$\le\sum_{i=m}^{\infty}\sum_{\nu=i}^{\infty}\frac{\nu}{i}|\Delta^2 a_\nu|=\sum_{\nu=m}^{\infty}\sum_{i=m}^{\nu}\frac{\nu}{i}|\Delta^2 a_\nu|=$$

$$=O\Big(\sum_{\nu=m}^{\infty}\nu\ln\frac{\nu+1}{m}|\Delta^2 a_\nu|\Big). \tag{57}$$

The proof of Corollary 4 is obtained from Theorem 2 and (53)–(57) by the reasoning similar to the proof of Corollary 3 by means of Theorem 1 and (53)–(56), but first one must pass to the limit in (33) as $\mu\to\infty$ and in (34) as $\nu\to\infty$.

Corollary 5. *Let the conditions of Corollary* 3 *be fulfilled. Then*

$$\omega^{(p,0)}\Big(\frac{1}{m},f\Big)=O(K_m),\quad \omega^{(0,q)}\Big(\frac{1}{n},f\Big)=O(H_n),$$
$$\omega^{(p,q)}\Big(\frac{1}{m},\frac{1}{n}\Big)=O(L_{m,n}).$$

Corollary 6. *Let the conditions of Corollary* 4 *be fulfilled. Then*

$$\omega^{(p,0)}\Big(\frac{1}{m},g\Big)=O(M_m),\quad \omega^{(0,q)}\Big(\frac{1}{n},g\Big)=O(N_n),$$
$$\omega^{(p,q)}\Big(\frac{1}{m},\frac{1}{n},g\Big)=O(R_{m,n}).$$

Corollaries 5 and 6 follow from Corollaries 3 and 4, respectively, if we use (45), (46), (47). Concerning these corollaries, see [4], pp. 6–86.

One can also consider the series

$$\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\lambda_{i,j}\cos ix\sin jy,\quad \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\lambda_{i,j}\sin ix\cos jy.$$

The interested reader will readily know how to formulate and simultaneously prove the corresponding theorems for these series by the repeated application of Theorems A_1 and A_2.

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