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ON THE INTEGRABILITY OF THE ERGODIC HILBERT TRANSFORM FOR A CLASS OF FUNCTIONS WITH EQUAL ABSOLUTE VALUES

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ABSTRACT. It is proved that for an arbitrary non-atomic finite measure space with a measure-preserving ergodic transformation there exists an integrable function f such that the ergodic Hilbert transform of any function equal in absolute values to f is non-integrable.

Let T be a one-to-one measure-preserving ergodic transformation of a finite measure space $(X, \mathbb{S}, \mu)$. For a real integrable function f, $f \in L(X)$, the ergodic Hilbert transform is defined by

$$H(f)(x) = {\sum_{k=-\infty}^{\infty}}' \frac{f(T^k x)}{k}, \quad x \in X$$

(the mark $'$ indicates that the 0-th term in the sum is omitted). It exists for almost all $x \in X$ (in the sense of measure μ) and, furthemore, the operator H satisfies a weak-type $(1,1)$ inequality

$$\mu(|H(f)| > \lambda) \leq \frac{c}{\lambda} \|f\|_1, \quad \lambda > 0, \ \ f \in L(X), \tag{1}$$

where c is an absolute constant (see, e.g., [2]).

It is well known that the condition $f \in L \lg L$ implies the integrability of $H(f)$ (see[1]) but, in general, it is rather difficult to give by means of any integral classes an exact description of the set of functions f for which $H(f) \in L(X)$. Therefore the theorem below seems to be interesting for investigation of this set.

Theorem. *Let $(X, \mathbb{S}, \mu)$ be a non-atomic finite measure space, and let T be a one-to-one measure-preserving ergodic transformation on X. Then*

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there exists $f \in L(X)$ such that for any function g with the same absolute values, $|g| = |f|$, we have

$$H(g) \notin L(X).$$

Analogous questions as a general problem of "kernel and hull" were formulated by Prof. O. Tsereteli (see [3],[4]). He also proved the following theorem on the conjugate operator: for any integrable function f on the unit circle there exists a function g with the same absolute values such that $\widetilde{g}$ is integrable (see [5]).

The proof of the following lemma is simple and we give it for the sake of completeness.

Lemma. *Let T be a one-to-one measure-preserving ergodic transformation of a non-atomic finite measure space $(X, \mathbb{S}, \mu)$. Then there exists a set A of positive measure, $\mu(A) > 0$, such that*

$$\mu\big(X \setminus \bigcup_{k=0}^{n} T^k A\big) > 0 \tag{2}$$

for all positive integers n.

Proof. Let us construct successively a sequence of imbedded measurable sets $(A_n)_{n=0}^{\infty}$. Let A_0 be a set of positive not full measure, i.e., $\mu(A_0) > 0$ and the inequality

$$\mu\big(X \setminus \bigcup_{k=0}^{n} T^k A_n\big) > 0 \tag{3}$$

holds for $n = 0$. Assume now that $A_0 \supset A_1 \supset \cdots \supset A_n$ have already been chosen, $\mu(A_n) > 0$, and inequality (3) holds for A_n. Choose $A_n^0 \subset X \setminus \bigcup_{k=0}^{n} T^k A_n$ such that

$$0 < \mu(A_n^0) < \frac{\mu(A_0)}{2^{n+2}}\,,$$

and assume $A_{n+1} = A_n \setminus T^{-(n+1)} A_n^0$. Then

$$A_n^0 \subset X \setminus \big(\bigcup_{k=0}^{n+1} T^k A_{n+1}\big). \tag{4}$$

Let

$$A = \bigcap_{n=0}^{\infty} A_n.$$

Obviously,

$$\begin{aligned}\mu(A) &= \mu(A_0) - \sum_{n=0}^{\infty} \mu(A_n \backslash A_{n+1}) \geq \\ &\geq \mu(A_0) - \sum_{n=0}^{\infty} \mu(T^{-(n+1)} A_n^0) = \mu(A_0) - \sum_{n=0}^{\infty} \mu(A_n^0) \geq \\ &\geq \mu(A_0) - \sum_{n=0}^{\infty} \frac{\mu(A_0)}{2^{n+2}} = \frac{\mu(A_0)}{2} > 0,\end{aligned}$$

and (2) also holds for each positive integer n because of (4). $\square$

Proof of the Theorem. Let A be a set of positive measure with property (2), and let us represent the space X as a skyscraper construction with the base A, i.e., assume

$$X = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{n} T^k B_n,$$

where $\cup_{n=0}^{\infty} B_n = A$ and $T^i B_n \cap T^j B_m = \varnothing$ when $n \neq m$, $0 \leq i \leq n, 0 \leq j \leq m$. Because of (2), there exists arbitrarily large n for which $\mu(B_n) > 0$.

Let us choose successively an increasing sequence of positive integers m_n, for which $\mu(B_{m_n}) > 0$, and decreasing to 0 sequences δ_n and ε_n, $n = 1, 2, \ldots$, which satisfy the inequalities

$$\begin{gathered}\delta_1 < \frac{1}{6}\,, \quad \sum_{n=1}^{\infty} \varepsilon_n < 1, \\ \delta_n \cdot \max_{1 \leq i \leq n-1} \left(\frac{\varepsilon_i}{\mu(B_{m_i})} \right) < \frac{1}{4}\,, \\ \varepsilon_n \cdot \log(\delta_n \cdot m_n) > 3, \\ \frac{\varepsilon_n}{\mu(B_{m_n})} \cdot c \sum_{k=n+1}^{\infty} \varepsilon_k < 1, \quad n = 1, 2, \ldots\,,\end{gathered} \tag{5}$$

where c is the constant appearing in inequality (1). After a suitable choice of ε_n, δ_{n-1} and m_{n-1} one can take δ_n small enough, m_n large enough and ε_{n+1} small enough again to satisfy the second, the third, and the fourth inequalities, respectively.

To simplify our expressions let us use the following notations: $\overline{n} = [\frac{m_n}{2}]$ ($[a]$ is the nearest integer less than or equal to a) and $\beta_n = \mu(B_{m_n})$. Define f_n, $n = 1, 2, \ldots$, by the equalities

$$f_n(x) = \begin{cases} \dfrac{\varepsilon_n}{\beta_n}\,, & \text{when} \quad x \in T^{\overline{n}} B_{m_n} \\ 0, & \text{when} \quad x \in X \backslash T^{\overline{n}} B_{m_n} \end{cases}.$$

Note that $\|f_n\| = \varepsilon_n$. Thus the function f,

$$f = \sum_{n=1}^{\infty} f_n,$$

is integrable and let us show that it has the property required in the theorem.

Let g be any function equal in absolute value to f. Assume

$$g = \sum_{n=1}^{\infty} g_n,$$

where $|g_n| = f_n$, $n = 1, 2, \dots$. Partial sums of this series will be denoted by h_n,

$$h_n = \sum_{k=1}^{n} g_k.$$

The relations below are assumed to hold for almost all x, in particular, for those x for which the Hilbert transform of the corresponding functions exists.

Temporarily, suppose n to be fixed. Let us show that the inequality

$$\big|H(h_n)(T^i x) - H(h_n)(T^{-i}x)\big| \geq \frac{2\varepsilon_n}{i\beta_n} - \frac{12\varepsilon_n}{m_n\beta_n} - 8\delta_n\|h_{n-1}\|_\infty, \tag{6}$$

$x \in T^{\overline{n}} B_{m_n}$, holds for each i, $0 < i < \delta_n m_n$. Indeed, because of the linearity of the Hilbert transform,

$$\begin{gathered}\big|H(h_n)(T^i x) - H(h_n)(T^{-i}x)\big| \geq \\ \geq \big|H(g_n)(T^i x) - H(g_n)(T^{-i}x)\big| - \big|H(h_{n-1})(T^i x) - H(h_{n-1})(T^{-i}x)\big| = \\ = S_1 - S_2.\end{gathered}$$

Taking into account that $T^k x \in T^{\overline{n}} B_{m_n}$ implies $T^{k+j} x \notin T^{\overline{n}} B_{m_n}$, $j = 1, 2, \dots, m_n$, we have

$$\begin{aligned}S_1 &= \bigg| -\frac{g_n(x)}{i} + {\sum_{k\neq -i}}' \frac{g_n(T^{k+i}x)}{k} - \frac{g_n(x)}{i} - {\sum_{k\neq i}}' \frac{g_n(T^{k-i}x)}{k}\bigg| = \\ &= \bigg| -\frac{2g_n(x)}{i} + \sum_{|k|>m_n/2}\Big(\frac{g_n(T^{k+i}x)}{k} - \frac{g_n(T^{k+i}x)}{k+2i}\Big)\bigg| \geq \\ &\geq \frac{2f_n(x)}{i} - \sum_{|k|>m_n/2} f_n(T^{k+i}x)\Big(\frac{1}{k} - \frac{1}{k+2i}\Big) \geq \\ &\geq \frac{2f_n(x)}{i} - 2f_n(x)\sum_{k=0}^{\infty}\Big(\frac{1}{m_n/2 + km_n} - \frac{1}{m_n/2 + km_n + 2i}\Big) \geq\end{aligned}$$

$$\geq \frac{2f_n(x)}{i} - \frac{4if_n(x)}{m_n^2}\sum_{k=0}^{\infty}\frac{1}{(1/2+k)^2} \geq \frac{2f_n(x)}{i} - \frac{12\delta_n f_n(x)}{m_n} =$$
$$= \frac{2\varepsilon_n}{i\beta_n} - \frac{12\delta_n\varepsilon_n}{m_n\beta_n}$$

and

$$S_2 = \Big|\sum_{|k|>(1-\delta_n)m_n}\Big(\frac{h_{n-1}(T^{k+i}x)}{k} - \frac{h_{n-1}(T^{k+i}x)}{k+2i}\Big)\Big| \leq$$
$$\leq 2\|h_{n-1}\|_\infty \sum_{k>(1-\delta_n)m_n}\Big(\frac{1}{k} - \frac{1}{k+2i}\Big) \leq 4i\|h_{n-1}\|_\infty \sum_{k>(1-\delta_n)m_n}\frac{1}{k^2} \leq$$
$$\leq 4\delta_n m_n\|h_{n-1}\|_\infty \frac{1}{(1-\delta_n)m_n - 1} \leq 8\delta_n\|h_{n-1}\|_\infty.$$

Thus (6) holds and therefore the inequality

$$\max\big(|H(h_n)(T^ix)|, |H(h_n)(T^{-i}x)|\big) \geq$$
$$\geq \frac{\varepsilon_n}{i\beta_n} - \frac{6\delta_n\varepsilon_n}{m_n\beta_n} - 4\delta_n\|h_{n-1}\|_\infty \equiv \eta_n(i) \tag{7}$$

is true.

Denote by the same letter η_n the function on X defined by the equalities: $\eta_n(T^ix) = \eta_n(|i|)$ whenever $x \in T^{\overline{n}}B_{m_n}$ and $|H(h_n)(T^ix)| \geq \eta_n(|i|)$, $0 < |i| < \delta_n m_n$; $\eta_n(x) = 0$ otherwise.

Obviously,

$$|H(h_n)| \geq \eta_n.$$

Let $E_n = \bigcup_{\overline{n}-\delta_n m_n < i < \overline{n}+\delta_n m_n} T^iB_{m_n}$ and

$$M_n = \Big\{x \in X : \Big|H\Big(\sum_{k=n+1}^{\infty} g_k\Big)\Big| > 1\Big\}.$$

Because of inequality (1),

$$\mu(M_n) \leq c\cdot\Big\|\sum_{k=n+1}^{\infty} g_k\Big\| = c\cdot\sum_{k=n+1}^{\infty}\varepsilon_k.$$

Taking into account (7), we have

$$\int_{E_n}|H(g)|d\mu \geq \int_{E_n}\Big|\,|H(h_n)| - \Big|H\Big(\sum_{k=n+1}^{\infty} g_k\Big)\Big|\,\Big|d\mu \geq$$
$$\geq \int_{E_n\setminus M_n}(|H(h_n)|-1)d\mu \geq \int_{E_n\setminus M_n}\eta_n d\mu - \mu(E_n) \geq$$

$$\geq \int_{E_n} \eta_n d\mu - \|\eta_n\|_\infty \mu(M_n) - \mu(E_n) \geq$$

$$\geq \sum_{i=1}^{[\delta_n m_n]} \eta_n(i)\beta_n - \frac{\varepsilon_n}{\beta_n}\,\mu(M_n) - \mu(E_n) \geq$$

$$\geq \varepsilon_n \log(\delta_n m_n) - \frac{6\delta_n\varepsilon_n[\delta_n m_n]}{m_n} - 4\delta_n\|h_{n-1}\|_\infty \mu(E_n) -$$

$$- \frac{\varepsilon_n}{\beta_n} c \sum_{k=n+1}^{\infty} \varepsilon_k - \mu(E_n).$$

Thus, by virtue of (5),

$$\int_{E_n} |H(g)| d\mu \geq 1 - 2\mu(E_n),$$

and since E_n, $n = 1, 2, \ldots$, are pairwise disjoint, we can conclude that

$$\|H(g)\| = \infty. \qquad \square$$

References

1. M. Cotlar, A unified theory of Hilbert transforms and ergodic theorems. *Rev. Mat. Cuyana* **1**(1955), No. 2, 105–167.

2. K. Petersen, Ergodic Theory. *Cambridge Univ. Press.,* 1983.

3. O. D. Tsereteli, On some problems arising in the theory of the Fourier series. (Russian) *Seminar Inst. Prikl. Mat. Tbilis. Univ. Annot. Dokladov* **6**(1972), 33–36.

4. O. D. Tsereteli, Metric properties of conjugate functions. *J. of Soviet Math.* **7**(1977), 317–343.

5. O. D. Tsereteli, On the integrability of conjugate functions. (Russian) *Mat. Zametki* **4**(1968), 461–465.

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