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ON THE BOUNDEDNESS OF CLASSICAL OPERATORS ON WEIGHTED LORENTZ SPACES

Y. RAKOTONDRATSIMBA

ABSTRACT. Conditions on weights $u(\cdot)$, $v(\cdot)$ are given so that a classical operator T sends the weighted Lorentz space $L^{rs}(vdx)$ into $L^{pq}(udx)$. Here T is either a fractional maximal operator M_α or a fractional integral operator I_α or a Calderón–Zygmund operator. A characterization of this boundedness is obtained for M_α and I_α when the weights have some usual properties and $\max(r,s) \leq \min(p,q)$.

§ 0. INTRODUCTION

Let $u(\cdot)$, $v(\cdot)$, $w_1(\cdot)$, $w_2(\cdot)$ be weight functions on $\mathbb{R}^n$, $n \in \mathbb{N}^*$, i.e., nonnegative locally integrable functions; and let T be a classical operator. The purpose of this paper is to determine when T is bounded from the weighted Lorentz space $L^{rs}_v(w_1)$ into $L^{pq}_u(w_2)$, i.e.,

$$\Big\|w_2(\cdot)(Tf)(\cdot)\Big\|_{L^{pq}_u} \leq C\Big\|w_1(\cdot)\, f(\cdot)\Big\|_{L^{rs}_v} \quad \text{for all functions} \quad f(\cdot). \tag{0.0}$$

Here $C > 0$ is a constant which depends only on n, p, q, r, s, and on the weight functions. Recall that

$$\|g(\cdot)\|^q_{L^{pq}_u} = q\int_0^\infty \Big(\int_{\{y\in\mathbb{R}^n;\, |g(y)|>\lambda\}} u(y)dy\Big)^{\frac{q}{p}} \lambda^{q-1}d\lambda,$$

for $1 \leq p < \infty$ and $1 \leq q < \infty$; and

$$\|g(\cdot)\|_{L^{p\infty}_u} = \sup_{\lambda>0} \lambda\Big(\int_{\{y\in\mathbb{R}^n;\, |g(y)|>\lambda\}} u(y)dy\Big)^{\frac{1}{p}}$$

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for $1 \leq p < \infty$. It is always assumed that $1 < r, s, p, q < \infty$. For convenience, the embedding defined by (0.0) will be denoted by $T : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$.

The classical operator under consideration is a fractional maximal operator or a fractional integral operator or a Calderón–Zygmund operator. The fractional maximal operator M_α of order α, $0 \leq \alpha < n$, is defined as

$$(M_\alpha f)(x) = \sup\Big\{|Q|^{\frac{\alpha}{n}-1} \int_Q |f(y)|dy; \quad Q \text{ a cube with } Q \ni x\Big\}.$$

Here Q is a cube with sides parallel to the coordinate planes. Thus $M = M_0$ is the well-known Hardy–Littlewood maximal operator. The fractional integral operator I_α, $0 < \alpha < n$, is given by

$$(I_\alpha f)(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)dy.$$

The Hilbert transform

$$(Hf)(x) = P.V. \int_{\mathbb{R}^1} \frac{f(y)}{x-y}dy = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}dy$$

is a particular case of the Calderón–Zygmund operator.

The boundedness $M : L_v^{rs}(1) \to L_u^{pq}(1)$ was considered and studied by many authors (see, for instance, [1], [2] and the references therein). However, as mentioned by Kokilashvili and Krbec [1], easy necessary and sufficient conditions on $v(\cdot)$, $u(\cdot)$ for which $M_\alpha : L_v^{rs}(1) \to L_u^{pq}(1)$, $0 \leq \alpha < n$, are not known. In this paper we find a sufficient condition for such a boundedness. For weight functions having some special properties (generally shared by usual weights), the condition found here is also a necessary one. One of the reasons which lead to considering $M_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$ is the fact that weights cannot be combined as in the Lebesgue case where, for instance, $\|f(\cdot)\|_{L_u^{pp}} = \|u^{\frac{1}{p}}(\cdot)f(\cdot)\|_{L_1^{pp}}$.

A weight function $w(\cdot)$ is constant on annuli if for a constant $c > 0$

$$\sup_{R<|y|\leq 64R} w(y) \leq c \inf_{R<|z|\leq 64R} w(z) \quad \text{for all} \quad R > 0.$$

This latter condition can be denoted by $w(\cdot) \in \mathcal{A}$. If $w(x) = |x|^\alpha \ln^\beta(e + |x|)$, with $\alpha \in \mathbb{R}$ and $\beta \geq 0$, then $w(\cdot) \in \mathcal{A}$. A large class of weight functions $w(\cdot)$ for which $w(\cdot) \in \mathcal{A}$ is given by those nondecreasing (resp. nonincreasing) radial $w(\cdot)$ which satisfy $w(64t) \leq Cw(t)$ (resp. $w(t) \leq Cw(64t)$) for all $t > 0$. In the proof of Lemma 1 below, it is observed that $M_\alpha : L_v^{rs}(1) \to L_u^{pq}(1)$ implies necessarily $\mathcal{H} : L_v^{rs}(1) \to L_u^{pq}(w)$ with $w(x) = |x|^{\alpha-n}$ and $(\mathcal{H}f)(x) = \int_{|y|<|x|} f(y)dy$. In view of this observation and also for convenience, it is always supposed that $w_1(\cdot) \in \mathcal{A}$ and $w_2(\cdot) \in \mathcal{A}$.

As in Lemma 1 below, the boundedness $M_\alpha : L_v^{rs}(1) \to L_u^{pq}(1)$ implies

$$\left\|(R+|cdot|)^{\alpha-n}\right\|_{L_u^{pq}} \left\|\frac{1}{v(\cdot)}1\!\!1_{|\cdot|<R}(\cdot)\right\|_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}} \le C \quad \text{for all } R>0(0.1)$$

and

$$|x|^{n[\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}]}(u(x))^{\frac{1}{p}} \le c(v(x))^{\frac{1}{r}} \quad \text{for almost every } x \tag{0.2}$$

for $\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}=0$. Here $1\!\!1_E(\cdot)$ denotes the characteristic function of the measurable set E. Since (0.2) is a pointwise inequality, this condition can be easily checked for given weights $u(\cdot)$ and $v(\cdot)$. Contrary to the well-known standard conditions (see [2], [1]), (0.1) is expressed neither in terms of the operator M_α itself nor in terms of arbitrary cubes. This test condition needs only integrations on balls centered at the origin, which are well adapted for radial weight functions (the most useful weights in applications). Consequently, our idea is to derive $M_\alpha : L_v^{rs}(1) \to L_u^{pq}(1)$ from conditions (0.1) and (0.2). However, when testing the problem in classical Lebesgue spaces, it is not reasonable to expect that the above embedding can be obtained only from these two conditions.

Roughly speaking, for $\max(r,s) \le \min(p,q)$ we will prove that $M_\alpha : L_v^{rs}(1) \to L_u^{pq}(1)$ whenever both (0.1) and a more stronger condition than (0.2) are satisfied (see Theorem 2). Eventually, for the Lebesgue case (i.e., $r=s$, $p=q$) the results we find are new. It is also of interest to note that the conditions used to get (0.0) are suitable for explicit computations.

The main results are presented in §1. The basic lemmas needed to prove them are given in §2. These latter are proved in §3. The final §5 is devoted to the proofs of the basic lemmas given in §2.

§ 1. The Results

Recall that our purpose is to study $T : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$, i.e.,

$$\left\|w_2(\cdot)(Tf)(\cdot)\right\|_{L_u^{pq}} \le C\left\|w_1(\cdot)f(\cdot)\right\|_{L_v^{rs}} \quad \text{for all functions } f(\cdot),$$

where T is a classical operator defined as above and $w_1(\cdot)$, $w_2(\cdot) \in \mathcal{A}$. Indeed, in considering this boundedness, restrictions on the range of r, s, p, q, and on the weight functions have to be done. To simplify the statement, consider the case of $T = M_\alpha$.

Lemma 1. *Let $0 \le \alpha < n$. Assume the embedding $M_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$ is satisfied. Then*

$$|Q|^{\frac{\alpha}{n}}\left\|w_2(\cdot)1\!\!1_Q(\cdot)\right\|_{L_u^{pq}} \le C_1\left\|w_1(\cdot)1\!\!1_Q(\cdot)\right\|_{L_v^{rs}} \quad \textit{for all cubes } Q. \tag{1.1}$$

Consequently if $w_1(\cdot) = w_2(\cdot) = 1$, then $\frac{1}{r} - \frac{1}{p} \le \frac{\alpha}{n}$. On the other hand,

$$\left\| \frac{1}{v(\cdot)w_1(\cdot)} 1\!\!1_Q(\cdot) \right\|_{L_v^{\frac{r}{r-1}\,\frac{s}{s-1}}} < \infty \quad \textit{for all cubes } Q. \tag{1.2}$$

The weight functions $u(\cdot)$, $v(\cdot)$ satisfy the Wheeden–Muckenhoupt condition

$$\left\| w_2(\cdot)\,(R + |\cdot|)^{\alpha-n} \right\|_{L_u^{pq}} \left\| \frac{1}{v(\cdot)w_1(\cdot)}\, 1\!\!1_{\{|\cdot|<R\}}(\cdot) \right\|_{L_v^{\frac{r}{r-1}\,\frac{s}{s-1}}} \le C_2 \tag{1.3}$$

for all $R > 0$.

Let $1 < r < \frac{n}{\alpha}$ and $\frac{1}{r^*} = \frac{1}{r} - \frac{\alpha}{n}$. If $p = r^*$ then

$$w_2(x)\,|x|^{n[\frac{1}{p} - \frac{1}{r^*}]}\,(u(x))^{\frac{1}{p}} \le c\, w_1(x)\,(v(x))^{\frac{1}{r}} \quad \textit{for almost every } x. \tag{1.4}$$

This inequality is also satisfied for $p \ne r^*$ if both $u(\cdot)$, $v(\cdot) \in \mathcal{A}$.

In view of this result, it will always be assumed that

$$v(\cdot) \quad \text{and} \quad w_1(\cdot) \quad \text{satisfy (1.2).}$$

So by (1.1), the study of $M_\alpha : L_v^{rs}(1) \to L_u^{pq}(1)$ for $0 \le \alpha < n$ and $1 < r < \frac{n}{\alpha}$ makes non-trivial sense only for the range $p \le r^*$. For this reason and also for technical motivation it will be supposed that

$$1 < r < \frac{n}{\alpha}, \quad p \le r^*, \quad \max(r, s) \le \min(p, q). \tag{1.5}$$

In dealing with M_α when $\max(r, s) = p < q$, it is useful to assume that

$$\sum_{m=-\infty}^{N-1} \left\| w_2(\cdot) 1\!\!1_{\{2^m < |\cdot| \le 2^{m+1}\}}(\cdot) \right\|_{L_u^{pq}}^p \le C \left\| w_2(\cdot) 1\!\!1_{\{|x| < 2^N\}}(\cdot) \right\|_{L_u^{pq}}^p \tag{1.6}$$

for all $N \in \mathbb{Z}$. Such an inequality is always satisfied when $q \le p$ (see Lemma 2 in §2). For the range $p < q$, (1.6) is true for some weight functions as in the case of $w_2(\cdot) = 1$, or for power weights (see Proposition 8 below).

A stronger condition than (1.4) is

$$\begin{aligned} w_2(x)\,|x|^{n[\frac{\alpha}{n} + \frac{1}{p} - \frac{1}{r}]} \Big(\sup_{4^{-1}|x| < |z| < 4|x|} u(z) \Big)^{\frac{1}{p}} \le \\ \le c\, w_1(x)\,(v(x))^{\frac{1}{r}} \quad \text{for a.e. } x, \end{aligned} \tag{1.7}$$

or

$$\begin{aligned} w_2(x)\,|x|^{n[\frac{\alpha}{n} + \frac{1}{p} - \frac{1}{r}]}\,(u(x))^{\frac{1}{p}} \Big(\sup_{4^{-1}|x| < |z| < 4|x|} v(z) \Big)^{-\frac{1}{r}} \le \\ \le c\, w_1(x) \quad \text{for a.e. } x, \end{aligned} \tag{1.7$'$}$$

We are now in the position to state our first main result for the fractional maximal operator.

Theorem 2. (*The fractional maximal operator M_α with $0 \le \alpha < n$*)
(A) *Suppose $M_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$. Then the Wheeden–Muckenhoupt condition* (1.3) *is satisfied.*
(B) *For the converse assume restrictions* (1.5) *and* (1.6) *hold. Then condition* (1.3) *implies $M_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$ whenever the pointwise inequality* (1.7) (*or* (1.7′)) *is satisfied.*

Remarks 3. (1) For the Hardy–Littlewood maximal operator $M = M_0$, this results deals with the embedding $M : L_v^{rs}(w_1) \to L_u^{rq}(w_2)$, since by restriction (1.5) we have $\alpha = 0$, $r^* = r = p$, and $s \le r = p < q$. For M_α with $0 < \alpha < n$, and in the Lebesgue case, i.e., $p = q$ and $r = s$, restriction (1.5) means $r \le p \le r^*$.
(2) Theorem 2 and Lemma 1 yield the following conclusion: *With restrictions* (1.5) *and* (1.6), *both conditions* (1.3) *and* (1.4) *characterize the embedding $M_\alpha : L_v^{rs}(w_1) \to L_u^{r^*q}(w_2)$ whenever either $u(\cdot)$ or $v(\cdot)$ is constant on annuli.* Similarly, if $p \ne r^*$ and both $u(\cdot)$ and $v(\cdot)$ are constant on annuli, then $M_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$ if and only if (1.3) and (1.4) are satisfied. Indeed, in the latter result condition (1.4) can be dropped by virtue of Proposition 9 and Remarks 11 below.
(3) The Wheeden–Muckenhoupt condition (1.3) is equivalent both to

$$R^{\alpha-n}\left\|w_2(\cdot)\,\mathbb{1}_{\{|\cdot|<R\}}(\cdot)\right\|_{L_u^{pq}}\left\|\frac{1}{v(\cdot)w_1(\cdot)}\,\mathbb{1}_{\{|\cdot|<R\}}(\cdot)\right\|_{L_v^{\frac{r}{r-1}\,\frac{s}{s-1}}} \le A \quad (1.8)$$

and to

$$\left\|w_2(\cdot)\,|\cdot|^{\alpha-n}\,\mathbb{1}_{\{|\cdot|>R\}}(\cdot)\right\|_{L_u^{pq}}\left\|\frac{1}{v(\cdot)w_1(\cdot)}\,\mathbb{1}_{\{|\cdot|<R\}}(\cdot)\right\|_{L_v^{\frac{r}{r-1}\,\frac{s}{s-1}}} \le H \qquad (1.9)$$

for all $R > 0$. The latter inequality is also useful to get the boundedness of some Hardy type operators in weighted Lorentz spaces (see Lemma 3).

It is of interest to identify some situations where the extra-condition (1.7) (or (1.7′)) can be obtained from the Wheeden–Muckenhoupt condition (1.3). Such a question will be discussed below.

But for the moment we state the main result for the fractional integral operator.

Theorem 4. (*The fractional integral operator I_α with $0 < \alpha < n$*)
(A) *Suppose $I_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$. Then the Hardy condition* (1.9) *is satisfied and so is its dual version*

$$\left\|\frac{1}{v(\cdot)w_1(\cdot)}\,|\cdot|^{\alpha-n}\mathbb{1}_{|\cdot|>R}(\cdot)\right\|_{L_v^{\frac{r}{r-1}\,\frac{s}{s-1}}}\left\|w_2(\cdot)\,\mathbb{1}_{\{|\cdot|<R\}}(\cdot)\right\|_{L_u^{pq}} \le H^*. \quad (1.9^*)$$

(B) *For the converse, assume restriction* (1.5) *holds. Then both conditions* (1.9) *and* (1.9*) *imply* $I_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$ *whenever the pointwise inequality* (1.7) (*or* (1.7′)) *is satisfied.*

As in Remark 3(2), by Theorem 4 and Lemma 1 we see that, *with restriction* (1.5), *both conditions* (1.9), (1.9*) *and* (1.4) *characterize the embedding* $I_\alpha : L_v^{rs}(w_1) \to L_u^{r^*q}(w_2)$ *whenever either* $u(\cdot)$ *or* $v(\cdot)$ *is constant on annuli.*

Next, the weighted inequalities for Calderón–Zygmund operators T are considered. Each T is a linear operator which sends $C_c^\infty(\mathbb{R}^n)$ into $L^1_{loc}(\mathbb{R}^n, dx)$, is bounded on $L^2(\mathbb{R}^n, dx)$, and has the representation

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \qquad \text{a.e.} \quad x \notin \operatorname{supp} f$$

for every $f \in L_c^\infty(\mathbb{R}^n)$. The kernel $K(x,y)$ is a continuous function defined on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n;\, x \neq y\}$ and satisfying the standard estimates

$$|K(x,y)| \leq C|x-y|^{-n} \quad \text{for all} \quad x \neq y,$$

$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C\left(\frac{|x-x'|}{|x-y|}\right)^\epsilon |x-y|^{-n}$ whenever $2|x-x'| \leq |x-y|$. Here $C > 0$ and $\epsilon \in]0,1]$ are fixed constants. These operators were introduced by Coifman and Meyer in [3] and were known to be bounded on each space L^p for $1 < p < \infty$.

Now we are in the position to state the sufficient conditions for these operators to be bounded on weighted Lorentz spaces.

Theorem 5. (*The Calderon–Zygmund operator* T)
Let $s \leq r \leq q$. *Then conditions* (1.9) *and* (1.9*) (*with* $\alpha = 0$, $p = r$) *imply* $T : L_v^{rs}(w_1) \to L_u^{rq}(w_2)$ *whenever the pointwise inequality* (1.7) *is satisfied.*

For the Hilbert transform some of the above conditions become also necessary.

Proposition 6. (*The Hilbert transform* H)
Suppose $H : L_v^{rs}(w_1) \to L_u^{rq}(w_2)$. *Then conditions* (1.9), (1.9*), *and* (1.4) (*with* $\alpha = 0$, $r^* = r = p$) *are satisfied.*

Next we deal with a result which yields cases where the Muckenhoupt condition (1.8), with $\alpha \geq 0$, implies the Hardy inequality (1.9). For this purpose, some weight conditions are needed. Thus $v(\cdot) \in RD_{\nu,r,s}(w_1)$, $\nu > 0$, when for a constant $c > 0$

$$\left\| \frac{1}{v(\cdot)w_1(\cdot)} 1\!\!1_{\{|\cdot| < \lambda R\}}(\cdot) \right\|_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}} \leq c\lambda^{n\nu(1-\frac{1}{r})} \left\| \frac{1}{v(\cdot)w_1(\cdot)} 1\!\!1_{\{|x| < R\}}(\cdot) \right\|_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}}$$

for all $0 < \lambda \le 1$ and $R > 0$. Similarly, $u(\cdot) \in D_{\varepsilon,p,q}(w_2)$, $\varepsilon \ge 1$, when

$$\Big\| w_2(\cdot)\, \mathbb{1}_{\{|\cdot|<\lambda R\}}(\cdot) \Big\|_{L^{pq}_u} \le c\, \lambda^{n\varepsilon \frac{1}{p}} \Big\| w_2(\cdot)\, \mathbb{1}_{\{|\cdot|<R\}}(\cdot) \Big\|_{L^{pq}_u}$$

for all $\lambda \ge 1$ and $R > 0$.

Proposition 7. *The Muckenhoupt condition* (1.8)*, with* $0 \le \alpha < n$*, implies the Hardy condition* (1.9) *whenever* $v(\cdot) \in RD_{\nu,r,s}(w_1)$ *for some* $\nu > 0$*. This implication is also true if* $u(\cdot) \in D_{\varepsilon,p,q}(w_2)$ *for* $1 \le \varepsilon < (1-\frac{\alpha}{n})p$.

After some tedious computations, we obtain

Proposition 8. *Let* $w(x) = |x|^{\beta-n}$, $w_1(x) = |x|^{\beta_1-n}$, $w_2(x) = |x|^{\beta_2-n}$, $u(x) = |x|^{\gamma-n}$, $v(x) = |x|^{\delta-n}$ *where* β, β_1, β_2, γ, δ *are nonnegative reals.*

(A) *If* $0 < (\beta - n) + \frac{1}{p}\gamma$*, then*

$$\Big\| w(\cdot)\mathbb{1}_{|\cdot|<R}(\cdot) \Big\|_{L^{pq}_u} \approx R^{(\beta-n)+\frac{1}{p}\gamma} \approx \Big(\frac{1}{R^n}\int_{|y|<R} w(y)dy\Big)\Big(\int_{|y|<R} u(y)dy\Big)^{\frac{1}{p}}$$

for all $R > 0$.

(B) *The extra-assumption* (1.6) *is satisfied with* $w_2(\cdot) = w(\cdot)$.

(C) *Let* $0 \le \alpha < n$, $0 < (\beta_2 - n) + \frac{1}{p}\gamma$ *and* $(\beta_1 - n) + \frac{1}{r}\delta < n$*. Then the pointwise inequality* (1.4) *and the Muckenhoupt condition* (1.8) *are satisfied if and only if*

$$\alpha + (\beta_2 - n) + \frac{1}{p}\gamma \;=\; (\beta_1 - n) + \frac{1}{r}\delta. \tag{1.10}$$

Moreover, since $v(\cdot) \in RD_{\nu,r,s}(w_1)$ *with* $\nu = \frac{r}{r-1}\frac{1}{n}\,[(n-\beta_1)+(n-\frac{1}{r}\delta)] > 0$*, then by Proposition* 7 *the Wheeden condition* (1.3) *is equivalent to* (1.10).

We will end this section by studying some cases where the pointwise inequality (1.7) becomes a necessary condition for $M_\alpha : L^{rs}_v(w_1) \to L^{rq}_u(w_2)$. For this purpose two weight conditions are introduced. Therefore we write that $u(\cdot) \in \mathcal{H}$ whenever for some $C > 0$ and $N \in \mathbb{N}^*$:

$$\sup_{4^{-1}|x|<|y|<4|x|} u(y) \le C|x|^{-n} \int_{2^{-N}|x|<|y|<2^N|x|} u(y)dy, \tag{1.11}$$

and $v(\cdot) \in \widetilde{\mathcal{H}}(r', s')$, $r' = \frac{r}{r-1}$, $s' = \frac{s}{s-1}$, whenever

$$|x|^n \Big[\frac{1}{v(x)}\Big]^{r'} v(x) \le C \Big\| \frac{1}{v(\cdot)}\, \mathbb{1}_{2^{-N}|x|<|\cdot|<2^N|x|}(\cdot) \Big\|^{r'}_{L^{r's'}_u}.$$

Without any difficulty we get

Proposition 9. *The Muckenhoupt condition* (1.8) *implies the pointwise inequality* (1.7) *whenever* $u(\cdot) \in \mathcal{H}$ *and* $v(\cdot) \in \widetilde{\mathcal{H}}(r', s')$.

In this result the condition constant on annuli for $w_1(\cdot)$ and $w_2(\cdot)$ is taken in the sense that $\sup_{R<|y|\le 2^{2N}R} w(y) \le c \inf_{R<|z|\le 2^{2N}R} w(z)$ with $N \ge 3$. An immediate consequence of Proposition 9 can be stated as

Corollary 10. *Let $u(\cdot) \in \mathcal{H}$ and $v(\cdot) \in \widetilde{\mathcal{H}}(r', s')$; then*
– Condition (1.7) (or (1.7′)) in Theorem 2 can be dropped;
– Condition (1.7) (or (1.7′)) in Theorems 4 and 5 can be replaced by the Muckenhoupt condition (1.8).

Remarks 11. (1) Property (1.11) holds for a large class of weight functions. For instance, $w(\cdot)$ satisfies (1.11) whenever $w(\cdot) \in \mathcal{A}$. Condition (1.11) is also true for any radial and monotone weight. But there also exists $w(\cdot)$ not necessarily monotone for which (1.11) is satisfied (take, for instance, $w(x) = |x|^{\delta-n}\mathbb{1}_{|x|<1}(x) + |x|^{\gamma-n}\mathbb{1}_{|x|>1}(x)$).

(2) The condition $v(\cdot) \in \widetilde{\mathcal{H}}(r', r')$ holds if $v^{1-r'}(\cdot)$ satisfies condition (1.11). For general r and s, we have $v(\cdot) \in \widetilde{\mathcal{H}}(r', s')$ whenever there is $C > 0$ such that

$$|x|^{-n}\int_{2^{-N}|x|<|y|<2^N|x|} v(y)dy \le cv(x). \tag{1.12}$$

Indeed, using the Hölder inequality and (1.12), with $\mathcal{C}(x, N) = \{2^{-N}|x| < |y| < 2^N|x|\}$, we obtain

$$\begin{aligned}
|x|^{\frac{n}{r'}}\Big[\frac{1}{v(x)}\Big]v^{\frac{1}{r'}}(x) &\approx \Big(\int_{\mathbb{R}^n}\frac{1}{v(y)}\mathbb{1}_{\mathcal{C}(x,N)}(y)v(y)dy\Big) \times \Big(|x|^n v(x)\Big)^{-\frac{1}{r}} \le \\
&\le c_1\Big\|\frac{1}{v(\cdot)}\mathbb{1}_{\mathcal{C}(x,N)}(\cdot)\Big\|_{L_v^{r's'}}\Big(\int_{\mathcal{C}(x,N)} v(z)dz\Big)^{\frac{1}{r}} \times \Big(|x|^n v(x)\Big)^{-\frac{1}{r}} \le \\
&\le c_2\Big\|\frac{1}{v(\cdot)}\mathbb{1}_{\mathcal{C}(x,N)}(\cdot)\Big\|_{L_v^{r's'}}.
\end{aligned}$$

Any Muckenhoupt A_1-weight function $v(\cdot)$ satisfies condition (1.12). The same is true for $v(\cdot) \in \mathcal{A}$.

(3) Theorem 2, Proposition 9 and Remark 11(2) yield the following conclusions: *With restrictions (1.5) and (1.6), the Wheeden–Muckenhoupt condition (1.3) characterizes the embedding $M_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$ whenever both $u(\cdot)$ and $v(\cdot)$ are constants on annuli.* Similarly, *if $u(\cdot)$ and $v(\cdot)$ are constant on annuli, then $I_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$ if and only if both (1.8), (1.9) and (1.9*) are satisfied.*

§ 2. Basic Lemmas

In this section we prove Lemma 1 and give some basic lemmas needed for the proofs of our results.

Proof of Lemma 1. Assume that $M_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$, i.e., $\Big\|w_2(\cdot)(M_\alpha f)(\cdot)\Big\|_{L_u^{pq}} \le C\Big\|w_1(\cdot)f(\cdot)\Big\|_{L_v^{rs}}$ for all functions $f(\cdot)$. Let Q be a cube and $f(\cdot)\ge 0$ with Q as its support. Since $|Q|^{\frac{\alpha}{n}-1}\Big(\int_Q f(y)dy\Big)\mathbb{1}_Q(x)\le (M_\alpha f)(x)$, therefore

$$|Q|^{\frac{\alpha}{n}-1}\left(\int_Q f(y)dy\right)\Big\|w_2(\cdot)\mathbb{1}_Q(\cdot)\Big\|_{L_u^{pq}} \le C\Big\|w_1(\cdot)(f\mathbb{1}_Q)(\cdot)\Big\|_{L_v^{rs}}. \quad (2.1)$$

This is the key inequality for the inequalities of this lemma (except for (1.3)).

Taking $f(\cdot) = \mathbb{1}_Q(\cdot)$ in (2.1), we obtain (1.1). In particular, for $w_1(\cdot) = w_2(\cdot) = 1$ we have $|Q|^{\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}}\Big(|Q|^{-1}\int_Q u(y)dy\Big)^{\frac{1}{p}} \le C\Big(|Q|^{-1}\int_Q v(y)dy\Big)^{\frac{1}{r}}$. The latter inequality implies $\frac{s}{n}+\frac{1}{p}-\frac{1}{r}\ge 0$. Indeed, if this is not the case, then by the Lebesgue differentiation theorem and letting $|Q|\to 0$ we necessarily have $u(\cdot)=0$ a.e..

To prove (1.2), suppose the contrary, i.e., $\Big\|\frac{1}{v(\cdot)w_1(\cdot)}\mathbb{1}_Q(\cdot)\Big\|_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}} = \infty$ for the cube Q. Then there is $g(\cdot)\ge 0$ for which $\|g(\cdot)\mathbb{1}_Q(\cdot)\|_{L_v^{rs}} < \infty$ and $\infty = \int_Q g(y)(\frac{1}{v(y)w_1(y)})v(y)dy = \int_Q g(y)w_1^{-1}(y)dy$. Consequently inequality (2.1) cannot hold for the function $f(\cdot) = g(\cdot)w_1^{-1}(\cdot)$ unless $u(\cdot)=0$ a.e. (since the quantity on the right is finite).

The Wheeden–Muckenhoupt condition (1.3) can be derived from an inequality similar to (2.1) which is

$$\Big(\int_{|y|<R} f(y)dy\Big)\Big\|(R+|\cdot|)^{\alpha-n}\,w_2(\cdot)\Big\|_{L_u^{pq}} \le C\Big\|w_1(\cdot)f(\cdot)\Big\|_{L_v^{rs}}, \quad (2.2)$$

for each $f(\cdot)\ge 0$ and whose support is the ball $B = B(0,R) = \{y;\, |y|<R\}$ centered at the origin and with radius R. Inequality (2.2) can be obtained immediately from $M_\alpha : L_v^{rs}(w_1)\to L_u^{pq}(w_2)$ and

$$(R+|x|)^{\alpha-n}\Big(\int_{|y|<R} f(y)dy\Big) \le c\,(M_\alpha f)(x).$$

Here $c = c(\alpha,n) > 0$ depends only on α and n. This inequality is valid, since for $|x|\le R$ we have $B\subset B(x,2R)$ and $(R+|x|)^{\alpha-n}\int_B f(y)dy \le R^{\alpha-n}\int_{B(x,2R)} f(y)dy \le c(M_\alpha f)(x)$, and for $R<|x|$ we obtain $B\subset B(x,2|x|)$ and $(R+|x|)^{\alpha-n}\int_B f(y)dy \le |x|^{\alpha-n}\int_{B(x,2|x|)} f(y)dy \le c(Mf)(x)$. Our purpose is to bound the quantity

$$\mathcal{T} = \Big\|w_2(\cdot)\,(R+|\cdot|)^{\alpha-n}\Big\|_{L_u^{pq}}\;\Big\|\frac{1}{v(\cdot)w_1(\cdot)}\,\mathbb{1}_{\{|\cdot|<R\}}(\cdot)\Big\|_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}}$$

by a constant which does not depend on $R > 0$. Since it can be assumed that $0 < \left\| \frac{1}{v(\cdot)w_1(\cdot)} \mathbb{1}_{\{|\cdot|<R\}} \right\|_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}} < \infty$, there is $g(\cdot) \geq 0$ such that $\left\| g(\cdot)\mathbb{1}_{|y|<R}(\cdot) \right\|_{L_v^{rs}} \leq 1$ and

$$\left\| \frac{1}{v(\cdot)w_1(\cdot)} \mathbb{1}_{\{|\cdot|<R\}}(\cdot) \right\|_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}} =$$
$$= \int_{|y|<R} \frac{1}{v(y)w_1(y)} g(y)v(y)dy = \int_{|y|<R} w_1^{-1}(y)g(y)dy.$$

Finally, condition (1.3) appears by taking $f(\cdot) = w_1^{-1}(\cdot)g(\cdot)$ in (2.2). Indeed,

$$\mathcal{T} = \Big(\int_{|y|<R} w_1^{-1}(y)g(y)dy \Big) \left\| (R+|\cdot|)^{\alpha-n}\, w_2(\cdot) \right\|_{L_u^{pq}} \leq C \left\| g(\cdot)\mathbb{1}_{|y|<R}(\cdot) \right\|_{L_v^{rs}} \leq C.$$

To prove (1.4), an inequality similar to (2.1), with cubes replaced by balls, is used. First consider the case $p = r^*$. Let $x \neq 0$ and $B = B(x, R)$ be the ball centered at x and with a small radius R, i.e., $R < \frac{1}{2}|x|$. Since $w_1(\cdot)$, $w_2(\cdot) \in \mathcal{A}$, for each $y \in B$: $w_1(x) \approx w_1(y)$ (in the sense that $c^{-1}w_1(y) \leq w_1(x) \leq cw_1(y)$) and $w_2(x) \approx w_2(y)$. Indeed, $\frac{1}{2}|x| < |y| < 4\frac{1}{2}|x|$ and $w_1(y) \leq \sup_{\frac{1}{2}|x| \leq |z| < 64\frac{1}{2}|x|} w_1(z) \leq c \inf_{\frac{1}{2}|x| \leq |z| < 64\frac{1}{2}|x|} w_1(z) \leq c\, w_1(x)$. Analogously, $w_1(x) \leq cw_1(y)$. Taking $f(\cdot) = \mathbb{1}_B(\cdot)$ in (2.1) (with balls instead of cubes) and using the above equivalences we obtain

$$w_2(x)\, |B|^{\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}} \Big(|B|^{-1} \int_B u(y)dy \Big)^{\frac{1}{p}} \leq C w_1(x) \Big(|B|^{-1} \int_B v(y)dy \Big)^{\frac{1}{r}}. \quad (2.3)$$

Here $\frac{\alpha}{n} + \frac{1}{p} - \frac{1}{r} = 0$ and $|B| = R^n$. Thus by (2.3) and the Lebesgue differentiation theorem (by letting $R \to 0$) we have $w_2(x)\, (u(x))^{\frac{1}{r^*}} \leq C\, w_1(x)\, (v(x))^{\frac{1}{p}}$.

Next suppose $p \neq r^*$ ($\frac{\alpha}{n} + \frac{1}{p} - \frac{1}{r} \neq 0$), and assume both $u(\cdot), v(\cdot) \in \mathcal{A}$. The purpose is to estimate $\mathcal{I} = \mathcal{I}(x) = w_2(x)\, |x|^{n[\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}]}(u(x))^{\frac{1}{p}}$. For the present case, the ball $B = B(x, R)$ is taken with radius $R = \frac{1}{9}|x|$. Observe that $\frac{8}{9}|x| < |y| < 64\frac{8}{9}|x|$ whenever $y \in B$. The conclusion appears as follows:

$$\mathcal{I} = w_2(x)\, |x|^{n[\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}]} \Big(u(x) \Big)^{\frac{1}{p}} \leq$$
$$\leq c_1\, w_2(x)\, R^{n[\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}]} \Big(\sup_{\frac{8}{9}|x|<|y|\leq 64\frac{8}{9}|x|} u(y) \Big)^{\frac{1}{p}} \leq$$
$$\leq c_1 G(u)\, w_2(x)\, R^{n[\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}]} \Big(R^{-n} \int_B [\inf_{\frac{8}{9}|x|<|z|\leq 64\frac{8}{9}|x|} u(z)]\, dy \Big)^{\frac{1}{p}} \leq$$

$$\leq c_1 G(u)\, w_2(x)\, R^{n[\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}]}\Big(R^{-n}\int_B u(y)dy\Big)^{\frac{1}{p}} \leq$$
$$\leq c_2 G(u)\, w_1(x)\,\Big(R^{-n}\int_B v(y)dy\Big)^{\frac{1}{r}} \quad \leq \text{(by (2.3))}$$
$$\leq c_3 G(u)\, w_1(x)\,\Big(\sup_{\frac{8}{9}|x|<|z|\leq 64\frac{8}{9}|x|} v(z)\Big)^{\frac{1}{r}} \leq$$
$$\leq c_3 G(u) G(v)\, w_1(x)\,\Big(\inf_{\frac{8}{9}|x|<|y|\leq 64\frac{8}{9}|x|} v(z)\Big)^{\frac{1}{r}} \leq$$
$$\leq c_3 G(u) G(v)\, w_1(x)\,\Big(v(x)\Big)^{\frac{1}{r}}.$$

In the proofs of our results we will have to perform some summations as stated in the following

Lemma 2. *Suppose* $\sum_k \mathbb{1}_{E_k}(\cdot) \leq C\,\mathbb{1}_{\cup E_k}(\cdot)$ *for a fixed constant* $C>0$, *where* E_k*'s are measurable sets* (*so these sets are quasi-disjoint*).

(A) *Then*

$$\sum_k \Big\| f(\cdot)\mathbb{1}_{E_k}(\cdot)\Big\|^{\lambda}_{L^{rs}_w} \leq c_1 \Big\| f(\cdot)\mathbb{1}_{\cup E_k}(\cdot)\Big\|^{\lambda}_{L^{rs}_w} \quad \textit{for all functions} \quad f(\cdot)$$

whenever $\max(r,s)\leq\lambda$.

(B) *For a constant* $c>0$, *which depends only on* C,

$$\Big\|\sum_k f(\cdot)\mathbb{1}_{E_k}(\cdot)\Big\|^{\gamma}_{L^{pq}_u} \leq c\sum_k \Big\| f(\cdot)\mathbb{1}_{E_k}(\cdot)\Big\|^{\gamma}_{L^{pq}_u}$$

whenever $0<\gamma\leq\min(p,q)$.

Also, the proofs of our results will depend much on the boundedness of generalized Hardy type operators on weighted Lorentz spaces which are already introduced and studied by Edmunds, Gurka, and Pick [4]. The Hardy type operators under consideration are of the forms

$$(\mathcal{H}f)(x) = (\mathcal{H}_{a,b}f)(x) = a(x)\int_{|y|\leq|x|} f(y)\, b(y)dy$$

and

$$(\mathcal{H}^* g)(x) = (\mathcal{H}^*_{a,b}g)(x) = b(x)\int_{|x|\leq|y|} g(y)\, a(y)dy,$$

where $a(\cdot)$ and $b(\cdot)$ are measurable nonnegative functions. It is supposed that

$$r=s=1 \;\text{ or }\; 1<r<\infty \;\text{ and };1\leq s\leq\infty,$$
$$p=q=1 \;\text{ or }\; 1<p<\infty \;\text{ and }\; 1\leq q\leq\infty.$$

Lemma 3. *Let r, s, p, q be as above and, moreover,* $\max(r,s) \leq \min(p,q)$. *Then*

$$\Big\|(\mathcal{H}f)(\cdot)\Big\|_{L^{pq}_u} \leq C\, \Big\|f(\cdot)\Big\|_{L^{rs}_v} \quad \textit{for all} \quad f(\cdot) \geq 0$$

if and only if

$$\sup_{R>0} \Big\|\, a(\cdot)\, 1\!\!1_{R<|\cdot|}(\cdot)\, \Big\|_{L^{pq}_u} \Big\| \frac{1}{v(\cdot)}\, b(\cdot)\, 1\!\!1_{|\cdot|<R}(\cdot)\, \Big\|_{L_v^{\frac{r}{r-1}\, \frac{s}{s-1}}} < \infty.$$

Similarly,

$$\Big\|(\mathcal{H}^* g)(\cdot)\Big\|_{L^{pq}_u} \leq C\, \Big\|g(\cdot)\Big\|_{L^{rs}_v} \quad \textit{for all} \quad g(\cdot) \geq 0$$

if and only if

$$\sup_{R>0} \Big\|\, b(\cdot)\, 1\!\!1_{|\cdot|<R}(\cdot)\, \Big\|_{L^{pq}_u} \Big\| \frac{1}{v(\cdot)}\, a(\cdot)\, 1\!\!1_{R<|x|}(\cdot)\, \Big\|_{L_v^{\frac{r}{r-1}\, \frac{s}{s-1}}} < \infty.$$

In order to get the weighted inequality for the maximal operators M_α, $0 \leq \alpha < n$, the following cutting lemma is needed.

Lemma 4. *Let $0 < \lambda \leq \min(p,q)$. Then for some constant $C > 0$ and for all $f(\cdot) \geq 0$:*

$$\Big\|\, w_2(\cdot)\, (M_\alpha f)(\cdot)\Big\|^{\lambda}_{L^{pq}_u} \leq C\left(\mathcal{S}_1^\lambda + \mathcal{S}_2^\lambda + \mathcal{S}_3^\lambda\right),$$

where

$$\mathcal{S}_1^\lambda = \Big\|\, w_2(\cdot)|\cdot|^{\alpha-n}\left(\int_{|y|\leq|\cdot|} f(y)dy\right)\Big\|^{\lambda}_{L^{pq}_u},$$

$$\mathcal{S}_2^\lambda = \sum_{k\in\mathbb{Z}} \Big\|\, w_2(\cdot)\left(M_\alpha f 1\!\!1_{G_k}\right)(\cdot)\, 1\!\!1_{E_k}(\cdot)\Big\|^{\lambda}_{L^{pq}_u},$$

$$\mathcal{S}_3^\lambda = \sum_{m\in\mathbb{Z}} \left[2^{(\alpha-n)m}\left(\int_{E_m} f(y)dy\right)\right]^{\lambda} \Big\|\, w_2(\cdot)\, 1\!\!1_{|x|<2^m}(\cdot)\Big\|^{\lambda}_{L^{pq}_u}$$

and $E_k = \{2^k < |x| \leq 2^{k+1}\}$, $G_k = \{2^{k-1} < |x| \leq 2^{k+2}\}$.

In order to state in a condensed form a similar result for any fractional integral and any Calderón–Zygmund operators, define the linear operator T_α, $0 \leq \alpha < n$, as sending $C_c^\infty(\mathbb{R}^n)$ into $L^1_{loc}(\mathbb{R}^n, dx)$ and such that

$$(T_\alpha f)(x) = \int_{\mathbb{R}^n} K_\alpha(x,y) f(y)dy \qquad \text{a.e.} \quad x \notin \operatorname{supp} f$$

for every $f(\cdot) \in L_c^\infty(\mathbb{R}^n)$, and with the kernel $K_s(x,y)$ satisfying

$$|K_\alpha(x,y)| \leq C|x-y|^{(\alpha-n)} \quad \text{for all} \quad x \neq y.$$

It is also assumed that $(T_\alpha f)(\cdot)$ is well defined almost everywhere for all bounded functions with compact supports. This is the case for $0 < \alpha < n$ when T_α is the fractional integral operator I_α. For $\alpha = 0$ this assumption will be realized if $T_0 : L^p \to L^p$ for some $p > 1$ (which is the case for a Calderón–Zygmund operator).

Lemma 5. *Let $0 \le \alpha < n$ and $0 < \lambda \le \min(p,q)$. Then for a constant $C > 0$ and for all functions $f \in C_c^\infty(\mathbb{R}^n)$:*

$$\Big\| w_2(\cdot)\,(T_\alpha f)(\cdot)\Big\|^\lambda_{L^{pq}_u} \le C\left(\mathcal{S}_1^\lambda + \mathcal{S}_2^\lambda + \mathcal{S}_3^\lambda\right)$$

where

$$\begin{aligned}
\mathcal{S}_1^\lambda &= \Big\| w_2(\cdot)|\cdot|^{\alpha-n}\Big(\int_{|y|\le|\cdot|}|f(y)|dy\Big)\Big\|^\lambda_{L^{pq}_u},\\
\mathcal{S}_2^\lambda &= \sum_{k\in\mathbb{Z}}\Big\| w_2(\cdot)\left(T_\alpha f\,\mathbb{1}_{G_k}\right)(\cdot)\,\mathbb{1}_{E_k}(\cdot)\Big\|^\lambda_{L^{pq}_u},\\
\mathcal{S}_3^\lambda &= \Big\| w_2(\cdot)\Big(\int_{|\cdot|\le|y|}|f(y)||y|^{\alpha-n}dy\Big)\Big\|^\lambda_{L^{pq}_u}
\end{aligned}$$

and E_k and G_k are defined as in Lemma 4.

The proofs of these lemmas will be given in §4, and now we proceed to proving our main results.

§ 3. Proofs of the Main Results

Proof of Theorem 2. The real problem is to prove Part B. Since $\max(r,s) \le \min(p,q)$ (see (1.5)), therefore one can find $\lambda > 0$ for which $\max(r,s) \le \lambda \le \min(p,q)$. In view of cutting Lemma 4, we have to estimate each of $\mathcal{S}_1^\lambda$, $\mathcal{S}_2^\lambda$, and $\mathcal{S}_3^\lambda$ by

$$C\,\Big\| w_1(\cdot)f(\cdot)\Big\|^\lambda_{L^{rs}_v}$$

for a fixed constant $C > 0$ which, in general, depends on α, n, p, q, r, s, $u(\cdot)$, $v(\cdot)$, $w_1(\cdot)$ and $w_2(\cdot)$.

Estimate of $\mathcal{S}_1^\lambda$. By taking $g(\cdot) = w_1(\cdot)f(\cdot)$, we obtain

$$\Big\| w_2(\cdot)|\cdot|^{\alpha-n}\Big(\int_{|y|\le|\cdot|}\frac{1}{w_1(y)}g(y)dy\Big)\Big\|_{L^{pq}_u} \le C\,\Big\|g(\cdot)\Big\|_{L^{rs}_v}.$$

Such an inequality can be considered as $\mathcal{H} : L^{rs}_v \to L^{pq}_u$ where $\mathcal{H} = \mathcal{H}_{a,b}$ is a Hardy type operator given by $a(x) = w_2(x)|x|^{\alpha-n}$ and $b(y) = \frac{1}{w_1(y)}$. In view of Lemma 3, this boundedness of $\mathcal{H}$ is equivalent to

$$\sup_{R>0}\Big\| a(\cdot)\,\mathbb{1}_{R<|\cdot|}(\cdot)\Big\|_{L^{pq}_u}\Big\|\frac{b(\cdot)}{v(\cdot)}\,\mathbb{1}_{|\cdot|<R}(\cdot)\Big\|_{L_v^{\frac{r}{r-1}\,\frac{s}{s-1}}} < \infty$$

or to

$$\left\| w_2(\cdot)|\cdot|^{\alpha-n}\, \mathbb{1}_{R<|\cdot|}(\cdot)\right\|_{L^{pq}_u} \left\| \frac{1}{v(\cdot)w_1(\cdot)}\, \mathbb{1}_{|\cdot|<R}(\cdot)\right\|_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}} < C$$

for all $R > 0$. It is the Hardy condition (1.9) which is an immediate consequence of the Wheeden–Muckenhoupt condition (1.3).

Estimate of $\mathcal{S}_3^\lambda$. Here the Muckenhoupt condition (1.8) (also an immediate consequence of (1.3)) is used. Now by the Hölder inequality and (1.8) we have

$$\begin{aligned}
\mathcal{S}_3^\lambda &= \sum_{m\in\mathbb{Z}} \left[2^{(\alpha-n)m}\left(\int_{E_m} f(y)dy\right)\right]^\lambda \left\|w_2(\cdot)\mathbb{1}_{|\cdot|<2^m}(\cdot)\right\|^\lambda_{L^{pq}_u} \le \\
&\le C^\lambda \sum_{m\in\mathbb{Z}}\left[2^{(\alpha-n)m}\left\| w_2(\cdot)\, \mathbb{1}_{|\cdot|<2^m}(\cdot)\right\|_{L^{pq}_u}\times\right. \\
&\times \left.\left\| \frac{1}{v(\cdot)w_1(\cdot)}\, \mathbb{1}_{|\cdot|<2^m}(\cdot)\right\|_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}}\right]^\lambda \left\|w_1(\cdot)(f\mathbb{1}_{E_m})(\cdot)\right\|^\lambda_{L^{rs}_u} \le \\
&\le (CA)^\lambda \sum_{m\in\mathbb{Z}} \left\|w_1(\cdot)(f\mathbb{1}_{E_m})(\cdot)\right\|^\lambda_{L^{rs}_v} \le \\
&\le (C'A)^\lambda \left\|w_1(\cdot)f(\cdot)\right\|^\lambda_{L^{rs}_v} \quad \text{by Part A in Lemma 2.}
\end{aligned}$$

Estimate of $\mathcal{S}_2^\lambda$. To estimate $\mathcal{S}_2^\lambda$it is sufficient to get

$$\left\|w_2(\cdot)\Big(M_\alpha f\mathbb{1}_{G_k}\Big)(\cdot)\mathbb{1}_{E_k}(\cdot)\right\|_{L^{pq}_u} \le C\left\|w_1(\cdot)(f\mathbb{1}_{G_k})(\cdot)\right\|_{L^{rs}_v}. \tag{3.1}$$

Indeed, since $\sum_k \mathbb{1}_{G_k}(\cdot) \le 3$ and $\max(r,s) \le \lambda$, therefore by Part A in Lemma 2:

$$\mathcal{S}_2^\lambda \le C^\lambda \sum_{k\in\mathbb{Z}} \left\|w_1(\cdot)(f\mathbb{1}_{G_k})(\cdot)\right\|^\lambda_{L^{rs}_u} \le (cC)^\lambda \left\|w_1(\cdot)f(\cdot)\right\|^\lambda_{L^{rs}_u}.$$

To get (3.1) we will use the fact that $M_\alpha : L_1^{rs}(1) \to L_1^{r^*s}(1)$ (see [1], Theorem 5.2.2, p. 155), where $1 < r < \infty$, $1 \le s \le \infty$ and $\frac{1}{r^*} = \frac{1}{r} - \frac{\alpha}{n}$. The following three properties of Lorentz spaces (see [5]) are also used:

$$\|\mathbb{1}_E(\cdot)\|_{L^{ps}_w} = \left(\int_E w(y)dy\right)^{\frac{1}{p}} \quad \text{for all measurable sets } E;$$
$$\|f\|_{L^{ps_1}} \le \|f\|_{L^{ps_2}} \quad \text{for a fixed } p, \text{ and } s_2 \le s_1;$$
$$\|f_1f_2\|_{L^{ps}} \le c\|f_1\|_{L^{p_1s_1}}\, \|f_2\|_{L^{p_2s_2}} \ \text{ with } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \ \text{ and } \frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}.$$

For convenience, set

$$\mathcal{W}_{1,k} = \sup_{x \in G_k} w_1(x), \quad \mathcal{W}_{2,k} = \sup_{y \in E_k} w_2(y), \quad \text{and} \quad \mathcal{U}_k = \sup_{z \in E_k} u(z).$$

Recall that $w_1(\cdot), w_2(\cdot) \in \mathcal{A}$ and the pointwise condition (1.7) is assumed. The chain of computations which leads to inequality (3.1) is as follows:

$$\begin{aligned}
\mathcal{T}_k \;&=\; \left\| w_2(\cdot)\Big(M_\alpha f \mathbb{1}_{G_k}\Big)(\cdot)\mathbb{1}_{E_k}(\cdot)\right\|_{L_u^{pq}} \le \\
&\le \left\| w_2(\cdot)\Big(M_\alpha f \mathbb{1}_{G_k}\Big)(\cdot)\mathbb{1}_{E_k}(\cdot)\right\|_{L_u^{ps}} \le \\
&\qquad (\text{here } \; s \le q \; \text{ since } \; \max(r,s) \le \min(p,q)) \\
&\le c_0 \mathcal{W}_{2,k}\, \mathcal{U}_k^{\frac{1}{p}} \left\| \Big(M_\alpha f \mathbb{1}_{G_k}\Big)(\cdot)\mathbb{1}_{E_k}(\cdot)\right\|_{L^{ps}} \le \\
&\le c_1 \mathcal{W}_{2,k}\, \mathcal{U}_k^{\frac{1}{p}} \left\| \Big(M_\alpha f \mathbb{1}_{G_k}\Big)(\cdot)\right\|_{L^{r^* s}} \times \left\| \mathbb{1}_{E_k}(\cdot)\right\|_{L^{\tilde{r}\infty}} \le \\
&\qquad \Big(\text{where } \frac{1}{\tilde{r}} = \frac{1}{p} - \frac{1}{r^*} = \frac{1}{p} + \frac{\alpha}{n} - \frac{1}{r}\Big) \\
&\le c_2\, 2^{nk[\frac{1}{p} + \frac{\alpha}{n} - \frac{1}{r}]} \mathcal{W}_{2,k}\, \mathcal{U}_k^{\frac{1}{p}} \left\| (f \mathbb{1}_{G_k})(\cdot)\right\|_{L^{rs}} \approx \\
&\qquad (\text{since } \; M_\alpha : L_1^{rs}(1) \to L_1^{r^* s}(1)) \\
&\approx c_2\, 2^{nk[\frac{\alpha}{n} + \frac{1}{p} - \frac{1}{r}]} \mathcal{W}_{2,k}\, \mathcal{U}_k^{\frac{1}{p}} \left[\sum_j 2^{js} \left(\int_{G_k \cap \{f(\cdot) > 2^j\}} dx\right)^{\frac{s}{r}}\right]^{\frac{1}{s}} = \\
&= c_2 \left[\sum_j 2^{js} \left(\int_{G_k \cap \{f(\cdot) > 2^j\}} \left[2^{nk[\frac{\alpha}{n} + \frac{1}{p} - \frac{1}{r}]}\, \mathcal{W}_{2,k}\, \mathcal{U}_k^{\frac{1}{p}}\right]^r dx\right)^{\frac{s}{r}}\right]^{\frac{1}{s}} \le \\
&\le c_3 \left[\sum_j 2^{js} \left(\int_{G_k \cap \{f(\cdot) > 2^j\}} \left[|x|^{n[\frac{\alpha}{n} + \frac{1}{p} - \frac{1}{r}]}\, w_2(x) \; \times \right.\right.\right. \\
&\qquad \left.\left.\left. \times \left(\sup_{4^{-1}|x| < |z| < 4|x|} u(z)\right)^{\frac{1}{p}}\right]^r dx\right)^{\frac{s}{r}}\right]^{\frac{1}{s}} \le \\
&\quad \Big(\text{here } \mathcal{W}_{2,k} \le \sup_{(4^{-1}|x|) < |y| < 16(4^{-1}|x|)} w_2(y) \le c w_2(x) \text{ since } w_2(\cdot) \in \mathcal{A}\Big) \\
&\le c_4 \left[\sum_j 2^{js} \left(\int_{G_k \cap \{f(\cdot) > 2^j\}} (w_1(x))^r\, v(x) dx\right)^{\frac{s}{r}}\right]^{\frac{1}{s}} \le \\
&\qquad \big(\text{by hypothesis } \; (1.7)\big) \\
&\le c_4 \left[\sum_j \mathcal{W}_{1,k}^s 2^{js} \left(\int_{G_k \cap \{f(\cdot) > 2^j\}} v(x) dx\right)^{\frac{s}{r}}\right]^{\frac{1}{s}} \le
\end{aligned}$$

$$\leq c_5 \left[\sum_j 2^{(j+N_k)s} \left(\int_{G_k \cap \{2^{N_k} f(\cdot) > 2^{(j+N_k)}\}} v(x)dx \right)^{\frac{s}{r}} \right]^{\frac{1}{s}} \leq$$

$$\big(\text{since } 2^{N_k} \leq \mathcal{W}_{1,k} < 2^{N_k+1} \text{ for some } N_k \in \mathbb{Z}\big)$$

$$\leq c_5 \left[\sum_j 2^{(j+N_k)s} \left(\int_{G_k \cap \{\mathcal{W}_{1,k} f(\cdot) > 2^{(j+N_k)}\}} v(x)dx \right)^{\frac{s}{r}} \right]^{\frac{1}{s}} \leq$$

$$\leq c_6 \left[\sum_j 2^{ls} \left(\int_{G_k \cap \{cw_1(\cdot) f(\cdot) > 2^l\}} v(x)dx \right)^{\frac{s}{r}} \right]^{\frac{1}{s}} \leq$$

$$\Big(\text{here } \mathcal{W}_{1,k} \leq \sup_{(8^{-1}|x|)<|z|<64(8^{-1}|x|)} w_1(z) \leq cw_1(x) \text{ since } w_1(\cdot) \in \mathcal{A}\Big)$$

$$\leq c_7 \left\| w_1(\cdot)(f \mathbb{1}_{G_k})(\cdot) \right\|_{L_v^{rs}}.$$

Now we will study how to obtain the same local estimate (3.1) if instead of (1.7) we use condition (1.7′). The main point is the existence of $C > 0$ for which

$$2^{nk[\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}]}\ \mathcal{W}_{2,k}\,\mathcal{U}_k^{\frac{1}{p}} \leq Cw_1(x)\Big(v(x)\Big)^{\frac{1}{r}} \quad \text{for all } x \in G_k.$$

Indeed, by virtue of this inequality, a modification of the previous chain of computations leads to

$$\mathcal{T}_k = \left\| w_2(\cdot)\Big(M_\alpha f \mathbb{1}_{G_k}\Big)(\cdot)\mathbb{1}_{E_k}(\cdot) \right\|_{L_u^{pq}} \leq$$

$$\leq c_2 \left[\sum_j 2^{js} \left(\int_{G_k \cap \{f(\cdot) > 2^j\}} \left[2^{nk[\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}]} \mathcal{W}_{2,k} \mathcal{U}_k^{\frac{1}{p}} \right]^r dx \right)^{\frac{s}{r}} \right]^{\frac{1}{s}} \leq \big(\text{see above}\big)$$

$$\leq c_8 \left[\sum_j 2^{js} \left(\int_{G_k \cap \{f(\cdot) > 2^j\}} (w_1(x))^r\, v(x)dx \right)^{\frac{s}{r}} \right]^{\frac{1}{s}} \leq \big(\text{by this main point}\big)$$

$$\leq c_9 \left\| w_1(\cdot)(f \mathbb{1}_{G_k})(\cdot) \right\|_{L_v^{rs}} \ \big(\text{see again the details in the above estimate}\big).$$

To prove the main point, it is essential to observe that

$$\frac{1}{\sup_{4^{-1}|z|<|y|<4|z|}\left[\frac{1}{v(y)}\right]} \leq\ v(x) \quad \text{for all } x \in G_k \text{ and } z \in E_k.$$

This inequality is true, since for $x \in G_k$ and $z \in E_k$ we have $4^{-1}|z| < 2^{k-1} < |x| < 2^{k+2} < 4|z|$ and $1 = v(x)\frac{1}{v(x)} \leq v(x) \sup_{4^{-1}|z|<|y|<4|z|} \left[\frac{1}{v(y)}\right]$.

So the main point appears since

$$
\begin{aligned}
&2^{nk[\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}]}\mathcal{W}_{2,k}\,\mathcal{U}_k^{\frac{1}{p}} \le C_1 \sup_{z\in E_k}\Big\{|z|^{n[\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}]}\mathcal{W}_{2,k}\,\Big(u(z)\Big)^{\frac{1}{p}}\Big\} \le \\
&\le C_2 \sup_{z\in E_k}\Big\{|z|^{n[\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}]}w_2(z)\,\Big(u(z)\Big)^{\frac{1}{p}}\Big\} \le \quad \big(\text{since } w_2(\cdot)\in\mathcal{A}\big)\\
&\le C_3 \sup_{z\in E_k}\left\{w_1(z)\,\frac{1}{\sup_{4^{-1}|z|<|y|<4|z|}[\frac{1}{v(y)}]^{\frac{1}{r}}}\right\} \le \quad \big(\text{by condition } (1.7')\big)\\
&\le C_4\, w_1(x)\, \sup_{z\in E_k}\left\{\frac{1}{\sup_{4^{-1}|z|<|y|<4|z|}[\frac{1}{v(y)}]^{\frac{1}{r}}}\right\} \le \quad \big(\text{since } w_1(\cdot)\in\mathcal{A}\big)\\
&\le C_4\, w_1(x)\,\Big(v(x)\Big)^{\frac{1}{r}} \quad \big(\text{by the above observation}\big).
\end{aligned}
$$

Proof of Theorems 4 *and* 5. Part A of Theorem 4 is proved. Next a condensed proof for Part B of Theorem 4 and Theorem 5 is given.

Suppose $I_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$. Since $(M_\alpha f)(\cdot) \le c(s,n)(I_\alpha f)(\cdot)$, we have $M_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$ and the necessity of condition (1.3) (given by Lemma 1) implies the Hardy condition (1.9). On the other hand, observe that $I_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$ is equivalent to $I_\alpha : L_u^{p'q'}(\frac{1}{uw_2}) \to L_v^{r's'}(\frac{1}{vw_1})$ where $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$, ... So the dual condition (1.9^*) appears from the embedding $M_\alpha : L_u^{p'q'}(\frac{1}{uw_2}) \to L_v^{r's'}(\frac{1}{vw_1})$ as above.

To get Part B of Theorem 4 and Theorem 5, it is sufficient to derive the embedding $T_\alpha : L_v^{rs}(w_1) \to L_u^{pq}(w_2)$ with $0 \le \alpha < n$ by using conditions (1.9), (1.9^*) and hypothesis (1.7). As in the proof of Theorem 2 and in view of Lemma 5, the problem is reduced to estimating each of $\mathcal{S}_1^\lambda$, $\mathcal{S}_2^\lambda$, and $\mathcal{S}_3^\lambda$ by $C\Big\|w_1(\cdot)f(\cdot)\Big\|_{L_v^{rs}}^\lambda$. Here $C > 0$ is a constant which depends eventually on α, n, p, q, r, s, $u(\cdot)$, $v(\cdot)$, $w_1(\cdot)$, $w_2(\cdot)$, and λ is chosen such that $\max(r,s) \le \lambda \le \min(p,q)$.

The estimation of $\mathcal{S}_1^\lambda$ can be carried out as in the proof of Theorem 2 by using the Hardy condition (1.9) and Lemma 3.

Also, $\mathcal{S}_2^\lambda$ can be estimated as in the proof of Theorem 2. Indeed, in the present case for $\alpha > 0$ we have $I_\alpha : L_1^{rs}(1) \to L_1^{r^*s}(1)$ (see [1], Theorem 6.3.3, p. 191), where $1 < r < \infty$, $1 \le s \le \infty$ and $\frac{1}{r^*} = \frac{1}{r} - \frac{\alpha}{n}$; and, on the other hand, for $\alpha = 0$ we have $T_\alpha : L_1^{rs}(1) \to L_1^{rs}(1)$ which can be obtained by interpolation.

The estimate for $\mathcal{S}_3^\lambda$ is equivalent to $\mathcal{H}^* : L_v^{rs} \to L_u^{pq}$, with $\mathcal{H}^* = \mathcal{H}_{a,b}^*$ and $b(x) = w_2(x)$, $a(y) = \frac{1}{w_1(y)}|y|^{\alpha-n}$. By Lemma 3, this boundedness of $\mathcal{H}^*$ is equivalent to the dual Hardy condition (1.9^*).

Proof of Proposition 6. Suppose $H : L_v^{rs}(w_1) \to L_u^{rq}(w_2)$. Our purpose is just to get conditions (1.4) and (1.9) (with $\alpha = 0$, $r^* = r = p$), since (1.9*) can be obtained by using a duality argument. The main key to the proof is

$$\Big(\int_I f(y)dy\Big) \Big\| w_2(\cdot)\Big(|\cdot - x_I| + |I|\Big)^{-1}\Big\|_{L_u^{rq}} \le C \Big\| w_1(\cdot)(f1\!\!1_I)(\cdot)\Big\|_{L_v^{rs}} \quad (3.2)$$

for each interval I centered at x_I, and for each $f(\cdot) \ge 0$ whose support is I.

Indeed, (3.2) first implies $\Big\| w_2(\cdot)1\!\!1_I(\cdot)\Big\|_{L_u^{rq}} \le C' \Big\| w_1(\cdot)1\!\!1_I(\cdot)\Big\|_{L_v^{rs}}$. So using $w_1(\cdot)$, $w_2(\cdot) \in \mathcal{A}$, we obtain

$$w_2(x)\Big(|I|^{-1}\int_I u(y)dy\Big)^{\frac{1}{r}} \le c w_1(x)\Big(|I|^{-1}\int_I v(y)dy\Big)^{\frac{1}{r}},$$

for each interval I centered at $x \neq 0$ and with a length $|I|$ sufficiently small. Then by the Lebesgue differentiation theorem: $w_2(x)(u(x))^{\frac{1}{r}} \le c w_2(x)(v(x))^{\frac{1}{r}}$, which is actually condition (1.4).

On the other hand, applying (3.2) for intervalls $I =]-R, R[$ (i.e., $x_I = 0$) we obtain condition (1.9), since

$$\begin{aligned}
&\Big\| \frac{1}{v(\cdot)w_1(\cdot)} 1\!\!1_{|\cdot|<R}(\cdot)\Big\|_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}} \Big\| w_2(\cdot)\Big(|\cdot| + R\Big)^{-1}\Big\|_{L_u^{rq}} \le \\
&\le c\Big(\int_{|x|<R} \frac{1}{v(y)w_1(y)} g(y)v(y)dy\Big)\Big\| w_2(\cdot)\Big(|\cdot| + R\Big)^{-1}\Big\|_{L_u^{rq}} \le \\
&\qquad \Big(\text{where } \|g(\cdot)\|_{L_v^{rs}} \le 1\Big) \\
&\le cC \Big\| w_1(\cdot)\frac{1}{w_1(\cdot)}g(\cdot)\Big\|_{L_v^{rs}} = cC\,\|g(\cdot)\|_{L_v^{rs}} \le cC.
\end{aligned}$$

In view of Part B of Lemma 2, inequality (3.2) will be obtained immediately for some $N > 0$:

$$\Big(\int_I f(y)dy\Big) \Big\| w_2(\cdot)s_I(\cdot)1\!\!1_{]a+N,\infty[}(\cdot)\Big\|_{L_u^{rq}} \le C \Big\| w_1(\cdot)(f1\!\!1_I)(\cdot)\Big\|_{L_v^{rs}} \quad (3.3)$$

and

$$\Big(\int_I f(y)dy\Big) \Big\| w_2(\cdot)s_I(\cdot)1\!\!1_{]-\infty,a+N[}(\cdot)\Big\|_{L_u^{rq}} \le C \Big\| w_1(\cdot)(f1\!\!1_I)(\cdot)\Big\|_{L_v^{rs}}, \quad (3.4)$$

where $s_I(x) = \Big(|x - x_I| + |I|\Big)^{-1}$ and I is any intervall centered at x_I and having the form $I = [a, a+R]$, $R > 0$. Inequalities (3.3) and (3.4) will be immediate consequences of the pointwise estimates

$$\Big(Hf1\!\!1_{[a,a+N]}\Big)(\cdot)\, 1\!\!1_{]a+N,\infty[}(\cdot) \ge c\, s_I(\cdot)\Big(\int_I f(y)dy\Big)\, 1\!\!1_{]a+N,\infty[}(\cdot) \quad (3.5)$$

and

$$\Big(Hf\mathbb{1}_{[a+N,a+R]}\Big)(\cdot)\,\mathbb{1}_{]-\infty,a+N[}(\cdot)\geq c\,s_I(\cdot)\Big(\int_I f(y)dy\Big)\,\mathbb{1}_{]-\infty,a+N[}(\cdot)\quad(3.6)$$

for a some $N>0$ and $N<R$. So the problem of obtaining inequality (3.2) is just reduced to obtaining (3.5) and (3.6).

Let $f(\cdot)\geq 0$ be supported by $I=[a,a+R]$, and let $\mathcal{Q}=\int_I f(y)dy$. One can take $0<N<R$ such that $\frac{1}{2}\mathcal{Q}=\int_{[a,a+N]}f(y)dy$. Clearly, $0<x-y=|x-y|\leq|x-x_I|+|I|=s_I^{-1}(x)$ for each $x\in]a+N,\infty[$ and $y\in[a,a+N]$. Inequality (3.5) is satisfied, since

$$\Big(Hf\mathbb{1}_{[a,a+N]}\Big)(x)\,\mathbb{1}_{]a+N,\infty[}(x)\geq s_I(x)\Big(\int_{[a,a+N]}f(y)dy\Big)\,\mathbb{1}_{]a+N,\infty[}(x)=$$
$$=\;s_I(x)\,\frac{1}{2}\mathcal{Q}\,\mathbb{1}_{]a+N,\infty[}(x)\;=\;s_I(x)\,\frac{1}{2}\Big(\int_I f(y)dy\Big)\,\mathbb{1}_{]a+N,\infty[}(x).$$

If we note that

$$\int_{[a+N,a+R]}f(y)dy=\int_{[a,a+R]}f(y)dy-\int_{[a,a+N]}f(y)dy=\frac{1}{2}\mathcal{Q},$$

inequality (3.6) can be proved in the same way

Proof of Proposition 7. To get the Hardy condition (1.9) from the Muckenhoupt condition (1.8), first consider the case $v(\cdot)\in RD_{\nu,r,s}(w_1)$, $\nu>0$. Applying Lemma 2 with $0<\theta\leq\min(p,q)$, we have

$$\Big\|w_2(\cdot)|\cdot|^{\alpha-n}\mathbb{1}_{\{|\cdot|>R\}}(\cdot)\Big\|^{\theta}_{L^{pq}_u}\Big\|\frac{1}{v(\cdot)w_1(\cdot)}\mathbb{1}_{\{|\cdot|<R\}}(\cdot)\Big\|^{\theta}_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}}\leq$$
$$\leq c_1\sum_{k\geq 0}\Big\|w_2(\cdot)\,|\cdot|^{\alpha-n}\,\mathbb{1}_{\{2^kR<|\cdot|\leq 2^{k+1}R\}}(\cdot)\Big\|^{\theta}_{L^{pq}_u}\times$$
$$\times\Big\|\frac{1}{v(\cdot)w_1(\cdot)}\mathbb{1}_{\{|\cdot|<2^{-(k+1)}2^{(k+1)}R\}}(\cdot)\Big\|^{\theta}_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}}\leq$$
$$\leq c_2\sum_{k\geq 0}2^{-n\nu\theta k(1-\frac{1}{r})}(2^{k+1}R)^{\alpha-n}\Bigg(\Big\|w_2(\cdot)\,\mathbb{1}_{\{|\cdot|\leq 2^{k+1}R\}}(\cdot)\Big\|_{L^{pq}_u}\times$$
$$\times\Big\|\frac{1}{v(\cdot)w_1(\cdot)}\,\mathbb{1}_{\{|\cdot|<2^{(k+1)}R\}}(\cdot)\Big\|_{L_v^{\frac{r}{r-1}\frac{s}{s-1}}}\Bigg)^{\theta}\leq c_3A^{\theta}.$$

For $u(\cdot)\in D_{\varepsilon,p,q}(w_2)$, $1\leq\varepsilon<(1-\frac{\alpha}{n})p$ and with the same choice of θ, then

$$\Big\|w_2(\cdot)|\cdot|^{\alpha-n}\mathbb{1}_{\{|\cdot|>R\}}(\cdot)\Big\|^{\theta}_{L^{pq}_u}\leq c_4\sum_{k\geq 0}\Big\|w_2(\cdot)|\cdot|^{\alpha-n}\mathbb{1}_{\{2^kR<|\cdot|\leq 2^{k+1}R\}}(\cdot)\Big\|^{\theta}_{L^{pq}_u}\leq$$

$$\leq c52R^{(\alpha-n)\theta} \sum_{k\geq 0} 2^{k(\alpha-n)\theta} \left\| w_2(\cdot)\, 1\!\!1_{\{|\cdot|\leq 2^{k+1}R\}}(\cdot)\right\|^{\theta}_{L^{pq}_u} \leq$$

$$\leq c_6 R^{(\alpha-n)\theta} \sum_{k\geq 0} 2^{-kn\theta[(1-\frac{\alpha}{n})-\frac{1}{p}\varepsilon]} \left\| w_2(\cdot)\, 1\!\!1_{\{|\cdot|<R\}}(\cdot)\right\|^{\theta}_{L^{pq}_u} \leq$$

$$\leq c_7 \left(R^{(\alpha-n)} \left\| w_2(\cdot)\, 1\!\!1_{\{|\cdot|<R\}}(\cdot)\right\|_{L^{pq}_u} \right)^{\theta}.$$

Clearly, by the latter estimate, condition (1.8) implies (1.9).

§ 4. Proofs of the Basic Lemmas and Propositions 8 and 9

In this section we prove Lemmas 2, 3, and 4 (which were used for the proofs of our main results) and also Propositions 8 and 10.

Proof of Lemma 2. A proof of Part A with $C = 1$ was given in [6]. The present case can be easily obtained by using a duality argument.

To prove Part B, assume that $0 < \gamma \leq \min(p, q)$. For $p = q$ or $q < p$, the key is based on the fact that $\|\cdot\|_{L_u^{\frac{p}{\gamma}\frac{q}{\gamma}}}$ is equivalent to a norm. Hence

$$\left\| f(\cdot) \sum_k 1\!\!1_{E_k}(\cdot)\right\|^{\gamma}_{L^{pq}_u} = \left\| \left(f(\cdot) \sum_k 1\!\!1_{E_k}(\cdot)\right)^{\gamma} \right\|_{L_u^{\frac{p}{\gamma}\frac{q}{\gamma}}} \leq$$

$$\leq c_1 \left\| f^{\gamma}(\cdot) \sum_k 1\!\!1_{E_k}(\cdot)\right\|_{L_u^{\frac{p}{\gamma}\frac{q}{\gamma}}} \leq c_2 \sum_k \left\| f^{\gamma}(\cdot) 1\!\!1_{E_k}(\cdot)\right\|_{L_u^{\frac{p}{\gamma}\frac{q}{\gamma}}} = c_3 \sum_k \left\| f(\cdot) 1\!\!1_{E_k}(\cdot)\right\|^{\gamma}_{L^{pq}_u}.$$

Now consider the case $p < q$, so $\gamma \leq p < q$ or $\frac{\gamma}{p} \leq 1$ and $1 < \frac{q}{\gamma}$. Thus for a nonnegative sequence of reals $(b_j)_j \in l^{\theta}$, with $\sum_j b_j^{\theta} \leq 1$ and $\theta = \frac{q}{\gamma - q}$ we obtain

$$\left\| f(\cdot) \sum_k 1\!\!1_{E_k}(\cdot)\right\|^{\gamma}_{L^{pq}_u} \leq c_4 \left[\sum_j 2^{jq} \left(\sum_k \int_{E_k \cap \{f(\cdot) > 2^j\}} u(y)dy \right)^{\frac{\gamma}{p} \times \frac{q}{\gamma}} \right]^{\frac{\gamma}{q}} \leq$$

$$\leq c_4 \left[\sum_j \left[2^{j\gamma} \sum_k \left(\int_{E_k \cap \{f(\cdot) > 2^j\}} u(y)dy \right)^{\frac{\gamma}{p}} \right]^{\frac{q}{\gamma}} \right]^{\frac{\gamma}{q}} \leq \quad \left(\text{since } \frac{\gamma}{p} \leq 1\right)$$

$$\leq c_4 \sum_k \sum_j 2^{j\gamma}\, b_j \left(\int_{E_k \cap \{f(\cdot) > 2^j\}} u(y)dy \right)^{\frac{\gamma}{p}} \leq$$

$$\leq c_4 \sum_k \left[\sum_j 2^{jq} \left(\int_{E_k \cap \{f(\cdot) > 2^j\}} u(y)dy \right)^{\frac{q}{p}} \right]^{\frac{\gamma}{q}} \approx c_4 \sum_k \left\| f(\cdot) 1\!\!1_{E_k}(\cdot)\right\|^{\gamma}_{L^{pq}_u}.$$

The first part of this result was proved by Edmunds, Gurka, and Pick [4]. The second part can be derived by the first one with a duality argument.

Proof of Lemma 4. Since M_α is subadditive, for a fixed constant $c>0$

$$\| w_2(\cdot)\,(M_\alpha f)(\cdot)\|^\lambda_{L^{pq}_u} \le c(\mathcal{P}^\lambda_1 + \mathcal{P}^\lambda_2 + \mathcal{P}^\lambda_3) \quad \text{for all } f(\cdot)\ge 0$$

with

$$\mathcal{P}^\lambda_1 = \Big\| \sum_{k\in\mathbb{Z}} w_2(\cdot)\left(M_\alpha f \mathbb{1}_{\{|y|\le 2^{k-1}\}}\right)(\cdot)\,\mathbb{1}_{E_k}(\cdot)\Big\|^\lambda_{L^{pq}_u},$$

$$\mathcal{P}^\lambda_2 = \Big\| \sum_{k\in\mathbb{Z}} w_2(\cdot)\left(M_\alpha f \mathbb{1}_{G_k}\right)(\cdot)\,\mathbb{1}_{E_k}(\cdot)\Big\|^\lambda_{L^{pq}_u},$$

$$\mathcal{P}^\lambda_3 = \Big\| \sum_{k\in\mathbb{Z}} w_2(\cdot)\left(M_\alpha f \mathbb{1}_{\{2^{k+2}<|y|\}}\right)(\cdot)\,\mathbb{1}_{E_k}(\cdot)\Big\|^\lambda_{L^{pq}_u}.$$

We have only to estimate $\mathcal{P}^\lambda_1$ and $\mathcal{P}^\lambda_3$.

Estimate of $\mathcal{P}^\lambda_1$. The key is to prove

$$\left(M_\alpha f_{\{|\cdot|\le 2^{k-1}\}}\right)(x) \le c \quad |x|^{(\alpha-n)}\Big[\int_{\{|y|\le|x|\}} f(y)dy\Big] \quad \text{for all } x\in E_k. \tag{4.1}$$

Here $c>0$ is a constant which depends only on α and n. Indeed, using (4.1) we have

$$\mathcal{P}^\lambda_1 \le c_1 \Big\| \sum_{k\in\mathbb{Z}} w_2(\cdot)|\cdot|^{\alpha-n}\left(\int_{|y|\le|\cdot|} f(y)dy\right)\mathbb{1}_{E_k}(\cdot)\Big\|^\lambda_{L^{pq}_u} =$$
$$= c_1 \Big\|\, w_2(\cdot)|\cdot|^{\alpha-n}\left(\int_{|y|\le|\cdot|} f(y)dy\right)\Big\|^\lambda_{L^{pq}_u} = c_1\mathcal{S}^\lambda_1.$$

To get inequality (4.1) observe that $|y|\le|x|$, for $x\in E_k$ and $|y|\le 2^{k-1}$. Thus the support of $g(\cdot)=f(\cdot)\mathbb{1}_{\{|y|\le 2^{k-1}\}}(\cdot)$ is contained in the set $\{y;\,|y|\le|x|\}=\{|y|\le|x|\}$. On the other hand, the term $\int_{B(x,r)} g(y)dy$ does not vanish whenever $r\ge 2^{k+2}$. Consequently for such a real r

$$r^{\alpha-n}\int_{B(x,r)} g(y)dy \le c\,2^{k(\alpha-n)}\int_{B(x,r)} g(y)dy \le$$
$$\le c'\,|x|^{k(\alpha-n)}\int_{B(x,r)\cap supp\, g} g(y)dy \le c'\,|x|^{k(\alpha-n)}\int_{\{|y|\le|x|\}} f(y)dy;$$

so (4.1) is proved.

Estimate of $\mathcal{P}^\lambda_3$. To estimate $\mathcal{P}^\lambda_3$ we claim that

$$\left(M_\alpha f_{\{2^{k+2}\le|\cdot|\}}\right)(x) \le c\sup_{l\ge 2}\Big\{d^{(\alpha-n)}_{k+l}\Big(\int_{E_{k+l}} f(y)dy\Big)\Big\} \quad \text{for each } x\in E_k, \tag{4.2}$$

where $d_{k+l}=2^{k+l}$ and $c=c(\alpha,n)>0$. The proof of this claim is given below.

First, the cases $\lambda = p = q$, $\lambda \le q < p$, and $\lambda < p \le q$ are treated. Observe that

$$\Big[\sum_{k\in\mathbb{Z}} \Big(M_\alpha f \mathbb{1}_{2^{k+2}<|\cdot|}\Big)(\cdot)\, \mathbb{1}_{E_k}(\cdot)\Big]^\lambda \le \sum_{k} \Big(M_\alpha f \mathbb{1}_{2^{k+2}<|\cdot|}\Big)^\lambda(\cdot)\, \mathbb{1}_{E_k}(\cdot) \le$$

$$\le c\sum_{k\in\mathbb{Z}} \Big[\sup_{l\ge 2}\Big\{d_{k+l}^{(s-n)}\Big(\int_{E_{k+l}} f(y)dy\Big)\Big\}\Big]^\lambda \mathbb{1}_{E_k}(\cdot) \le$$

$$\le c\sum_{l=2}^{\infty}\sum_{k\in\mathbb{Z}} \Big[d_{k+l}^{(s-n)}\Big(\int_{E_{k+l}} f(y)dy\Big)\Big]^\lambda \mathbb{1}_{E_k}(\cdot) =$$

$$= c\sum_{m\in\mathbb{Z}} \Big[d_{m}^{(s-n)}\Big(\int_{E_{m}} f(y)dy\Big)\Big]^\lambda \sum_{k=-\infty}^{m-2}\mathbb{1}_{E_k}(\cdot) =$$

$$= c\sum_{m\in\mathbb{Z}} \Big[d_{m}^{(s-n)}\Big(\int_{E_{m}} f(y)dy\Big)\Big]^\lambda \mathbb{1}_{\{|\cdot|<2^m\}}(\cdot).$$

Since with the above hypotheses $\|\cdot\|_{L_u^{\frac{p}{\lambda}\frac{q}{\lambda}}}$ is equivalent to a norm, we have

$$\mathcal{P}_3^\lambda \le c^\lambda \Big\| w_2(\cdot)^\lambda \sum_{m\in\mathbb{Z}} \Big[d_{m}^{(s-n)}\Big(\int_{E_{m}} f(y)dy\Big)\Big]^\lambda \mathbb{1}_{\{|\cdot|<2^m\}}(\cdot)\Big\|_{L_u^{\frac{p}{\lambda}\frac{q}{\lambda}}} \le$$

$$\le C\sum_{m\in\mathbb{Z}} \Big[d_{m}^{(s-n)}\Big(\int_{E_{m}} f(y)dy\Big)\Big]^\lambda \Big\| w_2(\cdot)\, \mathbb{1}_{\{|\cdot|<2^m\}}(\cdot)\Big\|_{L_u^{pq}}^\lambda.$$

Next the case $\lambda = p < q$ is considered. The key point is hypothesis (1.6). Thus

$$\mathcal{P}_3^p \le c\Big\|\sum_{k\in\mathbb{Z}}\Big[\sup_{l\ge 2} d_{k+l}^{(s-n)}\Big(\int_{E_{k+l}} f(y)dy\Big)\Big] w_2(\cdot)\, \mathbb{1}_{E_k}(\cdot)\Big\|_{L_u^{pq}}^p \le$$

$$\le C\sum_{k\in\mathbb{Z}}\Big[\sup_{l\ge 2} d_{k+l}^{(s-n)}\Big(\int_{E_{k+l}} f(y)dy\Big)\Big]^p \Big\|w_2(\cdot)\, \mathbb{1}_{E_k}(\cdot)\Big\|_{L_u^{pq}}^p \le$$

$\big($by the second part of Lemma 2 $\quad (\lambda = p \le \min(p,q))\big)$

$$\le C\sum_{l=2}^{\infty}\sum_{k\in\mathbb{Z}}\Big[d_{k+l}^{(s-n)}\Big(\int_{E_{k+l}} f(y)dy\Big)\Big]^p \Big\|w_2(\cdot)\, \mathbb{1}_{E_k}(\cdot)\Big\|_{L_u^{pq}}^p \le$$

$$\le C\sum_{m}\Big[d_{m}^{(s-n)}\Big(\int_{E_{m}} f(y)dy\Big)\Big]^p \sum_{k=-\infty}^{m-2}\Big\|w_2(\cdot)\, \mathbb{1}_{E_k}(\cdot)\Big\|_{L_u^{pq}}^p \le$$

$$\le C'\sum_{m}\Big[d_{m}^{(s-n)}\Big(\int_{E_{m}} f(y)dy\Big)\Big]^p \Big\|w_2(\cdot)\, \mathbb{1}_{|\cdot|<2^m}(\cdot)\Big\|_{L_u^{pq}}^p \quad \big(\text{by } (1.6)\big).$$

Now claim (4.2) can be proved. It is assumed that

$$\mathcal{S} = \sup_{l\geq 2} 2^{(\alpha-n)(k+l)}\left[\int_{\{2^{k+l}<|y|\leq 2^{k+l+1}\}} f(y)dy\right] < \infty.$$

Let $x \in E_k$. The claim is reduced to finding a constant $c > 0$ for which

$$r^{\alpha-n}\int_{B(x,r)} f(y)\mathbb{1}_{\{2^{k+2}<|y|\}}(y)dy \leq c\,\mathcal{S} \tag{4.3}$$

whenever $\int_{B(x,r)} f(y)\mathbb{1}_{\{2^{k+2}<|y|\}}(y)dy$ is a non-vanishing term.

Consider $r > 0$ with $\int_{B(x,r)} f(y)\mathbb{1}_{\{2^{k+2}<|y|\}}(y)dy \neq 0$. There is an integer $m \geq 2$ for which $B(x,r) \cap \{2^{k+m} < |y| \leq 2^{k+m+1}\} \neq \varnothing$ and $B(x,r) \cap \{2^{k+m+1} < |y| \leq 2^{k+m+2}\} = \varnothing$. Since $2^{k+m} - 2^{k+1} < r \leq 2^{k+m+1}$ and $m \geq 2$, we obtain $\frac{1}{2}2^{k+m} \leq r < 22^{k+m}$. With these preliminaries

$$\begin{aligned}\int_{B(x,r)} f(y)\mathbb{1}_{\{2^{k+2}<|y|\}}(y)dy &= \sum_{l=2}^{m}\int_{B(x,r)\cap\{2^{k+l}<|y|\leq 2^{k+l+1}\}} f(y)dy \leq \\ &\leq \sum_{l=2}^{m}\int_{\{2^{k+l}<|y|\leq 2^{k+l+1}\}} f(y)dy \leq \mathcal{S}\sum_{l=2}^{m} 2^{(k+l)(n-\alpha)} \leq \\ &\leq \mathcal{S}\,\frac{2^{2(n-\alpha)}}{2^{(n-\alpha)}-1} \times 2^{[k+(m-1)](n-\alpha)} = c(\alpha,n)\mathcal{S}\,r^{(n-\alpha)}.\end{aligned}$$

The latter inequality immediately implies (4.3).

Proof of Lemma 5. First it is clear that for a fixed constant $c > 0$:

$$\|w_2(\cdot)(T_\alpha f)(\cdot)\|^{\lambda}_{L^{pq}_u} \leq c(\mathcal{P}_1^\lambda + \mathcal{P}_2^\lambda + \mathcal{P}_3^\lambda) \quad \text{for all } f(\cdot) \in C_c^\infty$$

with

$$\mathcal{P}_1^\lambda = \left\|\sum_{k\in\mathbb{Z}} w_2(\cdot)\left(T_\alpha f\mathbb{1}_{\{|y|\leq 2^{k-1}\}}\right)(\cdot)\,\mathbb{1}_{E_k}(\cdot)\right\|^{\lambda}_{L^{pq}_u},$$

$$\mathcal{P}_2^\lambda = \left\|\sum_{k\in\mathbb{Z}} w_2(\cdot)\left(T_\alpha f\mathbb{1}_{G_k}\right)(\cdot)\,\mathbb{1}_{E_k}(\cdot)\right\|^{\lambda}_{L^{pq}_u},$$

$$\mathcal{P}_3^\lambda = \left\|\sum_{k\in\mathbb{Z}} w_2(\cdot)\left(T_\alpha f\mathbb{1}_{\{2^{k+2}<|y|\}}\right)(\cdot)\,\mathbb{1}_{E_k}(\cdot)\right\|^{\lambda}_{L^{pq}_u},$$

where E_k and G_k are defined as in the proof of Lemma 4. By the assumption on T_α, the expressions $(T_\alpha f\mathbb{1}_{\{|\cdot|\leq 2^{k-1}\}})(\cdot)$, $(T_\alpha f\mathbb{1}_{G_k})(\cdot)$, $(T_\alpha f\mathbb{1}_{\{2^{k+2}<|\cdot|\}})(\cdot)$ are well defined.

As in the proof of Lemma 4, the estimate for $\mathcal{P}_1^\lambda$ will be obtained at once:

$$\left|(T_s f_{\{|\cdot|\leq 2^{k-1}\}})(x)\right| \leq c(\alpha,n)\,|x|^{(\alpha-n)}\left[\int_{\{|y|\leq|x|\}} |f(y)|\,dy\right] \quad \text{for each } x \in E_k.$$

To get this inequality observe that $|y| \leq \frac{1}{2}|x| < |x|$ and $\frac{1}{2}|x| \leq |x-y|$ for $x \in E_k$ and $|y| \leq 2^{k-1}$. So using the standard estimate for the kernel K, we have $|K(x,y)| \leq c|x-y|^{(\alpha-n)} \leq c'|x|^{(s-n)}$. Since x does not belong to the support of the function $(f_{\{|y|\leq 2^{k-1}\}})(\cdot)$, we obtain $\left|(T_s f 1\!\!1_{\{|\cdot|\leq 2^{k-1}\}})(x)\right| \leq \int_{\{|y|\leq 2^{k-1}\}} |K(x,y)||f(y)|dy \leq C\,|x|^{(\alpha-n)}\left[\int_{\{|y|<|x|\}} |f(y)|dy\right]$.

Similarly, the estimate for $\mathcal{P}_3$ is a consequence of

$$\left|(T_s f_{\{2^{k+2}<|\cdot|\}})(x)\right| \leq c(s,n)\left[\int_{\{|x|\leq|y|\}} |f(y)||y|^{(s-n)}\,dy\right] \quad \text{for all } x \in E_k.$$

Indeed, $|x| < 2|x| \leq |y|$ and $\frac{1}{2}|y| \leq |x-y|$, for $x \in E_k$ and $2^{k+2} < |y|$. So $|K(x,y)| \leq C|x-y|^{(s-n)} \leq C|y|^{(s-n)}$. On the other hand, since x does not belong to the support of the function $(f 1\!\!1_{\{2^{k+2}<|y|\}})(\cdot)$, we have

$$\left|(T_s f 1\!\!1_{\{2^{k+2}<|\cdot|\}})(x)\right| \leq C\left[\int_{\{|x|<|y|\}} |f(y)||y|^{(s-n)}dy\right].$$

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Author's address:
Institut Polytechnique St. Louis
Ecole de Physique et de Mathématiques Industrielles
13, boulevard de l'Hautil, 95 092 Cergy-Pontoise Cedex
France