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OPERATIONAL IDENTITIES FOR CIRCULAR AND HYPERBOLIC FUNCTIONS AND THEIR GENERALIZATIONS

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Abstract. Starting from the exponential, some classes of analytic functions of the derivative operator are studied, including pseudo-hyperbolic and pseudo-circular functions. Some formulas related to operational calculus are deduced, and the important role played in such a context by Hermite–Kampé de Fériet polynomials is underlined.

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1. Introduction

Operational calculus is often related mainly to the properties of exponential operators and the classical Laplace transform. However, in literature even hyperbolic or circular functions of the derivative operator are considered (see, e.g., [1, p. 250], [2, p. 232]). In a forthcoming paper [3] we show how to exploit such operators, and Hermite–Kampé de Fériet [4] (or Gould–Hopper [5]) polynomials in order to construct polynomial solutions of special Boundary Value Problems (BVP) in the half-plane. In this article we provide an elementary background to this subject, considering also differential operators related to generalizations of sin and cos functions. Namely, the so-called pseudo-hyperbolic and pseudo-circular functions, which are related to the components of an exponential with respect to the cyclic group of order r, come into play (see [6]–[10]). In the above-cited article [3], using relevant differential operators, we extend to higher order BVP polynomial solutions existing for second order classical operators.

2. Hermite–Kampé de Fériet Polynomials

2.1. **Definitions.** We recall the definitions of H-KdF polynomials, starting from the two-dimensional case.

Put $D := \frac{d}{dx}$, and consider the shift operator

$$e^{yD}f(x) = f(x+y) = \sum_{n=0}^{\infty} \frac{y^n}{n!}\, f^{(n)}(x)$$

(see, e.g., [2, p. 171]), the second equation being meaningful for analytic functions.

Note that

- if $f(x) = x^m$, then $e^{yD}x^m = (x+y)^m$;

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- if $f(x) = \sum_{m=0}^{\infty} a_m x^m$, then $e^{yD} f(x) = \sum_{m=0}^{\infty} a_m (x+y)^m$.

Definition 2.1. The Hermite polynomials in two variables $H_m^{(1)}(x,y)$ are then defined by

$$H_m^{(1)}(x,y) := (x+y)^m.$$

Consequently,

- if $f(x) = \sum_{m=0}^{\infty} a_m x^m$, then

$$e^{yD} f(x) = \sum_{m=0}^{\infty} a_m H_m^{(1)}(x,y).$$

Consider now the exponential containing the second derivative, defined for an analytic function f as follows:

$$e^{yD^2} f(x) = \sum_{n=0}^{\infty} \frac{y^n}{n!}\ f^{(2n)}(x),$$

assuming that the last series is convergent in a non-trivial set.

Note that

- if $f(x) = x^m$, then for $n = 0, 1, \dots, \left[\frac{m}{2}\right]$ we can write: $D^{2n} x^m = m(m-1)\cdots(m-2n+1)x^{m-2n} = \frac{m!}{(m-2n)!}\ x^{m-2n}$ and therefore

$$e^{yD^2} x^m = \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{y^n}{n!} \frac{m!}{(m-2n)!}\ x^{m-2n}\ ;$$

- if $f(x) = \sum_{m=0}^{\infty} a_m x^m$, then $e^{yD^2} f(x) = \sum_{m=0}^{\infty} a_m H_m^{(2)}(x,y)$.

Definition 2.2. The H-KdF polynomials in two variables $H_m^{(2)}(x,y)$ are then defined by

$$H_m^{(2)}(x,y) := m! \sum_{n=0}^{\left[\frac{m}{2}\right]} \frac{y^n x^{m-2n}}{n!(m-2n)!}\ .$$

Considering, in general, the exponential raised to the j-th derivative we have:

$$e^{yD^j} f(x) = \sum_{n=0}^{\infty} \frac{y^n}{n!}\ f^{(jn)}(x),$$

assuming again the convergence of the last series, and therefore

- if $f(x) = x^m$, then for $n = 0, 1, \dots, \left[\frac{m}{j}\right]$ it follows: $D^{jn}x^m = m(m-1)\cdots(m-jn+1)x^{m-jn} = \frac{m!}{(m-jn)!}\, x^{m-jn}$ so that

$$e^{yD^j}x^m = \sum_{n=0}^{\left[\frac{m}{j}\right]} \frac{y^n}{n!}\frac{m!}{(m-jn)!}\, x^{m-jn}\ ;$$

- if $f(x) = \sum_{m=0}^{\infty} a_m x^m$, then $e^{yD^j}f(x) = \sum_{m=0}^{\infty} a_m H_m^{(j)}(x,y)$.

Definition 2.3. The H-KdF polynomials in two variables $H_m^{(j)}(x,y)$ are then defined by

$$H_m^{(j)}(x,y) := m! \sum_{n=0}^{\left[\frac{m}{j}\right]} \frac{y^n x^{m-jn}}{n!(m-jn)!}\ .$$

Remark 2.1. In the last section (Section 6), suitable hypotheses on the coefficients of the above-considered analytic functions are assumed in order to guarantee the convergence of expansions containing Hermite–Kampé de Fériet polynomials. Of course, these very strong conditions are not always necessary. E.g., in the case $j = 2$ they can be considerably weakened (see [1]).

But for our purposes it is sufficient to show the consistence of the class of analytic functions for which the relevant procedure can be applied, leaving to particular cases a deeper investigation about the region of convergence.

2.2. **Properties.** In a number of articles by G. Dattoli et al. (see, e.g., [11], [12], [13]), by using the so-called *monomiality principle*, the following properties for the two-variable H-KdF polynomials $H_m^{(j)}(x,y)$, $j \geq 2$, have been recovered (the case $j = 1$ reduces to results about simple powers):

- Operational definition

$$H_m^{(j)}(x,y) = e^{y\frac{\partial^j}{\partial x^j}}x^m = \left(x + jy\frac{\partial^{j-1}}{\partial x^{j-1}}\right)^m (1).$$

- Generating function

$$\sum_{n=0}^{\infty} H_n^{(j)}(x,y)\,\frac{t^n}{n!} = e^{xt+yt^j}.$$

In the case $j = 2$ (see [1]), the H-KdF polynomials $H_m^{(2)}(x,y)$ admit the following

- Integral representation

$$H_m^{(2)}(x,y) = \frac{1}{2\sqrt{\pi y}} \int_{-\infty}^{+\infty} \xi^m e^{-\frac{(x-\xi)^2}{4y}} d\xi, \tag{2.1}$$

which is a particular case of the so-called Gauss–Weierstrass (or Poisson) transform.

3. Hyperbolic and Circular Functions of the Derivative Operator

The two-variable H-KdF polynomials allow us to define, in a constructive way, the hyperbolic and circular functions of the derivative operator. As a matter of fact, for an analytic function f we put, by definition

$$\cosh(yD^j)f(x) := \sum_{k=0}^{\infty} \frac{y^{2k}D^{2jk}}{(2k)!} f(x) = \sum_{k=0}^{\infty} \frac{y^{2k}}{(2k)!} f^{(2jk)}(x)$$

provided that the last series is convergent in a non-trivial set.

Then, if $f(x) = x^m$, and $k = 0, 1, \ldots, \left[\frac{m}{2j}\right]$, since $D^{2jk}x^m = \frac{m!}{(m-2jk)!}x^{m-2jk}$, we can write:

$$\cosh(yD^j)x^m = m! \sum_{k=0}^{\left[\frac{m}{2j}\right]} \frac{y^{2k}x^{m-2jk}}{(2k)!(m-2jk)!} =: \mathcal{K}_m^{(j)}(x,y)$$

and, in general, if $f(x) = \sum_{m=0}^{\infty} a_m x^m$:

$$\cosh(yD^j)f(x) = \sum_{k=0}^{\infty} a_m \mathcal{K}_m^{(j)}(x,y),$$

where

$$\mathcal{K}_m^{(j)}(x,y) = \mathcal{E}_y\left(H_m^{(j)}(x,y)\right)$$

and $\mathcal{E}_y(\cdot)$ denotes the *even part*, with respect to the y variable, of the considered H-KdF polynomial.

Proceeding in an analogous way,

$$\sinh(yD^j)f(x) := \sum_{k=0}^{\infty} \frac{y^{2k+1}D^{j(2k+1)}}{(2k+1)!} f(x) = \sum_{k=0}^{\infty} \frac{y^{2k+1}}{(2k+1)!} f^{j(2k+1)}(x),$$

assuming the convergence of the last series in a non-trivial set.

Then, if $f(x) = x^m$, and $k = 0, 1, \ldots, \left[\frac{m-j}{2j}\right]$, since $D^{j(2k+1)}x^m = \frac{m!}{(m-2jk-j)!}x^{m-2jk-j}$, we can write:

$$\sinh(yD^j)x^m = m! \sum_{k=0}^{\left[\frac{m-j}{2j}\right]} \frac{y^{2k+1}x^{m-2jk-j}}{(2k+1)!(m-2jk-j)!} =: \mathcal{S}_m^{(j)}(x,y)$$

and, in general, if $f(x) = \sum_{m=0}^{\infty} a_m x^m$:

$$\sinh(yD^j)f(x) = \sum_{k=0}^{\infty} a_m \mathcal{S}_m^{(j)}(x,y),$$

where

$$\mathcal{S}_m^{(j)}(x,y) = \mathcal{O}_y\left(H_m^{(j)}(x,y)\right)$$

and $\mathcal{O}_y(\cdot)$ denotes the *odd part*, with respect to y, of the considered H-KdF polynomial.

For circular functions the above formulas become:

$$\cos(yD^j)f(x) = \sum_{k=0}^{\infty}(-1)^k \frac{y^{2k}D^{2jk}}{(2k)!}f(x) = \sum_{k=0}^{\infty}\frac{(iy)^{2k}}{(2k)!}f^{(2jk)}(x),$$

$$\cos(yD^j)f(x) = \sum_{k=0}^{\infty} a_m \mathcal{K}_m^{(j)}(x,iy) = \cosh(iyD^j)f(x),$$

$$\sin(yD^j)f(x) = \sum_{k=0}^{\infty}(-1)^k \frac{y^{2k+1}D^{2jk+j}}{(2k+1)!}f(x) = \sum_{k=0}^{\infty}(-1)^k\frac{y^{2k+1}}{(2k+1)!}f^{(2jk+j)}(x),$$

$$\sin(yD^j)f(x) = \frac{1}{i}\sum_{k=0}^{\infty} a_m \mathcal{S}_m^{(j)}(x,iy) = \frac{1}{i}\sinh(iyD^j)f(x).$$

3.1. **Euler formulas.** Obviously the Euler formula for hyperbolic functions holds true:

$$\cosh yD^j = \frac{e^{yD^j} + e^{-yD^j}}{2} = \mathcal{E}_y(e^{yD^j}),$$

where $\mathcal{E}_y(\cdot)$ denotes the *even part*, with respect to the variable y, of the considered function,

$$\sinh yD^j = \frac{e^{yD^j} - e^{-yD^j}}{2} = \mathcal{O}_y(e^{yD^j}),$$

where $\mathcal{O}_y(\cdot)$ denotes the *odd part*, with respect to the variable y, of the considered function.

Note that

$$\cosh^2(yD^j) - \sinh^2(yD^j) \equiv I,$$

where I is the identity operator.

Analogously, for circular functions, the Euler formulas are given by

$$\cos yD^j = \frac{e^{iyD^j} + e^{-iyD^j}}{2} = \mathcal{E}_y(e^{iyD^j}),$$

$$\sin yD^j = \frac{e^{iyD^j} - e^{-iyD^j}}{2i} = \frac{1}{i}\mathcal{O}_y(e^{iyD^j}),$$

and consequently

$$\cos^2(yD^j) + \sin^2(yD^j) \equiv I.$$

Proposition 3.1. *All the classical hyperbolic and circular formulas involving the above-mentioned functions still hold true.*

Proof. Since all the considered operators commute, the proof is obtained in the same way as in the scalar case. For example, the equations involving circular functions

$$\sin(2yD^j) = 2\sin(yD^j)\cos(yD^j), \qquad \cos(2yD^j) = \cos^2(yD^j) - \sin^2(yD^j),$$

are easily deduced taking into account, in both sides, the definition of the circular functions in terms of the exponential. □

The tangent and cotangent functions of the derivative operator can be defined starting from the relevant Taylor expansions, in which the Bernoulli and Euler numbers appear.

4. A Decomposition Theorem

In [7], in a general form, the following theorem was shown:

Theorem 4.1. *Fix the integer* $r \geq 2$*, and denote by*

$$\omega_{h,r} := \exp\left(\frac{2\pi i h}{r}\right) \qquad (h = 0, 1, \ldots, r-1),$$

the complex roots of unity.

Consider an analytic function of the complex variable z*, defined in a circular neighborhood of the origin, by means of the series expansion*

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and the series

$$\Pi_{[h,r]}\varphi(z) = \varphi_h(z; r) := \sum_{n=0}^{\infty} a_{rn+h} z^{rn+h}, \tag{4.1}$$

called the components of φ with respect to the cyclic group of order r*, then the representation formula*

$$\varphi_h(z; r) = \frac{1}{r}\sum_{j=0}^{r-1} \frac{\varphi(z\,\omega_{j,r})}{\omega_{jh,r}}$$

holds true.

For $r = 2$ and $h = 0$ or $h = 1$ the functions (4.1) reduce to the even or odd components, respectively, of the function $\varphi(z)$. In general, the $\varphi_h(z; r)$ functions are characterized by the symmetry property

$$\varphi_h(z\,\omega_{1,r}; r) = \omega_{h,r}\,\varphi_h(z; r) \qquad (h = 0, 1, \ldots, r-1)$$

with respect to the roots of unity (see also [8], [10]).

In particular, assuming $\varphi(z) := e^z$, the well known decomposition of the exponential function in terms of the so-called *pseudo-hyperbolic functions* appear (see [6]). For shortness, omitting in the sequel the label r, i.e., assuming $\omega_h := \omega_{h,r}$, $\varphi_h(z) := \varphi_h(z; r)$ and so on, we can write

$$f_h(z) := \Pi_{[h,r]}e^z = \sum_{n=0}^{\infty} \frac{z^{rn+h}}{(rn+h)!},$$

so that the exponential is decomposed as the sum

$$e^z = \sum_{h=0}^{r} f_h(z),$$

and the following properties hold true:

$$\begin{gathered} f_0(0) = 1; \qquad f_h(0) = 0 \quad (\text{if } \ h \neq 0), \\ f_h(\omega_1 z) = \omega_h \, f_h(z) \qquad (\text{symmetry property}), \\ D_z f_h(z) = f_{h-1}(z) \qquad (\text{differentiation rule}), \end{gathered}$$

where the indices are assumed to be congruent $(\operatorname{mod} r)$ so that $\forall h$ the pseudo-hyperbolic functions are solutions of the differential equation

$$w^{(r)}(z) - w(z) = 0.$$

The *pseudo-circular functions* are obtained by introducing any complex r-th root σ_0 of the number -1 and putting

$$g_h(z) := \sigma_0^{-h} f_h(\sigma_0 \, z).$$

The pseudo-circular functions are given by the series expansions

$$g_h(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{rn+h}}{(rn+h)!},$$

and satisfy the following properties:

$$\begin{gathered} g_0(0) = 1; \qquad g_h(0) = 0 \quad (\text{if } \ h \neq 0), \\ g_h(\omega_1 z) = \omega_h \, g_h(z) \qquad (\text{symmetry property}), \\ D_z g_0(z) = -g_{r-1}(z), \quad D_z g_h(z) = g_{h-1}(z) \quad (\text{if } \ h \neq 0) \\ (\text{differentiation rule}), \end{gathered}$$

where the indices are assumed to be congruent $(\operatorname{mod} r)$ so that $\forall h$ the pseudo-circular functions are solutions of the differential equation

$$w^{(r)}(z) + w(z) = 0.$$

For further properties, generalizing the ordinary trigonometrical rules, see [7].

5. Pseudo-Hyperbolic or Pseudo-Circular Functions of the Derivative Operator

The properties of the pseudo-hyperbolic or pseudo-circular functions of the derivative operator (see [7]), can be easily extended to the operational case. We give in the following a list of the relevant results, which can be deduced in the same way as in the functional case, considering the commutative property of the powers of D.

The pseudo-hyperbolic functions of the derivative operator are defined by the series expansions

$$f_h(D^j) := \Pi_{[h,r]} e^{D^j} = \sum_{n=0}^{\infty} \frac{D^{j(rn+h)}}{(rn+h)!} \qquad (h = 0, 1, \ldots, r-1)$$

so that the exponential is decomposed as the sum

$$e^{D^j} = \sum_{h=0}^{r-1} f_h(D^j),$$

and the following properties hold true:

$$f_h(\omega_1 D^j) = \omega_h \, f_h(D^j) \qquad \text{(symmetry property)}.$$

Considering two independent variables x and y and denoting by D_x and D_y the relevant differentiations, it follows that

$$D_y f_h(yD_x^j) = D_x^j f_{h-1}(yD_x^j) \qquad \text{(differentiation rule)},$$

where the indices are assumed to be congruent $(\operatorname{mod} r)$ so that $\forall h$ the pseudo-hyperbolic functions are solutions of the abstract differential equation

$$D_y^r f_h(yD_x^j) - D_x^{rj} f_h(yD_x^j) = 0.$$

The pseudo-circular functions of the derivative operator are given by the series expansions

$$g_h(D^j) = \sum_{n=0}^{\infty} (-1)^n \frac{D^{j(rn+h)}}{(rn+h)!},$$

and satisfy the following properties:

$$g_h(\omega_1 D^j) = \omega_h \, g_h(D^j) \qquad \text{(symmetry property)},$$

$$D_y g_0(yD_x^j) = -D_x^j g_{r-1}(yD_x^j), \quad D_y g_h(yD_x^j) = D_x^j g_{h-1}(yD_x^j) \quad (\text{if } h \neq 0)$$
$$\text{(differentiation rule)},$$

where the indices are assumed to be congruent $(\operatorname{mod} r)$ so that $\forall h$ the pseudo-circular functions are solutions of the abstract differential equation

$$D_y^r g_h(yD_x^j) + D_x^{rj} g_h(yD_x^j) = 0.$$

5.1. **Connections with the H-KdF polynomials.** Fixing the integer r, we consider now the action of the pseudo-hyperbolic and pseudo-circular functions on analytic functions, showing relations with the H-KdF polynomials.

Theorem 5.1. *For any $h = 0, 1, \dots, r-1$, and for any positive integer j, denoting by x and y independent variables, and by $D := D_x$, the action of the pseudo-hyperbolic function $f_h(yD^j)$ on the power x^m $(m \in \mathbf{N})$ is given by*

$$f_h\left(yD^j\right) x^m = m! \sum_{n=0}^{\left[\frac{m-jh}{jr}\right]} \frac{y^{rn+h} x^{m-j(rn+h)}}{(rn+h)!(m-j(rn+h))!}$$
$$= \Pi_{[h,r]_y}\left[H_m^{(j)}(x,y)\right],$$

where $\Pi_{[h,r]_y}[K(x,y)]$ $(h = 0, 1, \dots, r-1)$ denote the components of the function K with respect to the cyclic group of order r, and relevant to the variable y (assuming x as a parameter).

Then, if $A(x) = \sum_{m=0}^{\infty} a_m x^m$, *we can write*

$$f_h\left(yD^j\right)A(x) = \sum_{m=0}^{\infty} a_m\ \Pi_{[h,r]_y}\left[H_m^{(j)}(x,y)\right]$$
$$= \sum_{m=0}^{\infty} m!a_m \sum_{n=0}^{\left[\frac{m-jh}{jr}\right]} \frac{y^{rn+h}x^{m-j(rn+h)}}{(rn+h)!(m-j(rn+h))!}.$$

For the pseudo-circular functions, using the above notation, we have

Theorem 5.2. *For any* $h = 0,1,\dots,r-1$, *and for any positive integer* j, *denoting by* x *and* y *independent variables, and by* $D := D_x$, *the action of the pseudo-circular function* $g_h(yD^j)$ *on the power* x^m $(m \in \mathbf{N})$, *is given by*

$$g_h\left(yD^j\right)x^m = m! \sum_{n=0}^{\left[\frac{m-jh}{jr}\right]} (-1)^n \frac{y^{rn+h}x^{m-j(rn+h)}}{(rn+h)!(m-j(rn+h))!}$$
$$= \frac{1}{\sigma_0^h}\Pi_{[h,r]_y}\left[H_m^{(j)}(x,\sigma_0 y)\right],$$

and furthermore, if $A(x) = \sum_{m=0}^{\infty} a_m x^m$, *then*

$$g_h\left(yD^j\right)A(x) = \frac{1}{\sigma_0^h}\sum_{m=0}^{\infty} a_m\ \Pi_{[h,r]_y}\left[H_m^{(j)}(x,\sigma_0 y)\right]$$
$$= \sum_{m=0}^{\infty} m!a_m \sum_{n=0}^{\left[\frac{m-jh}{jr}\right]} (-1)^h \frac{y^{rn+h}x^{m-j(rn+h)}}{(rn+h)!(m-j(rn+h))!}$$

so that, if the considered coefficients a_m *and variables* x,y *are real, the resulting expansions are also real.*

6. Convergence Results

In this section we will recall a uniform estimate, with respect to j, for the convergence of series involving the H-KdF polynomials $H_n^{(j)}(x,y)$:

$$\sum_{n=0}^{\infty} a_n\ H_n^{(j)}(x,y). \tag{6.1}$$

Theorem 6.1. *For every* $j \geq 2$, $-\infty < x < +\infty$, $-\infty < y < +\infty$, $n = 0,1,2,\dots$, *the following estimate holds true:*

$$|H_n^{(j)}(x,y)| \leq n!\ \exp\{|x|+|y|\}.$$

The proof is derived by the same method as used in the book of Widder [1, p. 166] for the case $j = 2$. A deep analysis of the convergence condition for the series (6.1), in the case $j = 2$, is performed in this book, however the relevent estimates can be only partially extended to the general case since many of them are based on the integral representation (2.1). Unfortunately, in the case $j > 2$, an integral representation generalizing the Gauss–Weierstrass transform is not known.

In the present article, we always consider, analytic functions $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for which the coefficients a_n tend to zero sufficiently fast, in order to guarantee the convergence of the series expansion (6.1). To this aim, we only use the following theorem.

Theorem 6.2. *Suppose there exists a number* $\alpha > 1$ *such that the coefficients* a_n *satisfy the estimate*

$$|a_n| = O\left(\frac{1}{n^\alpha n!}\right). \tag{6.2}$$

Then, for every j, *the series expansion* (6.1) *is absolutely and uniformly convergent in every bounded region of the* (x, y) *plane.*

Proof. The result immediately follows from the estimate

$$\left|\sum_{n=0}^{\infty} a_n \; H_n^{(j)}(x,y)\right| \leq \sum_{n=0}^{\infty} \; |a_n| \left|H_n^{(j)}(x,y)\right| \leq e^{|x|+|y|} \sum_{n=0}^{\infty} \; |a_n| n!,$$

considering that the last series is convergent by condition (6.2). □

Remark 6.1. The condition (6.2) includes analytic functions with polynomial growth at infinity, but not the exponential function e^x.

Note that when the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is decomposed with respect to the cyclic group of order r, the same estimate of Theorem 6.2 is sufficient to guarantee the convergence of the relevant expansions according to the property

$$\left| \Pi_{[h,r]}\left(\sum_{n=0}^{\infty} \; |a_n| \, x^n\right) \right| \leq \left|\sum_{n=0}^{\infty} \; |a_n| \, x^n\right| \qquad (h = 0, 1, \ldots, r-1).$$

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Note Added in Proofs

Prof. Y. Ben Cheikh informed us that, in the particular case when $j = 2q$, q odd number, an integral representation generalizing the Gauss–Weierstrass transform can be found in the paper by D. T. Haimo and C. Market: A reprentation theory for solutions of a higher order heat equation, I. *J. Math. Anal. Appl.* **168**(1992), 89–107.

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