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On Schur‘s expansion of $\sin \pi x$

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Abstract

In this paper we study the sign of the sequence $(\Delta^n c_k)$ where (c_k) are the coefficients for the development

$$\sin ax \cdot \sin(a - ax) = \frac{1}{2}\sqrt{\frac{\pi}{2a}} \sum_{k=0}^{\infty} c_k [x(1-x)]^k \quad , a \in \left(0, \frac{\pi}{2}\right]. \tag{1}$$

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It is known the development (1) where $a \in \left(0, \frac{\pi}{2}\right]$, and

$$c_k = \frac{2^k}{k!} a^{k+1} J_{k-\frac{1}{2}}(a),$$

J_p are Bessel functions. The sequence (c_k) is strictly decreasing to zero(see [1]).

Firstly we consider the sequence

$$\Delta^1 c_0, \Delta^1 c_1, \Delta^1 c_2, ..., \Delta^1 c_n, ...$$

and we have

$$\Delta^1 c_n < 0, n = 0, 1, 2, ...$$

Now we consider

$$\Delta^2 c_0, \Delta^2 c_1, \Delta^2 c_2, ..., \Delta^2 c_n, ...,$$

where $\Delta^2 c_k = \Delta^1 c_{k+1} - \Delta^1 c_k$.

Using the Poisson formula

$$J_p(z) = \frac{2(\frac{z}{2})^p}{\Gamma(\frac{1}{2})\Gamma(p+\frac{1}{2})} \int\limits_0^1 (1-t^2)^{p-\frac{1}{2}} \cos tz \; dt$$

we have

$$c_{k+1} - c_k = \frac{2^k}{k!} a^{k+1} \left[\frac{2a}{k+1} J_{k+\frac{1}{2}}(a) - J_{k-\frac{1}{2}}(a) \right]$$

Hence

$$\Delta^2 c_k = \frac{2^k}{k!} a^{k+1} \left[\frac{4a^2}{(k+1)(k+2)} J_{k+\frac{3}{2}} - \frac{4a}{k+1} J_{k+\frac{1}{2}}(a) + J_{k-\frac{1}{2}}(a) \right]$$

whence

$$\frac{4a^2}{(k+1)(k+2)} \cdot \frac{\frac{a^2}{4}}{k(k+1)} (1-t^2)^2 - \frac{4a}{k+1} \cdot \frac{\frac{a}{2}}{k} (1-t^2) + 1 \geq 0 \tag{2}$$

if and only if

$$\Delta^2 c_k \geq 0.$$

We observe that (2) is true for $t \in [0,1]$ and $k \geq k_0$.

Next for

$$\Delta^3 c_k = \Delta^2 c_{k+1} - \Delta^2 c_k$$

we find that

$$(3) \quad \frac{8a^3}{(k+1)(k+2)(k+3)} \cdot \frac{(\frac{a}{2})^3}{k(k+1)(k+2)}(1-t^2)^3 - \frac{12a^2}{(k+1)(k+2)}$$

$$\cdot \frac{(\frac{a}{2})^2}{k(k+1)}(1-t^2)^2 + \frac{6a}{k+1} \cdot \frac{\frac{a}{2}}{k} \cdot (1-t^2) - 1 \leq 0.$$

if and only if

$$\Delta^3 c_k \leq 0.$$

We observe that 3 is true for $t \in [0,1]$ and $k \geq k_1$. In this manner, we have hence the following

Theorem *If* (c_k) *is the sequence of the (1), then* $\Delta^{2p} c_k \geq 0$ *for* $k \geq k'$ *and* $p = 1, 2, ...,$ *and* $\Delta^{2p+1} c_k \leq 0$ *for* $k \geq k''$ *and* $p = 0, 1, 2, ...$.

References

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