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ON THE SOLVABILITY OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Sufficient conditions are established for the solvability of the boundary value problem

$$\frac{dx(t)}{dt} = p(x,x)(t) + q(x)(t),$$
$$l(x,x) = c(x)\,,$$

where $p : C(I;R^n) \times C(I;R^n) \to L(I;R^n)$, $q : C(I;R^n) \to L(I;R^n)$, $l : C(I,R^n) \times C(I;R^n) \to R^n$, and $c_n : C(I,R^n) \to R^n$ are continuous operators, and $p(x,\cdot)$ and $l(x,\cdot)$ are linear operators for any fixed $x \in C(I;R^n)$.

1. Formulation of the Main Results

1.1. Formulation of the problem. Let n be a natural number, $I = [a,b]$, $-\infty < a < b + \infty$ and $p : C(I;R^n) \times C(I;R^n) \to L(I,R^n)$, $q : C(I;R^n) \to L(I;R^n)$, $l : C(I;R^n) \times C(I;R^n) \to R^n$ and $c : C(I;R^n) \to R^n$ be continuous operators. We consider the vector functional differential equation

$$\frac{dx(t)}{dt} = p(x,x)(t) + q(x)(t) \tag{1.1}$$

with the boundary condition

$$l(x,x) = c(x)\,. \tag{1.2}$$

By a solution of (1.1) we mean an absolutely continuous vector function $x : I \to R^n$ which satisfies it almost everywhere in I, and by a solution of problem (1.1), (1.2) a solution of (1.1) satisfying condition (1.2).

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In the present paper, we use the results proved in [1,2] to establish new sufficient conditions for the solvability of problem (1.1), (1.2). The results obtained are made more concrete for the boundary value problem

$$\frac{dx(t)}{dt} = \sum_{i=1}^{m} \mathcal{P}_i(x)(t)x\left(\tau_i(t)\right) + q(x)(t)\,, \tag{1.3}$$

$$\sum_{i=1}^{m_0} H_i(x)x\left(t_i\right) = c(x)\,, \tag{1.4}$$

where $\mathcal{P}_i : C(I, R^n) \to L(I; R^{n\times n})$ $(i = 1, \dots, m)$ and $H_i : C(I; R^n) \to R^n$ $(i = 1, \dots, m_0)$ are continuous operators, and $t_i \in I$ $(i = 1, \dots, m_0)$ and $\tau_j : I \to I$ $(i = 1, \dots, m)$ are measurable functions.

1.2. Basic notation and terms. Throughout this paper the following notation and terms are used.

$$R =]-\infty, +\infty[, R_+ = [0, +\infty[\ .$$

R^n is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in R$ $(i = 1, \dots, n)$ and the norm

$$\|x\| = \sum_{i=1}^{n} |x_i|\,.$$

$R^{n\times n}$ is the space of $n \times n$ matrices $X = (x_{ik})_{i,k=1}^n$ with elements $x_{ik} \in R$ $(i, k = 1, \dots, n)$ and the norm

$$\|X\| = \sum_{i,k=1}^{n} |x_{ik}|\,;$$

$$R_+^n = \{(x_i)_{i=1}^n \in R^n : x_i \geq 0 \quad (i = 1, \dots, n)\},$$

$$R_+^{n\times n} = \{(x_{ik})_{i,k=1}^n \in R^{n\times n} : x_{ik} \geq 0 \quad (i, k = 1, \dots, n)\}\,.$$

If $x, y \in R^n$ and $X, Y \in R^{n\times n}$, then

$$x \leq y \Leftrightarrow y - x \in R_+^n, \quad X \leq Y \Leftrightarrow Y - X \in R_+^{n\times n}\,.$$

If $x = (x_i)_{i=1}^n \in R^n$ and $X = (x_{ik})_{i,k=1}^n \in R^{n\times n}$, then

$$|x| = (|x_i|)_{i=1}^n, \quad |X| = (|x_{ik}|)_{i,k=1}^n\,.$$

X^{-1} is the inverse matrix of X; E is the unit matrix; $\det(X)$ is the determinant of the matrix X; $r(X)$ is the spectral radius of the matrix X.

A vector or a matrix function is said to be continuous, summable, etc. if all its components have such a property.

$C(I;R^n)$ is the space of continuous vector functions $x : I \to R^n$ with the norm

$$\|x\|_C = \max\{\|x(t)\| : t \in I\}\,.$$

If $x = (x_i)_{i=1}^n \in C(I;R^n)$, then $|x|_C = (\|x_i\|_C)_{i=1}^n$.

$C(I;R^{n\times n})$ is the space of continuous matrix functions $X : I \to R^{n\times n}$.

$L(I;R^n)$ is the space of summable vector functions $x : I \to R^n$ with the norm

$$\|x\|_L = \int_a^b \|x(t)\|\,dt\,.$$

$L(I;R^{n\times n})$ is the space of summable matrix functions $x : I \to R^{n\times n}$.

If $Y \in C(I;R^{n\times n})$ and $l_0 : C(I;R^n) \to R^n$ and $p_0 : C(I;R^n) \to L(I;R^n)$ are linear operators, then $l_0(Y) \in R^{n\times n}$ and $p_0(Y) \in L(I;R^{n\times n})$ are the matrix satisfying, for every $u \in R^n$, the equalities

$$l_0(Yu) = l_0(Y)u, \quad p_0(Yu)(t) = p_0(Y)(t)u \quad \text{for } t \in I\,.$$

$I_{t_0,t} = [t_0,t]$ for $t \ge t_0$ and $I_{t_0,t} = [t,t_0]$ for $t < t_0$.

A linear operator p is called a Volterra* operator with respect to $t_0 \in I$, if for arbitrary $t \in I$ and $x \in C(I;R^n)$ satisfying the condition $x(s) = 0$ for $s \in I_{t_0,t}$ we have $p(x)(s) = 0$ for almost all $s \in I_{t_0,t}$.

1.3. Problem (1.1), (1.2). We consider the case where:

(i) the operators $p : C(I;R^n)\times C(I;R^n) \to L(I;R^n)$ and $l : C(I;R^n)\times C(I;R^n) \to R^n$ are continuous and for arbitrary $x \in C(I;R^n)$ the operators $p(x,\cdot) : C(I;R^n) \to L(I;R^n)$ and $l(x,\cdot) : C(I;R^n) \to R^n$ are linear;

(ii) there exist a summable function $\alpha : [a,b] \to R_+$ and a positive number α_0 such that for arbitrary x and $y \in C(I;R^n)$ the following inequalities are satisfied:

$$\|p(x,y)(t)\| \le \alpha(t)\|y\|_C \ \text{ for almost all } t \in I, \ \ \|l(x,y)\| \le \alpha_0\|y\|_C\,;$$

(iii) the operators $q : C(I;R^n) \to L(I;R^n)$ and $c : C(I;R^n) \to R^n$ are continuous and

$$\lim_{\varrho\to\infty} \frac{1}{\varrho}\int_a^b \eta(t,\varrho)\,dt = 0, \quad \lim_{\varrho\to\infty} \frac{\eta_0(\varrho)}{\varrho} = 0\,,$$

where $\eta(t,\varrho) = \sup\{\|q(x)(t)\| : \|x\|_C \le \varrho\}$, $\eta_0(\varrho) = \sup\{\|c(x)\| : \|x\|_C \le \varrho\}$.

Following [2] we introduce

*See [3–5].

Definition 1.1. Let p and l be the operators satisfying conditions (i) and $p_0 : C(I;R^n) \to L(I;R^n)$ and $l_0 : C(I;R^n) \to R^n$ be the linear operators. We say that the pair of operators (p_0, l_0) belongs to the set $\mathcal{E}^n_{p,l}$ if there is a sequence $x_k \in C(I;R^n)$ $(k = 1, 2, \dots)$ such that for any $y \in C(I;R^n)$ we have

$$\lim_{k\to\infty} \int_a^t p(x_k, y)(s)\,ds = \int_a^t p_0(y)(s)\,ds \quad \text{uniformly on } I\,, \tag{1.5}$$

$$\lim_{k\to\infty} l(x_k, y) = l_0(y)\,. \tag{1.6}$$

Definition 1.2. We say that a pair of operators (p, l) belongs to the Opial class $\mathcal{O}^n_0$, if p and l satisfy conditions (i) and (ii), and for any $(p_0, l_0) \in \mathcal{E}^n_{p,l}$ the problem

$$\frac{dy(t)}{dt} = p_0(y)(t)\,, \tag{1.7}$$

$$l_0(y) = 0 \tag{1.8}$$

has only the trivial solution.

Theorem 1.1. *If*

$$(p, l) \in \mathcal{O}^n_0 \tag{1.9}$$

and the operators q and c satisfy conditions (iii)*, then problem* (1.1)*,* (1.2) *is solvable.*

The proof of this theorem is contained in [2] (see [2], Corollary 1.1).

Let t_0 be an arbitrary fixed point of I. We introduce a sequence of operators $Y_j : C(I;R^n) \to C(I;R^{n\times n})$ and $z_j : C(I;R^n) \times C(I;R^n) \to C(I;R^n)$ $(j = 1, 2, \dots)$ by

$$Y_1(x)(t) = E,\;\; Y_{j+1}(x)(t) = E + \int_{t_0}^t p(x, Y_j(x))(s) \;\; (j = 1, 2, \dots) \tag{1.10}$$

$$z_1(x,y)(t) = \int_{t_0}^t p(x,y)(s)\,ds\,, z_{j+1}(x,y)(t) =$$

$$= \int_{t_0}^t p(x, z_j(x,y))(s)\,ds \quad (i = 1, 2, \dots)\,. \tag{1.11}$$

If for some $x \in C(I;R^n)$ and natural j the matrix $l(x,Y_j(x))$ is nonsingular, we set

$$\begin{aligned} z_{j,k}(x,y)(t) = z_k(x,y)(t) - \\ -Y_k(x)(t)[l(x,Y_j(x))]^{-1} l(x,z_j(x,y)) \quad (k=1,2,\dots). \end{aligned} \tag{1.12}$$

Theorem 1.2. *Suppose that conditions* (i)–(iii) *are satisfied, and there exist $t_0 \in I$, $\delta > 0$, $A \in R_+^{n\times n}$ and natural numbers j_0 and k_0 such that*

$$r(A) < 1 \tag{1.13}$$

and for arbitrary x and $y \in C(I;R^n)$ the inequalities

$$|\det(l(x,Y_{j_0}(x))| > \delta \tag{1.14}$$

and

$$|z_{j_0,k_0}(x,y)|_C \le A|y|_C \tag{1.15}$$

hold, where Y_j and $z_{j,k}$ are the operators given by (1.10)–(1.12). *Then problem* (1.1), (1.2) *is solvable.*

If p satisfies conditions (i) and (ii) and for some $t_0 \in I$ and arbitrary fixed $x \in C(I;R^n)$ the operator $p(x,\cdot) : C(I;R^n) \to L(I;R^n)$ is a Volterra operator with respect to t_0, then for any $x \in C(I;R^n)$ the differential equation

$$\frac{dy(t)}{dt} = p(x,y)(t) \tag{1.16}$$

has a unique fundamental matrix* $Y(x) : I \to R^{n\times n}$ satisfying the initial condition

$$Y(x)(t_0) = E \tag{1.17}$$

(see [1], Lemma 1.2). In this case, Theorem 1.2 takes the following form.

Theorem 1.3. *Suppose that conditions* (i)–(iii) *are satisfied, and there is $t_0 \in I$ such that for any $x \in C(I;R^n)$ the operator $p(x,\cdot)$ is a Volterra operator with respect to t_0. Furthermore, let*

$$\inf\Big\{\Big|\det\big(l\big(x,Y(x)\big)\big)\Big| : x \in C(I;R^n)\Big\} > 0 \tag{1.18}$$

where $Y(x)$ is the fundamental matrix of system (1.16), *satisfying the initial condition* (1.17). *Then problem* (1.1), (1.2) *is solvable.*

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*That is, the matrix whole columns form a basis of the solution space of equation (1.16).

Remark 1.1. Theorem 1.3 is a generalization of a theorem of R. Conti ([6], Theorem 2) and Corollary 1.4′ from [2] to differential systems of form (1.1).

1.4. Problem (1.3), (1.4). Problem (1.3), (1.4) is obtained from (1.1), (1.2) for

$$p(x,y)(t)=\sum_{i=1}^{m}\mathcal{P}_i(x)(t)y(\tau_i(t)),\quad l(x,y)=\sum_{i=1}^{m_0}H_i(x)y(t_i)\,. \tag{1.19}$$

In order to guarantee that conditions (i) and (ii) be satisfied, we consider the case where:

(iv) the operators $\mathcal{P}_i : C(I;R^n)\to L(I;R^{n\times n})$ $(i=1,\dots,m)$ and $H_i : C(I;R^n)\to R^{n\times n}$ $(i=1,\dots,m_0)$ are continuous and there are a summable function $\alpha : I\to R_+$ and a positive number α_0 such that for any $x\in C(I;R^n)$ and $t\in I$ the following inequalities hold:
$$\sum_{i=1}^{m}\|\mathcal{P}_i(x)(t)\|\le\alpha(t)\,,\quad \sum_{i=1}^{m_0}\|H_i(x)\|\le\alpha_0\,.$$

From (1.19) and (1.10) we obtain

$$Y_1(x)(t)=E,\quad Y_{j+1}(x)(t)=$$
$$=E+\sum_{i=1}^{m}\int_{t_0}^{t}\mathcal{P}_i(x)(s)Y_j(\tau_i(s))\,ds\quad(j=1,2,\dots)\,. \tag{1.20}$$

Together with this we introduce the sequence of operators

$$\begin{gathered}Z_1(x)(t)=\sum_{i=1}^{m}\left|\int_{t_0}^{t}|\mathcal{P}_i(x)(s)|\,ds\right|,\\ Z_{j+1}(x)(t)=\sum_{i=1}^{m}\left|\int_{t_0}^{t}|\mathcal{P}_i(x)(s)|Z_j(x)(\tau_i(s))\,ds\right|\quad(j=1,2,\dots)\,.\end{gathered} \tag{1.21}$$

Corollary 1.1. *Suppose that conditions* (iii), (iv) *are satisfied, and there exist* $t_0\in I$, $\delta>0$, $A\in R_+^{n\times n}$, *and natural numbers* j_0 *and* k_0, *such that* $r(A)<1$ *and for arbitrary* $x\in C(I;R^n)$ *the inequalities*

$$\left|\det\left(\sum_{i=1}^{m_0}H_i(x)Y_{j_0}(x)(t)\right)\right|>\delta \tag{1.22}$$

and

$$\Big|Y_{k_0}(x)(t)\left(\sum_{i=1}^{m_0} H_i(x)Y_{j_0}(x)(t_i)\right)^{-1}\Big|\sum_{i=1}^{m_0}|H_i(x)|Z_{j_0}(x)(t_i)+ \\ +Z_{k_0}(x)(t)\le A \quad \text{for} \quad t\in I \tag{1.23}$$

hold, where Y_j *and* Z_k *are the operators given by* (1.20) *and* (1.21).

Then problem (1.3), (1.4) *is solvable.*

Corollary 1.2. *Suppose that conditions* (iii), (iv) *are satisfied, and there is* $t_0 \in I$ *such that*

$$(t-\tau_i(t))(t-t_0)\ge 0 \quad \text{for} \quad t\in I \quad (i=1,\dots,m)\,. \tag{1.24}$$

Furthermore, let

$$\inf\left\{\Big|\det\left(\sum_{i=1}^{m_0} H_i(x)Y(x)(t_i)\right)\Big| : x\in C(I;R^n)\right\}>0\,, \tag{1.25}$$

where $Y(x)$ *id the fundamental matrix of the differential system*

$$\frac{dy(t)}{dt}=\sum_{i=1}^{m}\mathcal{P}_i(x)(t)y(\tau_i(t))\,,$$

satisfying the initial condition

$$Y(x)(t_0)=E\,.$$

Then problem (1.3), (1.4) *is solvable.*

2. Auxiliary Statements

Lemma 2.1. *Let* $p_0 : C(I;R^n)\to L(I;R^n)$ *and* $l_0 : C(I;R^n)\to R^{n\times n}$ *be linear operators, and let there exist a summable function* $\alpha : I\to R_+$ *such that for any* $y\in C(I;R^n)$ *the inequality*

$$\|p_0(y)(t)\|\le \alpha(t)\|y\|_C \quad \text{for almost all} \quad t\in I \tag{2.1}$$

holds. Furthermore, let p_0 *be a Volterra operator with respect to some* $t_0\in I$. *Then equation* (1.7) *has a unique fundamental matrix* Y_0 *satisfying the initial condition*

$$Y(t_0)=E \tag{2.2}$$

and the condition

$$\det\left(l_0(Y_0)\right)\ne 0 \tag{2.3}$$

is necessary and sufficient for problem (1.7), (1.8) *to have only the trivial solution.*

Proof. According to Lemma 1.2 in [1], for any $c_0 \in R$, equation (1.7) has a unique solution satisfying the initial condition

$$y(t_0) = c_0 \,.$$

It follows that this equation has a unique fundamental matrix satisfying the initial condition (2.2) and each of its solutions admits a representation

$$y(t) = Y_0(t) c_0 \,.$$

It follows that problem (1.7), (1.8) has only the trivial solution if and only if the system of algebraic equations

$$l_0(Y_0) c_0 = 0$$

has only the trivial solution, i.e., condition (2.3) is satisfied. $\square$

Lemma 2.2. *Let p and l be the operators satisfying conditions* (i), (ii), *and $C(I;R^n) \to L(I;R^n)$ and $l_0 : C(I;R^n) \to R^n$ be linear operators. Furthermore, let there exist $t_0 \in I$ and $x_k \in C(I;R^n)$ $(k = 1, 2, \dots)$ such that for any natural k the operator $p(x_k, \cdot)$ is a Volterra operator with respect to t_0 and for any $y \in C(I;R^n)$ conditions* (1.5) *and* (1.6) *hold. Then p_0 is a Volterra operator with respect to t_0, satisfying condition* (2.1), *and*

$$\lim_{k\to\infty} l(x_k, Y(x_k)) = l_0(Y_0) \,, \tag{2.4}$$

where $Y(x_k)$ is the fundamental matrix of the equation

$$\frac{dy(t)}{dt} = p(x_k, y)(t) \,, \tag{2.5}$$

satisfying the condition

$$Y(x_k)(t_0) = E \,,$$

and Y_0 is the fundamental matrix of (1.7) *satisfying condition* (2.2).

Proof. By (1.5), for any s and $t \in I$ we have

$$\int_s^t p_0(y)(\xi)\, d\xi = \lim_{k\to\infty} \int_s^t p(x_k, y)(\xi)\, d\xi \,.$$

Since $p(x_k, \cdot)$ is a Volterra operator with respect to t_0 it clearly follows that p_0 is also Volterra operator with respect to t_0. On the other hand, using condition (ii), we get from the last equality

$$\Big\| \int_s^t p_0(y)(\xi)\, d\xi \Big\| \le \Big| \int_s^t \alpha(\xi)\, d\xi \Big| \, \|y\|_C \,.$$

Dividing both sides by $t - s$ and passing to the limit for $s \to t$ we get (2.1).

By Corollary 1.6 in [1], it follows from (i), (ii), (1.5), (2.1) and from the unique solvability of the Cauchy problem for (1.7) with the initial condition at t_0 that

$$\lim_{k\to\infty} \|Y(t) - Y_0\|_C = 0\,. \tag{2.6}$$

On the other hand, by (1.6)

$$\lim_{k\to\infty} \|l(x_k, Y_0) - l_0(Y_0)\| = 0\,. \tag{2.7}$$

Taking into account (ii), (2.6) and (2.7), we find

$$\begin{aligned} &\|l(x_k, Y(x_k)) - l_0(Y_0)\| \le \\ &\le \|l(x_k, Y(x_k) - Y_0)\| + \|l(x_k, Y_0) - l_0(Y)\| \le \\ &\le \alpha_0\|Y_k - Y_0\|_C + \|l(x_k, Y_0) - l_0(Y_0)\| \to 0 \quad \text{for} \quad k\to\infty\,. \end{aligned}$$

Consequently, equality (2.4) holds. □

Lemma 2.3. *Let conditions* (i), (ii) *hold and let there exist* $t_0 \in I$ *such that for any* $x \in C(I; R^n)$ *the operator* $p(x,\cdot)$ *is a Volterra operator with respect to* t_0. *Then inequality* (1.18) *is necessary and sufficient for condition* (1.9) *to hold.*

Proof. First we prove the necessity. Assume the contrary, that (1.9) holds, but (1.18) is violated. Then there is a sequence $x_k \in C(I; R^n)$ $(k = 1, 2, \dots)$ such that

$$\lim_{k\to\infty} \det\Big(l\big(x_k, Y(x_k)\big)\Big) = 0\,. \tag{2.8}$$

It follows from (i), (ii) and Lemma 2.1 in [1] that, without loss of generality, we can assume that conditions (1.5) and (1.6) hold for any $y \in C(I; R^n)$, where $(p_0, l_0) \in \mathcal{E}^n_{p,c}$. Then, by Lemma 2.2 and equality (2.8), the operator p_0 is a Volterra operator with respect to t_0, inequality (2.1) holds, and

$$\det\big(l_0(Y_0)\big) = 0\,,$$

where y_0 is the fundamental matrix of (1.7), satisfying condition (2.2). Therefore, it follows from Lemma 2.1 that problem (1.7), (1.8) has a non-trivial solution. This is a contradiction to (1.9), which proves that (1.18) holds.

Now we turn to proving the sufficiency. Let $(p_0, l_0) \in \mathcal{E}^n_{p,l}$. Then by Definition 1.1 there is a sequence $x_k \in C(I; R^n)$ $(k = 1, 2 \dots)$ such that for any $y \in C(I; R^n)$ conditions (1.5) and (1.6) hold. If we now apply Lemma 2.2, it becomes clear that the operator p_0 is a Volterra operator with respect to t_0, and conditions (2.1), (2.4) hold, where Y_0 is the fundamental matrix of equation (1.7), satisfying condition (2.2). On the other hand, by (1.18) condition (2.4) implies (2.3). By Lemma 2.1, this inequality implies that problem (1.7), (1.8) has only the trivial solution. Now since $(p_0, l_0) \in \mathcal{E}^n_{p,l}$ was arbitrary, condition (1.9) is obvious. □

3. Proofs of Main Results

Proof of Theorem 1.2. According to Theorem 1.1 it is enough to show that for any $(p_0, l_0) \in \mathcal{E}^n_{p,l}$ problem (1.7), (1.8) has only the trivial solution.

By Definition 1.1 there is a sequence $x_k \in C(I; R^n)$ $(k = 1, 2, \dots)$ such that for any $y \in C(I; R^n)$ conditions (1.5) and (1.6) hold. Using (ii), (1.5), and (1.6), we get from (1.10)

$$\lim_{k\to\infty} Y_j(x_k)(t) = Y_j^0(t) \quad \text{uniformly on } I \quad (j = 1, 2, \dots), \tag{3.1}$$

$$\lim_{k\to\infty} z_j(x_k, y)(t) = z_j^0(y)(t) \quad \text{uniformly on } I \quad (j = 1, 2, \dots) \tag{3.2}$$

for any $y \in C(I; R^n)$ and

$$\lim_{k\to\infty} l(x_k, Y_j(x_k)) = l_0(Y_j^0) \quad (j = 1, 2, \dots), \tag{3.3}$$

where

$$Y_1^0(t) = E, \ Y_{j+1}^0(t) = E + \int_{t_0}^{t} p_0(Y_j^0)(s)\, ds \quad (j = 1, 2, \dots),$$

$$z_1^0(y)(t) = \int_{t_0}^{t} p_0(y)(s)\, ds, \quad z_{j+1}^0(y)(t) = \int_{t_0}^{t} p_0(z_j^0(y))(s)\, ds \quad (j = 1, 2, \dots).$$

By (3.1)–(3.3) we get from (1.12), (1.14) and (1.15)

$$\det(l_0(Y_{j_0}^0)) \neq 0 \tag{3.4}$$

and

$$|z_{j_0,k_0}^0(y)|_C \leq A|y|_C\,, \tag{3.5}$$

where

$$z_{j_0,k_0}(y)(t) = z_{k_0}^0(y)(t) - Y_{k_0}^0(t)\big[l_0(Y_{j_0}^0)\big]^{-1} l_0(z_{j_0}^0)\,.$$

According to Theorem 1.2 in [1], it follows from (1.13), (3.4), and (3.5) that problem (1.7), (1.8) has only the trivial solution. □

Theorem 1.1 and Lemma 2.3 immediately imply Theorem 1.3.

Remark 3.1. In the case where $p(x, \cdot)$ is a Volterra operator with respect to t_0 for any $x \in C(I; R^n)$, then by Lemma 2.3, Theorem 1.1 and 1.3 are equivalent.

Proof of Corollary 1.1. As we already noted, problem (1.3), (1.4) is obtained from problem (1.1), (1.2) when the operators p and l are given by (1.19). In this case, obviously, conditions (1.14) and (1.22) are equivalent. On the other hand, using (1.19) and (1.21), it follows from (1.11) that

$$|z_j(x,y)(t)| \leq Z_j(x)(t)|y|_C \quad \text{for} \quad t \in I \quad (j=1,2,\dots)$$

and

$$|l(x,z_j(x,y))| \leq \sum_{i=1}^{m_0} |H_i(x)|Z_j(x)(t_i)|y|_C \quad (j=1,2,\dots)\,.$$

Using these inequalities, we get from (1.12) and (1.23)

$$|z_{j_0,k_0}(x,y)(t)| \leq Z_{k_0}(x)(t)|y|_C + |Y_{k_0}(x)(t)| \Big(\sum_{i=1}^{m_0} H_i(x)Y_{j_0}(x)(t_i)\Big)^{-1}\Big| \times$$
$$\times \sum_{i=1}^{m_0} |H_i(x)\Big| Z_{j_0}(x)(t_i)|y|_C \leq A|y|_C \ \text{ for } \ t \in I\,.$$

Consequently, inequality (1.15) holds. This argument proves that all conditions of Theorem 1.2 are satisfied, which guarantees the unique solvability of the problem under consideration. □

Corollary 1.2 is obtained from Theorem 1.3. It is enough to take into account that condition (1.24) and the equality

$$p(x,y)(t) = \sum_{i=1}^{m} \mathcal{P}_i(x)(t)y(\tau_i(t))$$

guarantee that $p(x,\cdot)$ is a Volterra operator with respect to t_0 for any $x \in C(I;R^n)$. □

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