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# BOUNDARY PROPERTIES OF SECOND-ORDER PARTIAL DERIVATIVES OF THE POISSON INTEGRAL FOR A HALF-SPACE $\mathbb{R}_+^{k+1}$ $(k > 1)$

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ABSTRACT. The boundary properties of second-order partial derivatives of the Poisson integral are studied for a half-space $\mathbb{R}_+^{k+1}$.

In this paper, the theorems generalizing the author's previous results [1–5] are proved. It is the continuation of [6] and uses the same notation. Let us recall some of them.

Let $M = \{1, 2, \dots, k\}$ $(k \geq 2)$ and $B \subset M$, $B' = M \setminus B$. For every $x \in \mathbb{R}^k$ and for an arbitrary $B \subset M$ the symbol $x_B$ denotes a point from $\mathbb{R}^k$ whose coordinates with indices from the set $B$ coincide with the respective coordinates of the point $x$, while the coordinates with indices from the set $B'$ are zeros ($x_M = x$, $B \setminus i = B \setminus \{i\}$); if $B = \{i_1, i_2, \dots, i_s\}$, $1 \leq s \leq k$ $(i_l < i_r$ for $l < r)$, then $\overline{x}_B = (x_{i_1}, x_{i_2}, \dots, x_{i_s}) \in \mathbb{R}^s$; $m(B)$ is the number of elements of the set $B$; $\widetilde{L}(\mathbb{R}^k)$ is the set of functions $f(x) = f(x_1, x_2, \dots, x_k)$ such that $\frac{f(x)}{(1+|x|)^{\frac{k+1}{2}}} \in L(\mathbb{R}^k)$; $\mathbb{R}_+^{k+1} = \{(x_1, x_2, \dots, x_{k+1}) \in \mathbb{R}^{k+1};\ x_{k+1} > 0\}$ (half-space); $\Delta_x = \Delta_{x_1 x_2 \cdots x_k} = \sum\limits_{i=1}^{k} \frac{\partial^2}{\partial x_1^2}$; the Poisson integral of the function $f(x)$ for the half-space $\mathbb{R}_+^{k+1}$ is

$$u(f; x, x_{k+1}) = \frac{x_{k+1}\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{f(t)\, dt}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+1}{2}}}.$$

As in [6], we use the following generalization of dihedral-angular limit introduced by O. P. Dzagnidze: if the point $N \in \mathbb{R}_+^{k+1}$ tends to the point

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$\mathcal{P}(x^0,0)$ by the condition $x_{k+1}(\sum\limits_{i\in B}(x_i-x_i^0)^2)^{-1/2}\ge C>0$, [1] then we shall write $N(x,x_{k+1})\underset{x_B}{\overset{\wedge}{\to}}\mathcal{P}(x^0,0)$.

If $B=M$, then we have an angular limit and write $N(x,x_{k+1})\overset{\wedge}{\to}\mathcal{P}(x^0,0)$. Finally, the notation $N(x,x_{k+1})\to\mathcal{P}(x^0,0)$ means that the point $N$ tends to $\mathcal{P}$ arbitrarily, remaining in the half-space $\mathbb{R}_+^{k+1}$.

Let $u\in\mathbb{R}$, $v\in\mathbb{R}$. We shall consider the following derivatives of the function $f(x)$:

1. The limit

$$\lim_{(u,\overline{x}_B)\to(0,\overline{x}_B^0)}\frac{f(x_B+x_{B'}^0+ue_i)+f(x_B+x_{B'}^0-ue_i)-2f(x_B+x_{B'}^0)}{u^2}$$

is denoted by:

(a) $\mathcal{D}_{x_i}^2f(x^0)$ if $B=\varnothing$;
(b) $\mathcal{D}_{x_i\,(\overline{x}_B)}^2f(x^0)$ if $i\in B'$;
(c) $\overline{\mathcal{D}}_{x_i\,(\overline{x}_B)}^2f(x^0)$ if $i\in B$.

2. The limit

$$\lim_{\substack{(u,v)\to(0,0)\\ \overline{x}_B\to\overline{x}_B^0}}\Big[\frac{f(x_B+x_{B'}^0+ue_i+ve_j)-f(x_B+x_{B'}^0+ue_i)}{uv}-\\ -\frac{f(x_B+x_{B'}^0+ve_j)-f(x_B+x_{B'}^0)}{uv}\Big]$$

is denoted by:

(a) $\mathcal{D}_{x_ix_j}f(x_0)$ if $B=\varnothing$;
(b) $\mathcal{D}_{x_ix_j\,(\overline{x}_B)}f(x^0)$ if $\{i,j\}\subset B'$;
(c) $\mathcal{D}_{[x_ix_j]\,(\overline{x}_B)}f(x^0)$ if $\{i,j\}\subset B$;
(d) $\mathcal{D}_{[x_i]x_j\,(\overline{x}_B)}f(x^0)$ if $i\in B$, $j\in B'$.

3. The limit

$$\lim_{\substack{(u,v)\to(0,0)\\ \overline{x}_B\to\overline{x}_B^0}}\Big[\frac{f(x_B+x_{B'}^0+ue_i+ve_j)-f(x_B+x_{B'}^0+ue_i-ve_j)}{4uv}-\\ -\frac{f(x_B+x_{B'}^0-ue_i+ve_j)-f(x_B+x_{B'}^0-ue_i-ve_j)}{4uv}\Big]$$

is denoted by:

(a) $\mathcal{D}_{x_ix_j}^*f(x_0)$ if $B=\varnothing$;
(b) $\mathcal{D}_{x_ix_j\,(\overline{x}_B)}^*f(x^0)$ if $\{i,j\}\subset B'$;
(c) $\mathcal{D}_{[x_ix_j]\,(\overline{x}_B)}^*f(x^0)$ if $\{i,j\}\subset B$;
(d) $\mathcal{D}_{[x_i]x_j\,(\overline{x}_B)}^*f(x^0)$ if $i\in B$, $j\in B'$.

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[1] Here and in what follows $C$ denotes absolute positive constants which, generally speaking, can be different in different relations.

4. The limit

$$\lim_{\substack{\rho\to 0\\ x_B+x^0_{B'}\to x^0}} \frac{\frac{1}{|S_\rho|}\int\limits_{S_\rho(x_B+x^0_{B'})} f(t)\,dS(t)-f(x_B+x^0_{B'})}{\rho^2/2k},$$

where $S_\rho(x)$ is the sphere in $\mathbb{R}^k$ with center at $x$, and radius $\rho$, and the $(k-1)$-dimensional surface area $|S_\rho(x)|$ is denoted by:

(a) $\overline{\Delta} f(x^0)$ if $B=\varnothing$;

(b) $\overline{\Delta}_x f(x^0)$ if $B=M$;

(c) $\overline{\Delta}_{x_B} f(x^0)$ if $B\neq\varnothing$ and $B\neq M$.

*Remark.* The $\lambda$-derivatives of a function of two variables have been studied by the author in [1, 2] and hence are not considered here.

The following propositions are valid:

(1) For any $B\subset M$, the existence of $\overline{\mathcal{D}}^2_{x_i\,(\overline{x}_B)} f(x^0)$ implies that there exists $\mathcal{D}_{x_i\,(\overline{x}_{B\setminus i})} f(x^0)$ and

$$\overline{\mathcal{D}}_{x_i\,(\overline{x}_B)} f(x^0)=\mathcal{D}^2_{x_i\,(\overline{x}_{B\setminus i})} f(x^0)=\mathcal{D}^2_{x_i} f(x^0).$$

If there exists $f''_{x_i^2}(x^0)$, then $\mathcal{D}^2_{x_i} f(x^0)$ exists too, and they have the same value.

(2) If in the neighborhood of the point $x^0$ there exists a partial derivative $f''_{x_i^2}(x)$ which is continuous at $x^0$, then $\overline{\mathcal{D}}^2_{x_i\,(x)} f(x^0)$ exists too and $\overline{\mathcal{D}}^2_{x_i\,(x)} f(x^0)=f''_{x_i^2}(x^0)$.

Indeed, if we apply the Cauchy formula for the function $f(x+ue_i)+f(x-ue_i)-2f(x^0)$ and for $u^2$ with respect to $u$, then we have

$$\frac{f(x+ue_i)+f(x-ue_i)-2f(x^0)}{u^2}=$$
$$=\frac{f'_{x_i}(x+\theta(x)ue_i)-f'_{x_i}(x-\theta(x)ue_i)}{2\theta(x)u},\quad 0<\theta<1.$$

Now applying the Lagrange formula, we obtain

$$\frac{f(x+ue_i)+f(x-ue_i)-2f(x^0)}{u^2}=$$
$$=\frac{2\theta u f''_{x_i^2}(x-\theta ue_i+2\theta_1\theta ue_i)}{2\theta u}=f''_{x_i^2}(x-\theta ue_i+2\theta_1\theta ue_i),\quad 0<\theta<1,$$

which by the continuity of $f''_{x_i^2}(x)$ implies that Proposition (2) is valid.

Note that the continuity of the partial derivative $f''_{x_i^2}(x)$ at $x^0$ is only a sufficient condition for the existence of the derivative $\overline{\mathcal{D}}_{x_i\,(\overline{x}_B)} f(x^0)$ for any $B\subset M$.

(3) If in the neighborhood of the point $x$ there exists a derivative $f''_{x_ix_j}(x)$ which is continuous at $x^0$, then there exists $\overline{\mathcal{D}}_{[x_ix_j]\,(x)}f(x^0)$ and $\overline{\mathcal{D}}_{[x_ix_j]\,(x)}f(x^0) = f''_{x_ix_j}f(x^0)$.

The continuity of $f''_{x_ix_j}(x)$ at the point $x^0$ is only a sufficient condition for the existence of $\mathcal{D}_{[x_ix_j]\,(x)}f(x^0)$.

(4) If for the function $f(x)$ at $x^0$ there exists $\mathcal{D}_{x_ix_j}f(x^0)$, then at the same point there exists $\mathcal{D}^*_{x_ix_j}f(x^0)$, and their values coincide.

(5) If the function $f(x)$ at the point $x^0$ has continuous partial derivatives up to second order inclusive, then at the same point there exists $\overline{\Delta}f(x^0)$, and $\overline{\Delta}f(x^0) = \Delta f(x^0)$ (see [7], p. 18).

In what follows it will always be assumed that $f \in \widetilde{L}(\mathbb{R}^k)$.

The next lemma is proved analogously to the lemma from [6].

**Lemma 1.** *For every* $(x_1, x_1, \dots, x_k)$ *the following equalities are valid:*

$$\int_{\infty}^{\infty} \frac{(k+3)(t_i - x_i)^2 - |t-x|^2 - x_{k+1}^2}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+5}{2}}}\, dt_i = 0,$$

$$\int_{\mathbb{R}^k} \frac{(k+3)(t_i - x_i)^2 - |t-x|^2 - x_{k+1}^2}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+5}{2}}}\, f(t - t_ie_i + x_ie_i)\, dt = 0,$$

$$\frac{(k+1)x_{k+1}\Gamma(\frac{k+1}{2})}{2\pi^{\frac{k+1}{2}}} \int_{\mathbb{R}^k} \frac{(k+3)(t_i - x_i)^2 - |t-x|^2 - x_{k+1}^2}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+5}{2}}}\, t_i^2\, dt = 1.$$

**Theorem 1.**

(a) *If at the point* $x^0$ *there exists a finite derivative* $\overline{\mathcal{D}}^2_{x_i\,(x)}f(x^0)$, *then*

$$\lim_{(x,x_{k+1})\to(x^0,0)} \frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i^2} = \mathcal{D}^2_{x_i}f(x^0). \tag{1}$$

(b) *There exists a continuous function* $f \in L(\mathbb{R}^k)$ *such that for any* $B \subset M$, $m(B) < k$ *all derivatives* $\overline{\mathcal{D}}^2_{x_i\,(\overline{x}_B)}f(0) = 0$, $i = \overline{1,k}$, *but the limits*

$$\lim_{x_{k+1}\to 0+} \frac{\partial^2 u(f;0,x_{k+1})}{\partial x_i^2}$$

*do not exist.*

*Proof of part* (a). Let $x^0 = 0$, $C_k = \frac{(k+1)\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}}$. It is easy to verify that

$$\frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i^2} = C_k x_{k+1} \int_{\mathbb{R}^k} \frac{(k+3)(t_i - x_i)^2 - |t-x|^2 - x_{k+1}^2}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+5}{2}}}\, f(t)\, dt =$$

$$= C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{(k+3)t_i^2 - |t|^2 - x_{k+1}^2}{(|t|^2 + x_{k+1}^2)^{\frac{k+5}{2}}} f(x+t)\, dt =$$

$$= C_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{(k+3)t_i^2 - |t|^2 - x_{k+1}^2}{(|t|^2 + x_{k+1}^2)^{\frac{k+5}{2}}} f(x+t-2t_i e_i)\, dt.$$

By Lemma 1 this gives

$$\frac{\partial^2 u(f; x, x_{k+1})}{\partial x_i^2} - \overline{\mathcal{D}}^2_{x_i\,(x)} f(0) = \frac{1}{2}\, c_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{[(k+3)t_i^2 - |t|^2 - x_{k+1}^2] t_i^2}{(|t|^2 + x_{k+1}^2)^{\frac{k+5}{2}}} \times$$
$$\times \left[ \frac{f(x+t) + f(x+t-2t_i e_i) - 2f(x+t-t_i e_i)}{t_i^2} - \overline{\mathcal{D}}^2_{x_i\,(x)} f(0) \right] dt,$$

which implies that equality (1) is valid.
*Proof of part* (b). Assume $D = [0 \le t_1 < \infty;\ 0 \le t_2 < \infty;\ 0 \le t_3 < \infty]$. Let

$$f(t) = \begin{cases} \sqrt[3]{t_1 t_2 t_3} & \text{if } (t_1, t_2, t_3) \in D, \\ 0 & \text{if } (t_1, t_2, t_3) \in CD. \end{cases}$$

We can easily find that $\overline{\mathcal{D}}^2_{x_i\,(x_j)} f(0) = 0$, $i, j = 1, 2, 3$, $i \ne j$. However, for the given function we have

$$\frac{\partial^2 u(f; 0, x_4)}{\partial x_1^2} = \frac{4x_4}{\pi^2} \int\limits_D \frac{6t_1^2 - |t|^2 - x_4^2}{(|t|^2 + x_4^2)^4} \sqrt[3]{t_1 t_2 t_3}\, dt =$$

$$= \frac{4x_4}{\pi^2} \int\limits_0^{\infty} \int\limits_0^{\frac{\pi}{2}} \int\limits_0^{\frac{\pi}{2}} \frac{6\rho^2 \sin^2\vartheta \cos^2\varphi - \rho^2 - x_4^2}{(\rho^2 + x_4^2)^4} \times$$
$$\times \sqrt[3]{\rho^3 \sin^2\vartheta \cos\vartheta \sin\varphi \cos\varphi}\; \rho^2 \sin\vartheta\, d\rho\, d\vartheta\, d\varphi =$$

$$= \frac{C}{x_4} \left( 4 \int\limits_0^{\frac{\pi}{2}} \sqrt[3]{\sin^2\vartheta \cos\vartheta}\; \sin^3\vartheta\, d\vartheta \int\limits_0^{\frac{\pi}{2}} \sqrt[3]{\sin 2\varphi}\; \sin^2\varphi\, d\varphi - \right.$$
$$\left. - \int\limits_0^{\frac{\pi}{2}} \sqrt[3]{\sin^2\vartheta \cos\vartheta}\; \sin\vartheta\, d\vartheta \int\limits_0^{\frac{\pi}{2}} \sqrt[3]{\sin 2\varphi}\, d\varphi \right) =$$

$$= \frac{4C}{9x_4} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) \int\limits_0^{\frac{\pi}{2}} \left( \sin^2\varphi - \frac{3}{8} \right) \sqrt[3]{\sin 2\varphi}\, d\varphi =$$

$$= \frac{C}{x_4}\Bigg( \int\limits_0^{\arcsin\sqrt{\frac{3}{8}}} + \int\limits_{\arcsin\sqrt{\frac{3}{8}}}^{\frac{\pi}{2}-\arcsin\sqrt{\frac{3}{8}}} + \int\limits_{\frac{\pi}{2}-\arcsin\sqrt{\frac{3}{8}}}^{\frac{\pi}{2}} \Bigg) =$$

$$= \frac{C}{x_4}(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3).$$

Clearly, $\mathcal{I}_1 < 0$, $\mathcal{I}_2 > 0$, $\mathcal{I}_3 > 0$. Further,

$$\mathcal{I}_3 = \int\limits_0^{\arcsin\sqrt{\frac{3}{8}}} \Big(\cos^2\varphi - \frac{3}{8}\Big)\sqrt[3]{\sin 2\varphi}\, d\varphi.$$

Since $\mathcal{I}_1 + \mathcal{I}_3 > 0$, we have $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 > 0$. Finally, we obtain

$$\frac{\partial^2 u(f;0,x_4)}{\partial x_1^2} \to +\infty \quad \text{as} \quad x_4 \to 0+. \qquad \square$$

**Corollary.** *If the function $f$ has a continuous partial derivative $\frac{\partial^2 f(x)}{\partial x_i^2}$ at the point $x^0$, then equality* (1) *is fulfilled.*

**Theorem 2.**
(a) *If at the point $x^0$ there exists a finite derivative $\mathcal{D}_{x_i\,(x_{M\setminus i})} f(x^0)$, then*

$$\lim_{(x-x_ie_i+x_i^0e_i,x_{k+1})\to(x^0,0)} \frac{\partial^2 u(f;x-x_ie_i+x_i^0e_i,x_{k+1})}{\partial x_i^2} = \mathcal{D}^2_{x_i} f(x^0).$$

(b) *There exists a function $f \in L(\mathbb{R}^k)$ such that $\mathcal{D}_{x_i\,(x_{M\setminus i})} f(x^0) = 0$, but the limit*

$$\lim_{(x,x_{k+1}\overset{\wedge}{\to}(x^0,0)} \frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i^2}$$

*does not exist.*

*Proof of part* (a). Let $x^0 = 0$. By Lemma 1 we easily obtain

$$\frac{\partial^2 u(f;x-x_ie_i,x_{k+1})}{\partial x_i^2} - \mathcal{D}^2_{x_i\,(x_{M\setminus i})} f(0) =$$

$$= \frac{1}{2}\, c_k x_{k+1} \int\limits_{\mathbb{R}^k} \frac{[(k+3)t_i^2 - |t|^2 - x_{k+1}^2]t_i^2}{(|t|^2 + x_{k+1}^2)^{\frac{k+5}{2}}} \times$$

$$\times \Bigg[ \frac{f(t+x-x_ie_i) + f(t-2t_ie_i+x-x_ie_i) - 2f(t-t_ie_i+x-x_ie_i)}{t_i^2} -$$

$$- \mathcal{D}^2_{x_i\,(x_{M\setminus i})} f(0) \Bigg]\, dt.$$

*Proof of part* (b). This will be given for the cases $i=1$ and $k=2$. We have

$$\frac{\partial^2 u(f;x_1,x_2,x_3)}{\partial x_1^2}=\frac{3x_3}{2\pi}\int\limits_{\mathbb{R}^2}\frac{4(t_1-x_1)^2-(t_2-x_2)^2-x_3^2}{(|t-x|^2+x_3^2)^{7/2}}\,f(t)\,dt=$$

$$=\frac{3x_3}{2\pi}\int\limits_0^{\infty}\int\limits_0^{2\pi}\frac{4(\rho\cos\varphi-x_1)^2-(\rho\sin\varphi-x_2)-x_3^2}{[(\rho\cos\varphi-x_1)^2+(\rho\sin\varphi-x_2)^2+x_3^2]^{7/2}}\,f(\rho\cos\varphi,\rho\sin\varphi)\rho\,d\rho\,d\varphi.$$

Let $N(x_1,x_2,x_3)\to(0,0,0)$ so that $x_2=0$ and $x_3=x_1$. Then

$$\frac{\partial^2 u(f;x_3,0,x_3)}{\partial x_1^2}=$$

$$=\frac{3x_3}{2\pi}\int\limits_0^{\infty}\int\limits_0^{2\pi}\frac{4\rho^2\cos^2\varphi-8\rho x_3\cos\varphi-\rho^2\sin^2\varphi+3x_3^2}{(\rho^2-2\rho x_3\cos\varphi+2x_3)^{7/2}}\,f(\rho\cos\varphi,\rho\sin\varphi)\rho\,d\rho\,d\varphi.$$

Using Lemma 1, for any $x_1,x_2$, and $x_3$ we have

$$\int\limits_{\mathbb{R}^2}\frac{4(t_1-x_1)^2-(t_2-x_2)^2-x_3^2}{(|t-x|^2+x+3^2)^{7/2}}\,dt_1\,dt_2=0,$$

which, in particular, yields

$$\int\limits_0^{\infty}\int\limits_0^{2\pi}\frac{4\rho^2\cos^2\varphi-8\rho x_3\cos\varphi-\rho^2\sin^2\varphi+3x_3^2}{(\rho^2-2\rho x_3\cos\varphi+2x_3)^{7/2}}\,\rho\,d\rho\,d\varphi=$$

$$=2\int\limits_0^{\infty}\int\limits_0^{\pi}\frac{4\rho^2\cos^2\varphi-8\rho x_3\cos\varphi-\rho^2\sin^2\varphi+3x_3^2}{(\rho^2-2\rho x_3\cos\varphi+2x_3)^{7/2}}f(\rho\cos\varphi,\rho\sin\varphi)\rho d\rho d\varphi=0.$$

Therefore

$$\int\limits_0^{\infty}\int\limits_0^{\pi}\frac{4\rho^2\cos^2\varphi-8\rho x_3\cos\varphi-\rho^2\sin^2\varphi+3x_3^2}{(\rho^2-2\rho x_3\cos\varphi+2x_3)^{7/2}}\,\rho\,d\rho\,d\varphi=0. \tag{2}$$

In the interval $\frac{\pi}{2}\le\varphi\le\pi$ we have

$$x_3\int\limits_0^{\infty}\int\limits_{\frac{\pi}{2}}^{\pi}\frac{4\rho^2\cos^2\varphi-8\rho x_3\cos\varphi-\rho^2\sin^2\varphi+3x_3^2}{(\rho^2-2\rho x_3\cos\varphi+2x_3)^{7/2}}\,\rho\,d\rho\,d\varphi>$$

$$>x_3\int\limits_0^{\infty}\int\limits_{\frac{\pi}{2}}^{\pi}\frac{5\rho^2\cos^2\varphi-\rho^2}{(\rho^2+2\rho x_3+4x_3)^{7/2}}\,\rho\,d\rho\,d\varphi=x_3\int\limits_0^{\infty}\int\limits_{\frac{\pi}{2}}^{\pi}\frac{5\rho^2\cos^2\varphi-\rho^2}{(\rho+2x_3)^7}\,\rho\,d\rho\,d\varphi=$$

$$= \frac{3\pi}{4} x_3 \int\limits_0^\infty \frac{\rho^3\, d\rho}{(\rho+2x_3)^7} > cx_3 \int\limits_0^{x_3} \frac{\rho^3\, rho}{(\rho+2x_3)^7} = \frac{c}{x_3^2}.$$

Hence

$$\lim_{x_3\to 0+} x_3 \int\limits_0^\infty \int\limits_{\frac{\pi}{2}}^{\pi} \frac{4\rho^2\cos^2\varphi - 8\rho x_3\cos\varphi - \rho^2\sin^2\varphi + 3x_3^2}{(\rho^2 - 2\rho x_3\cos\varphi + 2x_3)^{7/2}}\,\rho\, d\rho\, d\varphi = +\infty. \quad (3)$$

Equations (2) and (3) imply

$$\lim_{x_3\to 0+} x_3 \int\limits_0^\infty \int\limits_0^{\frac{\pi}{2}} \frac{4\rho^2\cos^2\varphi - 8\rho x_3\cos\varphi - \rho^2\sin^2\varphi + 3x_3^2}{(\rho^2 - 2\rho x_3\cos\varphi + 2x_3)^{7/2}}\,\rho\, d\rho\, d\varphi = -\infty. \quad (4)$$

Next, we define $f(t_1,t_2)$ as follows:

$$f(t_1,t_2) = \begin{cases} -1 & \text{for } t_1>0,\ t_2>0, \\ 1 & \text{for } t_1<0,\ t_2>0, \\ 0 & \text{for } -\infty<t_1<\infty,\ t_2\le 0, \\ 0 & \text{for } t_1=0,\ 0<t_2<\infty. \end{cases}$$

Clearly, for this function

$$\mathcal{D}^2_{x_1\,(x_2)} f(0,0) = \lim_{(t_1,x_2)\to(0,0)} \frac{f(t_1,x_1)+f(-t_1,x_2)-2f(0,x_2)}{t_1^2} = 0.$$

However, by (3) and (4), $\frac{\partial^2 u(f;x_1,0,x_3)}{\partial x_1^2} \to +\infty$, as $(x_1,x_2,x_3)\to(0,0,0)$. $\square$

**Lemma 2.** *For any $(x_1,x_2,\dots,x_{k+1})$ the following equalities are valid:*

$$\int\limits_{\mathbb{R}^k} \frac{(t_i-x_i)(t_j-x_j)f(t-t_ie_i+x_ie_i)}{(|t-x|^2+x_{k+1}^2)^{\frac{k+5}{2}}}\,dt = 0,$$

$$\int\limits_{\mathbb{R}^k} \frac{(t_i-x_i)(t_j-x_j)f(t-t_ie_i-t_je_j+x_ie_i+x_je_j)}{(|t-x|^2+x_{k+1}^2)^{\frac{k+5}{2}}}\,dt = 0,$$

$$\frac{(k+1)(k+3)x_{k+1}\Gamma(\frac{k+1}{2})}{2\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{(t_i-x_i)^2(t_j-x_j)^2}{(|t-x|^2+x_{k+1}^2)^{\frac{k+5}{2}}}\,t_i^2\,dt = 1.$$

**Theorem 3.**

(a) *If at the point $x^0$ there exists a finite derivative $\mathcal{D}_{[x_ix_j]\,(x)}f(x^0)$, $i\neq j$, then*

$$\lim_{(x,x_{k+1})\to(x^0,0)} \frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i\partial x_j} = \mathcal{D}_{x_ix_j}f(x^0).$$

(b) *There exists a continuous function* $f\in L(\mathbb{R}^k)$ *such that for any* $B\subset M$, $m(B)<k$ *all derivatives* $\mathcal{D}_{[x_ix_j]\,(x)}f(0)=0$, *but the limits*

$$\lim_{x_{k+1}\to 0+}\frac{\partial^2 u(f;0,x_{k+1})}{\partial x_i\partial x_j}$$

*do not exist.*

*Proof of part* (a). Let $x^0=0$ and $B_k=\frac{(k+1)(k+3)\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}}$. It is easy to verify that

$$\frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i\partial x_j}=B_kx_{k+1}\int\limits_{\mathbb{R}^k}\frac{(t_i-x_i)(t_j-x_j)f(t)\,dt}{(|t-x|^2+x_{k+1}^2)^{\frac{k+5}{2}}}.$$

By Lemma 2 we obtain

$$\frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i\partial x_j}-\mathcal{D}_{[x_ix_j]\,(x)}f(0)=B_kx_{k+1}\int\limits_{\mathbb{R}^k}\frac{t_i^2t_j^2}{(|t|^2+x_{k+1}^2)^{\frac{k+5}{2}}}\times$$
$$\times\left[\frac{f(x+t)-f(x+t-t_ie_i)+f(x+t-t_je_j)+f(x+t-t_ie_i-t_je_j)}{t_it_j}-\right.$$
$$\left.-\mathcal{D}_{[x_ix_j]\,(x)}f(0)\right]dt,$$

which implies that part (a) is valid.

*Proof of part* (b). Let $k=4$ and $D=(\sum\limits_{i=1}^{4}t_i^2\le 1,\ t_i\ge 0,\ i=\overline{1,4})$. We define $f$ as $f(t)=\sqrt[5]{t_1t_2t_3t_4}$ for $t\in D$, and extend it continuously to the set $\mathbb{R}^4\setminus D$ so that $f\in L(\mathbb{R}^4)$. It is easy verify that for any $B\subset M$, $m(B)<4$ we have $\mathcal{D}_{[x_ix_j]\,(x)}f(0)=0$. But

$$\frac{\partial^2 u(f;0,x_5)}{\partial x_1\partial x_1}=cx_5\int\limits_D\frac{t_1t_2\sqrt[5]{t_1t_2t_3t_4}}{(|t|^2+x_5^2)^{9/2}}\,dt+o(1)=$$
$$=cx_5\int\limits_0^1\frac{\rho^{29/5}d\rho}{(\rho^2+x_5^2)^{9/2}}+o(1)\to+\infty\quad\text{as}\quad x_5\to 0+.\qquad\square$$

**Corollary.** *If the function* $f$ *at the point* $x^0$ *has a continuous derivative* $f''_{x_ix_j}(x)$, *then*

$$\lim_{(x,x_{k+1})\to(x^0,0)}\frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i\partial x_j}=\frac{\partial^2 f(x^0)}{\partial x_i\partial x_j}.$$

The following theorem is valid by the corollaries of Theorems 1 and 3.

**Theorem 4.** *If the function $f$ is twice continuously differentiable at the point $x^0$, then* $\lim_{(x,x_{k+1})\to(x^0,0)} d_x^2 u(f;x,x_{k+1}) = d^2 f(x^0)$.

**Theorem 5.**

(a) *If at the point $x^0$ there exists a finite derivative $\mathcal{D}_{[x_i]x_j\,(x)} f(x^0)$, then*

$$\lim_{(x,x_{k+1}) \overset{\wedge}{\underset{x_j}{\to}} (x^0,0)} \frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i \partial x_j} = \mathcal{D}_{x_i x_j} f(x^0).$$

(b) *There exists a continuous function $f \in L(\mathbb{R}^k)$ such that for every $B \subset M$, $m(B) < k-1$ all derivatives $\mathcal{D}_{[x_i]x_j\,(\overline{x}_B)} f(0) = 0$, but the limits*

$$\lim_{x_{k+1}\to 0+} \frac{\partial^2 u(f;0,x_{k+1})}{\partial x_i \partial x_j}$$

*do not exist.*

**Theorem 6.**

(a) *If at the point $x^0$ there exists a finite derivative $\mathcal{D}_{x_i x_j\,(x_{M\setminus\{i,j\}})} f(x^0)$, then*

$$\lim_{(x,x_{k+1}) \overset{\wedge}{\underset{x_i x_j}{\to}} (x^0,0)} \frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i \partial x_j} = \mathcal{D}_{x_i x_j} f(x^0).$$

.

(b) *There exists a continuous function $f \in L(\mathbb{R}^k)$ such that for every $B \subset M$, $m(B) < k-2$ all derivatives $\mathcal{D}_{x_i x_j\,(\overline{x}_B)} f(0) = 0$, but the limits*

$$\lim_{x_{k+1}\to 0+} \frac{\partial^2 u(f;0,x_{k+1})}{\partial x_i \partial x_j}$$

*do not exist.*

**Theorem 7.**

(a) *If the function $f$ is twice differentiable at the point $x^0$, then*

$$\lim_{(x,x_{k+1}) \overset{\wedge}{\to} (x^0,0)} d_x^2 u(f;x,x_{k+1}) = d^2 f(x^0).$$

(b) *There exists a continuous function $f \in L(\mathbb{R}^k)$ such that it is differentiable at the point $x^0 = (0,0)$ and has at this point all partial derivatives of any order; however the limits*

$$\lim_{x_{k+1}\to 0+} \frac{\partial^2 u(f;0,x_{k+1})}{\partial x_i \partial x_j}, \quad i,j = \overline{1,k}$$

*do not exist.*

*Proof of part* (a). Let $x^0=0$. The validity of part (a) follows from the equalities

$$\frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i^2}-\frac{\partial^2 f(0)}{\partial x_i^2}=$$

$$=c_k x_{k+1}\int\limits_{\mathbb{R}^k}\frac{[(k+3)(t_i-x_i)^2-|t-x|^2-x_{k+1}^2]|t|^2}{(|t-x|^2+x_{k+1}^2)^{\frac{k+5}{2}}}\times$$

$$\times\Bigg[\frac{f(t)+f(0)-(\sum\limits_{\nu=1}^{k}t_\nu\frac{\partial}{\partial t_\nu})f(0)-\frac12(\sum\limits_{\nu=1}^{k}t_\nu\frac{\partial}{\partial t_\nu})^2f(0)}{|t|^2}\,dt,$$

$$\frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i\partial x_j}-\frac{\partial^2 f(0)}{\partial x_i\partial x_j}=$$

$$=B_k x_{k+1}\int\limits_{\mathbb{R}^k}\frac{(t_i-x_i)(t_j-x_j)|t|^2}{-}(|t-x|^2+x_{k+1}^2)^{\frac{k+5}{2}}\times$$

$$\times\Bigg[\frac{f(t)+f(0)-(\sum\limits_{\nu=1}^{k}t_\nu\frac{\partial}{\partial t_\nu})f(0)-\frac12(\sum\limits_{\nu=1}^{k}t_\nu\frac{\partial}{\partial t_\nu})^2f(0)}{|t|^2}\,dt.$$

*Proof of part* (b). Consider the function equal to $\sqrt[3]{(2t_1-t_2)^2(t_2-\frac12t_1)^2}$ when $(t_1,t_2)\in D=\{(t_1,t_2)\in\mathbb{R}^2\colon\ 0\le t_1<\infty;\ \frac12t_1\le t_2\le 2t_1\}$ and to 0 otherwise, which is continuous in $\mathbb{R}^2$, differentiable at the point $(0,0)$, and has all partial derivatives of any order equal to zero, but

$$\frac{\partial^2 u(f;0,0,x_3)}{\partial x_1\partial x_2}=\frac{15x_3}{2\pi}\int\limits_0^\infty dt_1\int\limits_{\frac12t_1}^{2t_1}\frac{t_1t_2\sqrt[3]{(2t_1-t_2)^2(t_2-\frac12t_1)^2}}{(t_1^2+t_2^2+x_3^2)^{7/2}}\,dt_2>$$

$$>cx_3\int\limits_{x_3}^{2x_3}t_1^2dt_1\int\limits_{t_1}^{\frac32t_1}\frac{\sqrt[3]{(2t_1-\frac32t_1)^2(t_1-\frac12t_1)^2}}{(t_1^2+4t_1^2+x_3^2)^{7/2}}=$$

$$=\frac{c}{x_3^6}\int\limits_{x_3}^{2x_3}t_1^{\frac{13}{3}}\,dt_1=\frac{c}{\sqrt[3]{x_3^2}}\to\infty\quad\text{as}\quad x_3\to0+.\qquad\square$$

**Theorem 8.**
(a) *If at the point $x^0$ there exists a finite derivative $\mathcal{D}^*_{x_ix_j\,(\overline{x}_{M\setminus\{i,j\}})}f(x^0)$, then*

$$\lim_{x_{k+1}\to0+}\frac{\partial^2u(f;x^0,x_{k+1})}{\partial x_i\partial x_j}=\mathcal{D}^*_{x_ix_j}f(x^0).$$

(b) *There exists a continuous function* $f \in L(\mathbb{R}^k)$ *such that we have* $\mathcal{D}^*_{x_i x_j\,(\overline{x}_{M\setminus\{i,j\}})} f(x^0) = 0$, *but the limits*

$$\lim_{(x,x_{k+1})\overset{\wedge}{\to}(x^0,0)} \frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i \partial x_j}$$

*do not exist.*

*Proof of part* (b). This is given for the case $k = 2$. Assume that $D_1 = [0,1;0,1]$, $D_2 = [-1,0;0,1]$. Let

$$f(t_1,t_2) = \begin{cases} \sqrt{t_1 t_2} & \text{for } (t_1,t_2) \in D_1, \\ \sqrt{-t_1 t_2} & \text{for } (t_1,t_2) \in D_2, \\ 0 & \text{for } t_2 \le 0 \end{cases}$$

and extend $f(t_1,t_2)$ continuously to the set $\mathbb{R}^2 \setminus (D_1 \cup D_2)$ so that $f \in L(\mathbb{R}^2)$. It is easy to verify that $D^* f(0,0)$. Let $x_1^0 = x_2^0 = 0$ and $(x_1,x_2,x_3) \to (0,0,0)$ so that $x_2 = 0$ and $x_3 = x_1$. Then for the constructed function we have

$$\begin{aligned}
\frac{\partial^2 u(f;x_1,x_2,x_3)}{\partial x_1 \partial x_2} &= \frac{15 x_3}{2\pi} \Bigg\{ \int_0^1 \int_0^1 \frac{(t_1 - x_1) t_2 \sqrt{t_1 t_2}\, dt_1\, dt_2}{[(t_1 - x_1)^2 + t_2^2 + x_3^2]^{7/2}} + \\
&\quad + \int_{-1}^0 \int_0^1 \frac{(t_1 - x_1) t_2 \sqrt{-t_1 t_2}\, dt_1\, dt_2}{[(t_1 - x_1)^2 + t_2^2 + x_3^2]^{7/2}} \Bigg\} + o(1) = \\
&= c x_1 \Bigg\{ \int_{-x_1}^{1-x_1} \int_0^1 \frac{t_1 t_2 \sqrt{t_2 (t_1 + x_1)}\, dt_1\, dt_2}{(t_1^2 + t_2^2 + x_1^2)^{7/2}} - \\
&\quad - \int_{x_1}^{1+x_1} \int_0^1 \frac{t_1 t_2 \sqrt{(t_1 - x_1) t_2}\, dt_1\, dt_2}{(t_1^2 + t_2^2 + x_1^2)^{7/2}} \Bigg\} + o(1) = \\
&= c x_1 \Bigg\{ \int_{-x_1}^{x_1} \int_0^1 \frac{t_1 t_2 \sqrt{t_2 (t_1 + x_1)}\, dt_1\, dt_2}{(t_1^2 + t_2^2 + x_1^2)^{7/2}} + \\
&\quad + \int_{x_1}^{1-x_1} \int_0^1 \frac{t_1 t_2 [\sqrt{t_2 (x_1 + t_1)} - \sqrt{t_2 (t_1 - x_1)}]}{(t_1^2 + t_2^2 + x_1^2)^{7/2}}\, dt_1\, dt_2 - \\
&\quad - \int_{1-x_1}^{1+x_1} \int_0^1 \frac{t_1 t_2 \sqrt{(t_1 - x_1) t_2}\, dt_1\, dt_2}{(t_1^2 + t_2^2 + x_1^2)^{7/2}} \Bigg\} + o(1) =
\end{aligned}$$

$$= Cx_1(\mathcal{I}_1+\mathcal{I}_2-\mathcal{I}_3)+o(1).$$

We can readily show that

$$\mathcal{I}_1=\int\limits_0^{x_1}\int\limits_0^{1}\frac{t_1t_2[\sqrt{t_2(t_1+x_1)}-\sqrt{t_2(x_1-t_1)}]}{(t_1^2+t_2^2+x_1^2)^{7/2}}\,dt_1\,dt_2>0,\ \ \mathcal{I}_3=O(1). \quad (5)$$

Next,

$$\mathcal{I}_2>\int\limits_{x_1}^{2x_1}\int\limits_0^{x_1}\frac{t_1t_2[\sqrt{t_2(x_1+t_1)}-\sqrt{t_2x_1}]}{(t_1^2+t_2^2+x_1^2)^{7/2}}\,dt_1\,dt_2=$$

$$=\int\limits_0^{x_1}t_2^{3/2}\int\limits_{x_1}^{2x_2}\frac{t_1(\sqrt{x_1+t_1}-\sqrt{x_1})}{(t_1^2+t_2^2+x_1^2)^{7/2}}\,dt_1>$$

$$>\int\limits_0^{x_1}t_2^{3/2}\int\limits_{x_1}^{2x_2}\frac{x_1(\sqrt{2x_1}-\sqrt{x_1})}{(4x_1^2+x_1^2+x_1^2)^{7/2}}\,dt_1>\frac{c}{x_1^2}. \quad (6)$$

Therefore for $x_2=0$ and $x_3=x_1$ (5) and (6) imply $\frac{\partial^2u(f;x_1,0,x_1)}{\partial x_1\partial x_2}>\frac{c}{x_1}$, from which we obtain $\frac{\partial^2u(f;x_1,x_2,x_3)}{\partial x_1\partial x_2}\to+\infty$, as $(x_1,x_2,x_3)\to(0,0,0)$. □

**Theorem 9.** *If at the point $x^0$ there exists a finite derivative $\mathcal{D}^*_{[x_ix_j]\,(x)}f(x^0)$, then*

$$\lim_{(x,x_{k+1})\to(x^0,0)}\frac{\partial^2u(f;x,x_{k+1})}{\partial x_i\partial x_j}=\mathcal{D}^*_{x_ix_j}f(x^0).$$

Part (a) of Theorem 3 is the corollary of Theorem 9.

**Theorem 10.** *If at the point $x^0$ there exists a finite derivative $\mathcal{D}^*_{[x_i]x_j\,(x)}f(x^0)$, then*

$$\lim_{(x-x_je_j+x_j^0e_j,x_{k+1})\to(x^0,0)}\frac{\partial^2u(f;x-x_je_j+x_j^0e_j,x_{k+1})}{\partial x_i\partial x_j}=\mathcal{D}^*_{x_ix_j}f(x^0).$$

**Lemma 3.** *The following equalities are valid:*

$$\int\limits_0^{\infty}\frac{2\rho^2-kx_{k+1}^2}{(\rho^2+x_{k+1}^2)^{\frac{k+5}{2}}}\,\rho^{k-1}d\rho=0,$$

$$\frac{(k+1)x_{k+1}\Gamma(\frac{k+1}{2})}{k\sqrt{\pi}\Gamma(\frac{k}{2})}\int\limits_0^{\infty}\frac{3\rho^2-kx_{k+1}^2}{(\rho^2+x_{k+1}^2)^{\frac{k+5}{2}}}\,\rho^{k+1}d\rho=1.$$

**Theorem 11.**

(a) *Let $B\subset M$. If at the point $x^0$ there exists a finite derivative $\overline{\Delta}_{x_B} f(x^0)$, then*

$$\lim_{x_B+x^0_{B'},x_{k+1})\to(x^0,0)} \Delta_x u(f;x_B+x^0_{B'},x_{k+1})=\overline{\Delta}_{x_B} f(x^0).$$

(b) *There exists a function $f\in L(\mathbb{R}^k)$ such that $\overline{\Delta} f(x^0)$ exists but the limit* $\lim_{x,x_{k+1})\overset{\wedge}{\to}(x^0,0)} \Delta_x u(f;x,x_{k+1})$ *does not.*

*Proof of part* (a). One can easily verify that

$$\Delta_x u(f;x,x_{k+1})=-\frac{\partial^2 u(f;x,x_{k+1})}{\partial x^2_{k+1}}=$$
$$=\frac{(k+1)x_{k+1}\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}}\int\limits_{\mathbb{R}^k}\frac{3|t-x|^2-kx^2_{k+1}}{(|t-x|^2+x^2_{k+1})^{\frac{k+5}{2}}}\,f(t)\,dt.$$

Assume that $\mathcal{D}_k=\frac{2(k+1)\Gamma(\frac{k+1}{2})}{\sqrt{\pi}\Gamma(\frac{k}{2})}$ and $\theta=[0,\pi]^{k-2}\times[0,2\pi]$. Passing to the spherical coordinates, we have

$$\Delta_x u(f;x_B+x^0_{B'},x_{k+1})=c_k x_{k+1}\int\limits_0^\infty\frac{3\rho^2-kx^2_{k+1}}{(\rho^2+x^2_{k+1})^{\frac{k+5}{2}}}\times$$
$$\times\int\limits_\Theta f(x_B+x^0_{B'}+t)\rho^{k-1}\sin^{k-2}\vartheta_1\cdots\sin\vartheta_{k-2}\,d\rho\,d\vartheta_1\cdots d\vartheta_{k-2}\,d\varphi=$$
$$=\mathcal{D}_k x_{k+1}\int\limits_0^\infty\frac{3\rho^2-kx^2_{k+1}}{(\rho^2+x^2_{k+1})^{\frac{k+5}{2}}}\,\rho^{k-1}\Bigg[\frac{1}{|S_\rho|}\int\limits_{S_\rho(x_B+x^0_{B'})}f(t)\,dS(t)\Bigg]d\rho.$$

Hence by Lemma 3 we obtain

$$\Delta_x u(f;x_B+x^0_{B'},x_{k+1})-\overline{\Delta}_{x_B} f(x^0)=\mathcal{D}_k x_{k+1}\int\limits_0^\infty\frac{3\rho^2-kx^2_{k+1}}{(\rho^2+x^2_{k+1})^{\frac{k+5}{2}}}\times$$
$$\times\Bigg[\frac{\frac{1}{|S_\rho|}\int_{S_\rho(x_B+x^0_{B'})}f(t)\,dS(t)-f(x_B+x^0_{B'})}{\rho^2/2k}-\overline{\Delta}_{x_B} f(x^0)\Bigg]\frac{\rho^2}{2k}\,d\rho.$$

For the proof of part (b) see [2], p. 16. □

**Corollary 1.** *If at the point $x^0$ there exists a finite derivative $\overline{\Delta} f(x^0)$, then*

$$\lim_{x_{k+1}\to 0+}\Delta_x u(f;x^0,x_{k+1})=\overline{\Delta} f(x^0).$$

**Corollary 2.** *If at the point $x^0$ there exists a finite derivative $\overline{\Delta} f(x^0)$, then*

$$\lim_{(x,x_{k+1})\to(x^0,0)} \Delta_x u(f;x,x_{k+1}) = \overline{\Delta}_x f(x^0).$$

Let $\delta > 0$ and $S_\delta = \prod\limits_{\nu=1}^{k} [x_\nu^0 - \delta; x_\nu^0 + \delta]$.

**Theorem 12.** *Let $f'_{t_i}(t) \in L(S_\delta)$. If $f'_{t_i}(t)$ has a finite derivative $\mathcal{D}_{x_i\,(\overline{x}_{M\setminus i})} f'_{x_i}(x^0)$ at the point $x^0$, then*

$$\lim_{(x,x_{k+1})\underset{x_i}{\overset{\wedge}{\to}}(x^0,0)} \frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i^2} = f''_{x_i^2}(x^0).$$

*Proof.* Let $x^0 = 0$. The validity of the theorem follows from the equality

$$\frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i^2} - \mathcal{D}_{x_i\,(\overline{x}_{M\setminus i})} f'_{x_i}(0) =$$
$$= c_k x_{k+1} \int\limits_{S_\delta} \frac{(t_i - x_i)t_i}{(|t-x|^2 + x_{k+1}^2)^{\frac{k+3}{2}}} \left[ \frac{f'_{t_i}(t) - f'_{t_i}(t - t_i e_i)}{t_i} - \right.$$
$$\left. - \mathcal{D}_{x_i\,(\overline{x}_{M\setminus i})} f'_{x_i}(0) \right] dt + o(1). \qquad \square$$

The theorems below are proved analogously.

**Theorem 13.** *Let $f'_{x_i}(t) \in L(S_\delta)$. If $f'_{x_i}(t)$ has a finite derivative $\mathcal{D}_{x_j\,(\overline{x}_{M\setminus j})} f'_{x_i}(x^0)$ at the point $x^0$, then*

$$\lim_{(x,x_{k+1})\underset{x_j}{\overset{\wedge}{\to}}(x^0,0)} \frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i \partial x_j} = \frac{\partial^2 f(x^0)}{\partial x_i \partial x_j}.$$

**Corollary.** *Let $f'_{x_i}$ and $f'_{x_j} \in L(S_\delta)$ for some $\delta > 0$. If at the point $x^0$ there exist finite derivatives $\mathcal{D}_{x_i\,(\overline{x}_{M\setminus i})} f'_{x_j}(x^0)$ and $\mathcal{D}_{x_j\,(\overline{x}_{M\setminus j})} f'_{x_i}(x^0)$, then*

$$\lim_{(x,x_{k+1})\underset{x_i x_j}{\overset{\wedge}{\to}}(x^0,0)} \frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i \partial x_j} = \frac{\partial^2 f(x^0)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x^0)}{\partial x_j \partial x_i}.$$

Note that by this corollary we have obtained the condition of the coincidence of mixed derivatives of the function $f$ at the point $x^0$.

**Theorem 14.** *Let $f'_{x_i} \in L(S_\delta)$ for some $\delta > 0$. If at the point $x^0$ there exists a finite derivative $\overline{\mathcal{D}}_{x_i\,(x)} f'_{x_i}(x^0)$, then*

$$\lim_{(x,x_{k+1})\to(x^0,0)} \frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i^2} = \frac{\partial^2 f(x^0)}{\partial x_i^2}.$$

**Theorem 15.** *Let $f'_{x_i} \in L(S_\delta)$. If $f'_{x_i}(t)$ has a finite derivative $\overline{\mathcal{D}}_{x_j\,(x)} f'_{x_i}(x^0)$ at the point $x^0$, then*

$$\lim_{(x,x_{k+1})\to(x^0,0)} \frac{\partial^2 u(f;x,x_{k+1})}{\partial x_i \partial x_j} = \frac{\partial^2 f(x^0)}{\partial x_i \partial x_j}.$$

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