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ON THE CORRECTNESS OF LINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. The sufficient conditions are established for the correctness of the linear boundary value problem

$$dx(t) = dA(t) \cdot x(t) + df(t), \quad l(x) = c_0,$$

where $A : [a, b] \to \mathbb{R}^{n\times n}$ and $f : [a, b] \to \mathbb{R}^n$ are matrix- and vector-functions of bounded variation, $c_0 \in \mathbb{R}^n$, and l is a linear continuous operator from the space of n-dimentional vector-functions of bounded variation into $\mathbb{R}^n$.

Let the matrix- and vector-functions, $A : [a, b] \to \mathbb{R}^{n\times n}$ and $f : [a, b] \to \mathbb{R}^n$, respectively, be of bounded variation, $c_0 \in \mathbb{R}^n$, and let $l : BV_n(a, b) \to \mathbb{R}^n$ be a linear continuous operator such that the boundary value problem

$$dx(t) = dA(t) \cdot x(t) + df(t), \tag{1}$$

$$l(x) = c_0 \tag{2}$$

has the unique solution x_0.

Consider the sequences of matrix- and vector-functions of bounded variation $A_k : [a, b] \to \mathbb{R}^{n\times n}$ $(k = 1, 2, \dots)$ and $f_k : [a, b] \to \mathbb{R}^n$ $(k = 1, 2, \dots)$, respectively, the sequence of constant vectors $c_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$ and the sequence of linear continuous operators $l_k : BV_n(a, b) \to \mathbb{R}^n$ $(k = 1, 2, \dots)$.

In this paper the sufficient conditions are given for the problem

$$dx(t) = dA_k(t) \cdot x(t) + df_k(t), \tag{3}$$

$$l_k(x) = c_k \tag{4}$$

to have a unique solution x_k for any sufficiently large k and

$$\lim_{k\to+\infty} x_k(t) = x_0(t) \quad \text{uniformly on } [a, b]. \tag{5}$$

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An analogous question is studied in [2–4] for the boundary value problem for a system of ordinary differential equations.

The theory of generalized ordinary differential equations enables one to investigate ordinary differential and difference equations from the common standpoint. Moreover, the convergence conditions for difference schemes corresponding to boundary value problems for systems of ordinary differential equations can be deduced from the correctness results for aprropriate boundary value problems for systems of generalized ordinary differential equations [1, 5, 6].

The following notations and definitions will be used throughout the paper:

$\mathbb{R} =]-\infty, +\infty[$;

$\mathbb{R}^n$ is a space of real column n-vectors $x = (x_i)_{i=1}^n$ with the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$\mathbb{R}^{n\times n}$ is a space of real $n \times n$-matrices $X = (x_{ij})_{i,j=1}^n$ with the norm

$$\|X\| = \max_{j=1,\dots,n} \sum_{i=1}^n |x_{ij}|;$$

If $X \in \mathbb{R}^{n\times n}$, then X^{-1} and $\det(X)$ are the matrix inverse to X and the determinant of X, respectively; E is the identity $n \times n$ matrix;

$\overset{b}{\underset{a}{\vee}} x$ and $\overset{b}{\underset{a}{\vee}} X$ are the sums of total variations of components of vector- and matrix-functions, $x : [a, b] \to \mathbb{R}^n$ and $X : [a, b] \to \mathbb{R}^{n\times n}$, respectively;

$BV_n(a, b)$ is a space of all vector-functions of bounded variation $x : [a, b] \to \mathbb{R}^n$ (i.e., such that $\overset{b}{\underset{a}{\vee}} x < +\infty$) with the norm

$$\|x\|_{\sup} = \sup\{\|x(t)\| : t \in [a, b]\}^1;$$

$x(t-)$ and $x(t+)$ $(x(a-) = x(a),\ x(b+) = x(b))$ are the left and the right limits of the vector-function $x : [a, b] \to \mathbb{R}^n$ at the point t;

$d_1x(t) = x(t) - x(t-)$, $d_2x(t) = x(t+) - x(t)$;

$BV_{n\times n}(a, b)$ is a set of all matrix-functions of bounded variation $X : [a, b] \to \mathbb{R}^{n\times n}$, i.e., such that $\overset{b}{\underset{a}{\vee}} X < +\infty$;

$d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$;

If $X = (x_{ij})_{i,j=1}^n \in BV_{n\times n}(a, b)$, then $V(X) : [a, b] \to \mathbb{R}^{n\times n}$ is defined by

$$V(X)(a) = 0, \quad V(X)(t) = \Big(\overset{t}{\underset{a}{\vee}} x_{ij}\Big)_{i,j=1}^n \quad (a < t \le b);$$

[1] $BV_n(a, b)$ is not the Banach space with respect to this norm.

If $\alpha \in BV_1(a,b)$, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int\limits_s^t x(\tau)\,d\alpha(\tau) = x(t)d_1\alpha(t) + x(s)d_2\alpha(s) + \int\limits_{]s,t[} x(\tau)\,d\alpha(\tau),$$

where $\int_{]s,t[} x(\tau)\,d\alpha(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ (if $s=t$, then $\int_s^t x(\tau)\,d\alpha(\tau)=0$);

If $A=(a_{ij})_{i,j=1}^n \in BV_{n\times n}(a,b)$, $X=(x_{ij})_{i,j=1}^n : [a,b]\to\mathbb{R}^{n\times n}$, $x=(x_i)_{i=1}^n : [a,b]\to\mathbb{R}^n$ and $a\le s\le t\le b$, then

$$\int\limits_s^t dA(\tau)\cdot X(\tau) = \Big(\sum_{k=1}^n \int\limits_s^t x_{kj}(\tau)\,da_{ik}(\tau)\Big)_{i,j=1}^n,$$

$$\int\limits_s^t dA(\tau)\cdot x(\tau) = \Big(\sum_{k=1}^n \int\limits_s^t x_{k}(\tau)\,da_{ik}(\tau)\Big)_{i=1}^n;$$

$\|l\|$ is the norm of the linear continuous operator $l : BV_n(a,b)\to\mathbb{R}^n$;

If $X\in BV_{n\times n}(a,b)$ is the matrix-function with columns $x_1,\dots,x_n$, then $l(X)$ is the matrix with columns $l(x_1),\dots,l(x_n)$.

A function $x\in BV_n(a,b)$ is said to be a solution of problem (1), (2) if it satisfies condition (2) and

$$x(t) = x(s) + \int\limits_s^t dA(\tau)\cdot x(\tau) + f(t) - f(s) \quad \text{for } a\le s\le t\le b.$$

Alongside with (1) and (3), we shall consider the corresponding homogeneous systems

$$dx(t) = dA(t)\cdot x(t) \tag{1_0}$$

and

$$dx(t) = dA_k(t)\cdot x(t), \tag{3_0}$$

respectively.

A matrix-function $Y\in BV_{n\times n}(a,b)$ is said to be a fundamental matrix of the homogeneous system (1_0) if

$$Y(t) = Y(s) + \int\limits_s^t dA(\tau)\cdot Y(\tau) \quad \text{for } a\le s\le t\le b$$

and $\det\big(Y(t)\big)\ne 0$ for $t\in[a,b]$.

Theorem 1. *Let the conditions*

$$\det\big(E+(-1)^j d_j A(t)\big)\neq 0 \quad \text{for } t\in[a,b] \ (j=1,2), \tag{6}$$

$$\lim_{k\to+\infty} l_k(y)=l(y) \quad \text{for } y\in BV_n(a,b), \tag{7}$$

$$\lim_{k\to+\infty} c_k=c_0, \tag{8}$$

$$\lim_{k\to+\infty}\sup\|l_k\|<+\infty, \tag{9}$$

$$\lim_{k\to+\infty}\sup\mathop{\vee}_a^b A_k<+\infty \tag{10}$$

be satisfied and let the conditions

$$\lim_{k\to+\infty}\big[A_k(t)-A_k(a)\big]=A(t)-A(a), \tag{11}$$

$$\lim_{k\to+\infty}\big[f_k(t)-f_k(a)\big]=f(t)-f(a) \tag{12}$$

be fulfilled uniformly on $[a,b]$. *Then for any sufficiently large* k *problem* (3), (4) *has the unique solution* x_k *and* (5) *is valid.*

To prove the theorem we shall use the following lemmas.

Lemma 1. *Let* $\alpha_k,\beta_k\in BV_1(a,b)$ $(k=0,1\dots)$,

$$\lim_{k\to+\infty}\|\beta_k-\beta_0\|_{\sup}=0, \tag{13}$$

$$r=\sup\Big\{\mathop{\vee}_a^b\alpha_k : k=0,1,\dots\Big\}<+\infty \tag{14}$$

and the condition

$$\lim_{k\to+\infty}\big[\alpha_k(t)-\alpha_k(a)\big]=\alpha_0(t)-\alpha_0(a) \tag{15}$$

be fulfilled uniformly on $[a,b]$. *Then*

$$\lim_{k\to+\infty}\int_a^t\beta_k(\tau)\,d\alpha_k(\tau)=\int_a^t\beta_0(\tau)\,d\alpha_0(\tau)$$

uniformly on $[a,b]$.

Proof. Let ε be an arbitrary positive number. We denote

$$\mathcal{D}_j(a,b,\varepsilon;g)=\big\{t\in[a,b]: d_j g(t)\geq\varepsilon\big\} \quad (j=1,2)$$

where

$$g(t)=V(\beta_0)(t) \quad \text{for } t\in[a,b].$$

By Lemma 1.1.1 from [5] there exists a finite subdivision $\{\alpha_0,\tau_1,\alpha_1,\dots,\tau_m,\alpha_m\}$ of $[a,b]$ such that

a) $a=\alpha_0<\alpha_1<\dots<\alpha_m=b$, $\alpha_0\leq\tau_1\leq\alpha_1\leq\dots\leq\tau_m\leq\alpha_m$;

b) If $\tau_i \notin \mathcal{D}_1(a,b,\varepsilon;g)$, then $g(\tau_i) - g(\alpha_{i-1}) < \varepsilon$;
If $\tau_i \in \mathcal{D}_1(a,b,\varepsilon;g)$, then $\alpha_{i-1} < \tau_i$ and $g(\tau_i-) - g(\alpha_{i-1}) < \varepsilon$;
c) If $\tau_i \notin \mathcal{D}_2(a,b,\varepsilon;g)$, then $g(\alpha_i) - g(\tau_i) < \varepsilon$;
If $\tau_i \in \mathcal{D}_2(a,b,\varepsilon;g)$, then $\tau_i < \alpha_i$ and $g(\alpha_i) - g(\tau_i+) < \varepsilon$.

We set

$$\eta(t) = \begin{cases} \beta_0(t) & \text{for } t \in \{\alpha_0, \tau_1, \alpha_1, \dots, \tau_m, \alpha_m\}; \\ \beta_0(\tau_i-) & \text{for } t \in]\alpha_{i-1}, \tau_i[, \ \tau_i \in \mathcal{D}_1(a,b,\varepsilon;g); \\ \beta_0(\tau_i) & \text{for } t \in]\alpha_{i-1}, \tau_i[, \ \tau_i \notin \mathcal{D}_1(a,b,\varepsilon;g) \text{ or} \\ & \text{for } t \in]\tau_i, \alpha_i[, \ \tau_i \notin \mathcal{D}_2(a,b,\varepsilon;g); \\ \beta_0(\tau_i+) & \text{for } t \in]\tau_i, \alpha_i[, \ \tau_i \in \mathcal{D}_2(a,b,\varepsilon;g); \\ & \qquad (i = 1, \dots, m). \end{cases}$$

It can be easily shown that $\eta \in BV_1(a,b)$ and

$$|\beta_0(t) - \eta(t)| < 2\varepsilon \quad \text{for } \ t \in [a,b]. \tag{16}$$

For every natural k and $t \in [a,b]$ we assume

$$\gamma_k(t) = \int\limits_a^t \beta_k(\tau)\, d\alpha_k(t) - \int\limits_a^t \beta_0(\tau)\, d\alpha_0(\tau)$$

and

$$\delta_k(t) = \int\limits_a^t \eta(t)\, d[\alpha_k(\tau) - \alpha_0(\tau)].$$

It follows from (15) that

$$\lim_{k \to +\infty} \|\delta_k\|_{\sup} = 0. \tag{17}$$

On the other hand, by (14) and (16) we have

$$\|\gamma_k\|_{\sup} \le 4r\varepsilon + r\|\beta_k - \beta_0\|_{\sup} + \|\delta\|_{\sup} \quad (k = 1, 2, \dots).$$

Hence in view of (13) and (17) $\lim_{k \to +\infty} \|\gamma_k\|_{\sup} = 0$ since ε is arbitrary. □

Lemma 2. *Let condition* (6) *be fulfilled and*

$$\lim_{k \to +\infty} Y_k(t) = Y(t) \quad \textit{uniformly on } [a,b], \tag{18}$$

where Y and Y_k are the fundamental matrices of the homogeneous systems (1_0) *and* (3_0)*, respectively. Then*

$$\inf\big\{\big|\det(Y(t))\big| : t \in [a,b]\big\} > 0, \tag{19}$$

$$\inf\big\{\big|\det(Y^{-1}(t))\big| : t \in [a,b]\big\} > 0 \tag{20}$$

and

$$\lim_{k\to+\infty} Y_k^{-1}(t) = Y^{-1}(t) \quad \textit{uniformly on } [a,b]. \tag{21}$$

Proof. It is known ([6], Theorem III.2.10) that $d_j Y(t) = d_j A(t)\cdot Y(t)$ for $t\in[a,b]$ $(j=1,2)$. Therefore (6) implies

$$\det\big(Y(t-)\cdot Y(t+)\big) = \big[\det(Y(t))^2\big]\cdot\prod_{j=1}^{2}\det\big(E+(-1)^j d_j A(t)\big)\neq 0 \\ \text{for} \quad t\in[a,b]. \tag{22}$$

Let us show that (19) is valid. Assume the contrary. Then it can be easily shown that there exists a point $t_0\in[a,b]$ such that $\det\big(Y(t_0-)\cdot Y(t_0+)\big) = 0$. But this equality contradicts (22). Inequality (19) is proved.

The proof of inequality (20) is analogous.

In view of (18) and (19) there exists a positive number q such that $\inf\big\{\big|\det(Y_k(t))\big| : t\in[a,b]\big\} > q > 0$ for any sufficiently large k. From this and (18) we obtain (21). $\square$

Proof of the Theorem. Let us show that

$$\det\big(E+(-1)^j d_j A_k(t)\big)\neq 0 \quad \text{for } t\in[a,b] \ (j=1,2) \tag{23}$$

for any sufficiently large k.

By (11)

$$\lim_{k\to+\infty} d_j A_k(t) = d_j A(t) \quad (j=1,2) \tag{24}$$

uniformly on $[a,b]$. Since $\overset{b}{\underset{a}{\vee}} A < +\infty$, the series $\sum_{t\in[a,b]}\|d_j A(t)\|$ $(j=1,2)$ converges. Thus for any $j\in\{1,2\}$ the inequality

$$\|d_j A(t)\| \geq \frac{1}{2}$$

may hold only for some finite number of points $t_{j1},\dots,t_{j\,m_j}$ in $[a,b]$. Therefore

$$\|d_j A(t)\| < \frac{1}{2} \quad \text{for } t\in[a,b], \ t\neq t_{ji} \ (i=1,\dots,m_j). \tag{25}$$

It follows from (6), (24) and (25) that for any sufficiently large k and for $j\in\{1,2\}$

$$\det\big(E+(-1)^j d_j A_k(t_{ji})\big)\neq 0 \quad (i=1,\dots,m_j) \tag{26}$$

and

$$\|d_j A_k(t)\| < \frac{1}{2} \quad \text{for } t\in[a,b], \ t\neq t_{ji} \ (i=1,\dots,m_j). \tag{27}$$

The latter inequality implies that the matrices $E+(-1)^j d_j A_k(t)$ $(j=1,2)$ are invertible for $t\in[a,b]$, $t\neq t_{ji}$ $(i=1,\dots,m_j)$ too. Therefore (23) is proved.

Besides, by (26) and (27) there exists a positive number r_0 such that for any sufficiently large k

$$\big\|\big[E+(-1)^j d_j A_k(t)\big]^{-1}\big\|\le r_0 \quad\text{for}\quad t\in[a,b]\ \ (j=1,2). \tag{28}$$

Let k be a sufficiently large natural number. In view of (6) and (23) there exist ([6], Theorem III.2.10) fundamental matrices Y and Y_k of systems (1_0) and (3_0), respectively, satisfying $Y(a)=Y_k(a)=E$. Moreover, $Y_k^{-1}\in BV_{n\times n}(a,b)$.

Let us prove (18). We set $Z_k(t)=Y_k(t)-Y(t)$ for $t\in[a,b]$ and $B_k(t)=A_k(t-)$ for $t\in[a,b]$. Obviously, for every $t\in[a,b]$

$$d_1\big[B_k(t)-A_k(t)\big]=-d_2\big[B_k(t)-A_k(t)\big]=-d_1A_k(t)$$

and

$$\int_a^t d\big[B_k(\tau)-A_k(\tau)\big]\cdot Z_k(\tau)=-d_1A_k(t)\cdot Z_k(t).$$

Consequently,

$$Z_k(t)\equiv\big[E-d_1A_k(t)\big]^{-1}\Big[\int_a^t d\big[A_k(\tau)-A(\tau)\big]\cdot Y(\tau)+\int_a^t dB_k(\tau)\cdot Z_k(\tau)\Big].$$

From this and (28) we get

$$\|Z_k(t)\|\le r_0\Big(\varepsilon_k+\int_a^t d\|V(B_k)(\tau)\|\cdot\|Z_k(\tau)\|\Big)\quad\text{for}\quad t\in[a,b],$$

where

$$\varepsilon_k=\sup\Big\{\Big\|\int_a^t d\big[A_k(\tau)-A(\tau)\big]\cdot Y(\tau)\Big\|:t\in[a,b]\Big\}.$$

Hence, according to the Gronwall inequality ([6], Theorem I.4.30),

$$\|Z_k(t)\|\le r_0\varepsilon_k\exp\big(r_0\bigvee_a^b B_k\big)\le r_0\varepsilon_k\exp\big(r_0\bigvee_a^b A_k\big)\quad\text{for}\quad t\in[a,b].$$

By (10), (11) and Lemma 1 this inequality implies (18).

It is known ([6], Theorem III.2.13) that if x_k is the solution of (3), then

$$x_k(t)\equiv Y_k(t)x_k(a)+f_k(t)-f_k(a)-Y_k(t)\int_a^t dY_k^{-1}(\tau)\cdot\big[f_k(\tau)-f_k(a)\big].$$

Thus problem (3), (4) has the unique solution if and only if

$$\det\big(l_k(Y_k)\big) \neq 0. \tag{29}$$

Since problem (1), (2) has the unique solution x_0, we have

$$\det\big(l(Y)\big) \neq 0. \tag{30}$$

Besides, by (7), (9) and (18)

$$\lim_{k\to+\infty} l_k(Y_k) = l(Y).$$

Therefore, in view of (30), there exists a natural number k_0 such that condition (29) is fulfilled for every $k \geq k_0$. Thus problem (3), (4) has the unique solution x_k for $k \geq k_0$ and

$$x_k(t) \equiv Y_k(t)\big[l_k(Y_k)\big]^{-1}\big[c_k - l_k(F_k(f_k))\big] + F_k(f_k)(t), \tag{31}$$

where

$$F_k(f_k)(t) = f_k(t) - f_k(a) - Y_k(t)\int_a^t dY_k^{-1}(\tau)\cdot\big[f_k(\tau) - f_k(a)\big].$$

According to Lemma 2 condition (21) is fulfilled and

$$\rho = \sup\big\{\|Y_k^{-1}(t)\| + \|Y_k(t)\| : t \in [a,b],\ k \geq k_0\big\} < +\infty. \tag{32}$$

The equality

$$Y_k^{-1}(t) - Y_k^{-1}(s) = Y_k^{-1}(s)\int_t^s dA_k(\tau)\cdot Y_k(\tau)\ Y_k^{-1}(t)$$

implies

$$\|Y_k^{-1}(t) - Y_k^{-1}(s)\| \leq \rho^3 \bigvee_s^t A_k \quad \text{for} \ \ a \leq s \leq t \leq b \ \ (k \geq k_0).$$

This inequality, together with (10) and (32), yields

$$\lim_{k\to+\infty} \sup \bigvee_a^b Y_k^{-1} < +\infty.$$

By this, (12) and (21) it follows from Lemma 1 that

$$\lim_{k\to+\infty}\int_a^t dY_k^{-1}(\tau)\cdot\big[f_k(\tau) - f_k(a)\big] =$$

$$= \int_a^t dY^{-1}(\tau)\cdot\big[f(\tau) - f(a)\big] \tag{33}$$

uniformly on $[a,b]$.

Using (7)-(9), (12), (18), (29), (30) and (33), from (31) we get $\lim\limits_{k\to+\infty} x_k(t) = z(t)$ uniformly on $[a,b]$, where

$$z(t) = Y(t)\big[l(Y)\big]^{-1}\big[c_0 - l\big(F(f)\big)\big] + F(f)(t),$$

$$F(f)(t) = f(t) - f(a) - Y(t)\int\limits_a^t dY^{-1}(\tau)\cdot\big[f(\tau) - f(a)\big].$$

It is easy to verify that the vector-function $z : [a,b] \to \mathbb{R}^n$ is the solution of problem (1), (2). Therefore

$$x_0(t) = z(t) \quad \text{for} \quad t \in [a,b]. \qquad \square$$

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