General Mathematics Vol. 8, No. 1–2 (2000)

On an integral operator and a subclass of starlike functions

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Abstract

We consider the following integral operator:

$$F(z) = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt, \quad \gamma \geq 0. \tag{1}$$

It is well-known that if f is starlike, then F is also starlike. In this paper we will obtain some conditions on f such that F satisfies the inequality

$$\left|\frac{zF'(z)}{F(z)} - 1\right| < 1.$$

1991 Mathematical Subject Classification 30C45

1 Introduction and preliminaries

Let U be the unit disc of the complex plane:

$$U = \{z \in \mathbf{C},\ |z| < 1\},$$

let $\mathcal{H}[U]$ be the space of holomorphic functions in U. We let:

$$A = \{f \in \mathcal{H}[U],\ f(z) = z + a_2 z^2 + a_{n+2} z^{n+2} + \ldots,\ z \in U\}$$

and

$$S^* = \left\{ f \in A,\ \text{Re}\ \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\},$$

the class of starlike functions in U.

If f and g are analytic functions in U, then we say that f is subordinated to g, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$ for all $z \in U$ such that $f(z) = g(w(z))$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma A. [1,2] *Let q be univalent in U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$, when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi[q(z)],$$

$$h(z) = \theta[q(z)] + Q(z)$$

and suppose that

(i) Q is starlike

and

(ii) $\text{Re}\ \dfrac{zh'(z)}{Q(z)} = \text{Re}\ \left[\dfrac{\theta'[q(z)]}{\phi[q(z)]} + \dfrac{zQ'(z)}{Q(z)}\right] > 0.$

If p is analytic in U, with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)] = h(z),$$

then $p \prec q$, and q is the best dominant.

2 Main results

Theorem. *Let $\gamma \geq 0$ and let*

$$h(z) = 1 + z + \frac{z}{1+\gamma+z}, \quad z \in U. \tag{2}$$

If $f \in A$ and

$$\frac{zf'(z)}{f(z)} \prec h(z), \tag{3}$$

then

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| < 1, \tag{4}$$

where F is given by (1).

Proof. From (1) we deduce

$$\gamma F(z) + zF'(z) = (\gamma + 1)f(z). \tag{5}$$

If we let $p(z) = \dfrac{zF'(z)}{F(z)}$, then (5) becomes

$$\gamma F(z) + p(z)F(z) = (\gamma + 1)f(z),$$

and we obtain

$$\frac{zp'(z)}{p(z)+\gamma} + p(z) = \frac{zf'(z)}{f(z)}.$$

Then (3) becomes

$$p(z) + \frac{zp'(z)}{p(z)+\gamma} \prec h(z).$$

To show that F satisfies (4), we will use Lemma A.

If we let

$$q(z) = 1 + z,$$

$$\theta(w) = w,$$

$$\theta[q(z)] = 1 + z,$$

$$\phi[q(z)] = \frac{1}{q(z)+\gamma} = \frac{1}{1+\gamma+z}$$

then we have

$$Q(z) = zq'(z)\phi[q(z)] = \frac{z}{1+\gamma+z},$$

$$h(z) = \theta[q(z)] + Q(z) = 1 + z + \frac{z}{1+\gamma+z}.$$

Since Q is starlike (even convex) and Re $\phi(q(z)) > 0$, by Lemma A we deduce $p \prec q$, which shows that F satisfies

$$\left|\frac{zF'(z)}{F(z)} - 1\right| < 1.$$

References

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