# Polyhedra Inscribed and Circumscribed to Convex Bodies

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**Abstract**

In this talk we give a survey of older results and some new results about the following question: what type of polyhedra can be inscribed or circumscribed to convex bodies in $\mathbb{R}^n$.

## 1 Introduction

We begin with an unsolved

**Problem** Let $C \subset \mathbb{R}^2$ be a Jordan curve (a closed curve without self-intersections, i.e., a homeomorphic image of the unit circle $S^1$). Then does $C$ contain all four vertices of some square?

We will formulate the above question also as follows: does $C$ have an inscribed square. Let us observe, that a square in the plane is determined by four parameters, and we have to satisfy four conditions (namely that the four vertices should lie on $C$), therefore it makes sense to expect a positive answer to the question.

**Definition.** Let $S \subset \mathbb{R}^n$ be a surface, or $K \subset \mathbb{R}^n$ a convex body (i.e., a compact, convex set with non-empty interior). We say that $P$ is an *inscribed polyhedron of $S$, or of $K$,* if each vertex of $P$ lies on $S$, or on the boundary $\partial K$ of $K$, respectively.

Observe that, if $S$ bounds some domain, we do not require that $P$ should lie in that domain.

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**Definition.** Let $K \subset \mathbb{R}^n$ be a convex body. We say that $P$ is a *circumscribed polyhedron of* $K$, if $K \subset P$, and each $(n-1)$-face of $P$ has a common point with $\partial K$.

First we cite two old theorems.

**Theorem.** *([Em 13], [Em 15], [Zi], [Ch]) Let $K \subset \mathbb{R}^2$ be a convex body. Then $K$ admits an inscribed square.*

**Theorem.** *([Ka], [YY]) Let $K \subset \mathbb{R}^n$ be a convex body. Then $K$ admits a circumscribed cube.*

The proof of the first of these theorems is essentially geometrical. The continuity arguments it uses are trivial: if a continuous function on an interval assumes both positive and negative values, then it assumes 0 as well. On the other hand, the second of this theorems is proven in essentially a topological way. Namely for any $n$-frame in $\mathbb{R}^n$ at the origin there exists a rectangular parallelepiped, circumscribed to $K$, and having edges parallel to the vectors of the $n$-frame. Then with a suitable rotation of the $n$-frame we have to achieve that the edge-lengths of this rectangular parallelepiped become equal. These edge-lengths are the widths of the thinnest parallel strips containing the convex body, bounded by hyperplanes with normals the vectors of the $n$-frame. The proof of this theorem then goes in the following way: any continuous function $S^{n-1} \to \mathbb{R}$ assumes equal values for the vectors of a suitably rotated copy of our $n$-frame.

In the following we first give planar results, then higher dimensional ones, and afterwards we turn to an analogous covering problem. Finally we say some words about the techniques used in the proofs of the new results.

## 2 The planar case

**Theorem.** *Let $C \subset \mathbb{R}^2$ be a Jordan curve. Then we have the following statements:*

(1) *([Me 80], [KK]) $C$ admits an inscribed (positively) similar copy of any triangle.*
(2) *([Me 84], attributed to H. E. Vaughan) $C$ admits an inscribed rectangle.*
(3) *([Mi], [Ni]) $C$ admits an inscribed rhomb, even with one side of any prescribed direction.*
(4) *([Ni]) $C$ admits an inscribed parallelogram, with one side of any prescribed direction, and with any prescribed ratio of the length of this side to the length of the other side.*

Under stronger hypotheses one can assert more.

**Theorem.** *Let $C \subset \mathbb{R}^2$ be a Jordan curve, which is a $C^1$-submanifold of $\mathbb{R}^2$. Then we have the following statements.*

(1) *([Šn], [St]) $C$ admits an inscribed square.*
(2) *([Gri]) $C$ admits an inscribed rectangle, with any prescribed ratio of the lengths of the sides.*

In the converse direction, we may ask what can be said about a polygon, if it admits inscribed similar copies to any plane convex body.

**Theorem.** *([Me 81]) Let $P \subset \mathbb{R}^2$ be a convex polygon, such that each plane convex body $K \subset \mathbb{R}^2$ admits an inscribed similar copy of $P$. Then $P$ is a triangle, or a quadrangle having a circumcircle.*

By (1) of the first theorem in §2, all triangles admit inscribed similar copies to any plane convex body. However, by [KW], Exercise 11.5, not all quadrangles, having circumcircles, admit inscribed similar copies to each plane convex body (while squares do admit, in virtue of the first theorem of §1). Therefore they raised the following

**Question.** ([KW], Problem 11.1) Let $P \subset \mathbb{R}^2$ be a convex quadrangle, having a circumcircle, such that $P$ is not a square. Then does there exist a plane convex body $K \subset \mathbb{R}^2$, such that $K$ does not admit an inscribed similar copy of $P$?

To this question answers our first theorem, in the negative:

**Theorem 1.** *Let $K \subset \mathbb{R}^2$ be a plane convex body. Then $K$ admits an inscribed rectangle, with any prescribed ratio of the side lengths.*

## 3 The case of $\mathbb{R}^3$ and $\mathbb{R}^4$

**Theorem.** *([BR]) Let $K \subset \mathbb{R}^3$ be a convex body. Then $K$ admits an inscribed parallelepiped.*

**Theorem.** *([BGKvL]) Let $K \subset \mathbb{R}^4$ be a centrally symmetric convex body. Then $K$ admits an inscribed parallelepiped.*

Now there follows a negative result.

**Theorem.** *([Bi]) There exists a convex body $K \subset \mathbb{R}^3$, that admits no inscribed rectangular parallelepiped (in particular, no inscribed cube).*

This theorem is the more interesting, because a rectangular parallelepiped in $\mathbb{R}^3$ is determined by nine parameters, while only eight conditions are to be satisfied (namely that all vertices should lie on $\partial K$).

However, in the centrally symmetric case we have a positive statement for rectangular parallelepipeds, thereby supplementing the above given theorems. This is given in our next

**Theorem 2.** *Let $K \subset \mathbb{R}^3$ be a centrally symmetric convex body. Then $K$ admits an inscribed cube, and even an inscribed similar copy of any prescribed rectangular parallelepiped.*

## 4 The case of $\mathbb{R}^n$

First we will consider the $n$-dimensional analogue of the triangle, i.e., the simplex.

**Theorem.** *([Ma 92]) Let $K \subset \mathbb{R}^n$ be a convex body. Then $K$ admits an inscribed similar copy of any prescribed simplex.*

Under stronger hypotheses one can assert more.

**Theorem.** *([Gro], [KN 73], [KN 75], [Ma 87]) Let $K \subset \mathbb{R}^n$ be a convex body, with $\partial K$ a $C^1$-submanifold of $\mathbb{R}^n$, further let $\Sigma \subset \mathbb{R}^n$ be a simplex. Then there exist $\lambda > 0$ and $x \in \mathbb{R}^n$ such that the homothetic copy $\lambda\Sigma + x$ of $\Sigma$ is inscribed to $K$.*

Now we turn to an $n$-dimensional analogue of the parallelogram, namely to the parallelepiped. Since parallelepipeds in $\mathbb{R}^n$ are determined by $n^2 + n$ parameters, while have $2^n$ vertices, one cannot expect that a generic convex body admits an inscribed parallelepiped (namely then $2^n$ conditions ought to be satisfied). This led [BGKvL] and [KW] to pose the following

**Question.** ([BGKvL], [KW] Problem 11.7) Does there exist an $n_0$ such that for all $n \geq n_0$ there exists a (centrally symmetric) convex body $K \subset \mathbb{R}^n$ admitting no inscribed parallelepiped? Is it even true, that for $n \geq n_0$ "most" (centrally symmetric) convex bodies in $\mathbb{R}^n$ do not admit an inscribed parallelepiped?

Now we define what is meant by "most" convex bodies. Before this we recall that the set of all convex bodies in $\mathbb{R}^n$, endowed with the Hausdorff metric (Blaschke metric) is a locally compact metric space $\mathcal{C}^n$, thus satisfies the Baire category theorem (i.e., in $\mathcal{C}^n$ a countable union of nowhere dense sets cannot have an interior point). Then one can consider as "large" sets in $\mathcal{C}^n$ the residual sets, i.e., those whose complement is a countable union of nowhere dense sets.

**Definition.** We say that a property is enjoyed by *"most" convex bodies in $\mathbb{R}^n$*, if the set of convex bodies in $\mathbb{R}^n$ having that property is a residual set in $\mathcal{C}^n$.

Our next theorem answers this question in the expected, positive way:

**Theorem 3.** *There exists an $n_0$, such that for any $n \geq n_0$ "most" (centrally symmetric) convex bodies in $\mathbb{R}^n$ do not admit inscribed parallelepipeds. Actually, $n_0 = 5$ ($n_0 = 7$ in the centrally symmetric case) will do.*

Here the concept of "most" centrally symmetric convex bodies is defined in analogy with the case of all convex bodies. Still we note that in both cases the given value of $n_0$ is the smallest dimension, such that the number of conditions to be satisfied is larger than the number of parameters (so here the heuristic works).

## 5 Universal covers

The concept "dual" to inscribed polyhedra is the concept of circumscribed polyhedra. Here we will depart from this concept a bit further, and will consider polyhedra containing some sets.

**Definition.** We say that a set $C \subset \mathbb{R}^n$ is a *universal cover in $\mathbb{R}^n$* if, for each set $X \subset \mathbb{R}^n$, of diameter at most 1, there exists a positively congruent copy $C'$ of $C$, such that $X \subset C'$.

Of course, a unit ball is a universal cover in $\mathbb{R}^n$. However, one can assert more:

**Theorem.** *([Ju 01], [Ju 10]) The circumball of a regular simplex in $\mathbb{R}^n$, of unit edges (that has radius $\sqrt{n/(2n+2)}$), is a universal cover in $\mathbb{R}^n$ (and is the smallest ball that is a universal cover in $\mathbb{R}^n$).*

Another universal cover is given by the following

**Theorem.** *([Pá]) A regular hexagon in $\mathbb{R}^2$, with distance of opposite sides being equal to* 1*, is a universal cover in $\mathbb{R}^2$.*

Since this regular hexagon is inscribed to the circle given in the previous theorem for $n = 2$, this theorem is a sharpening of the planar case of the previous theorem.

A generalization of the second theorem to $\mathbb{R}^n$ has been conjectured by [Ma 94]. To state this conjecture, we have to introduce some notations. Let $\Sigma^n \subset \mathbb{R}^n$ be a regular simplex of unit edges. Let its vertices be denoted by $v_1, ..., v_{n+1}$. Let $S_{ij}$ $(1 \leq i < j \leq n+1)$ be the parallel strip of width 1, bounded by the $(n-1)$-planes orthogonal to the edge $[v_i, v_j]$ and passing through $v_i$ and $v_j$, respectively. Finally, let us put $C_n = \bigcap_{1 \leq i < j \leq n+1} S_{ij}$. Clearly, for $n = 2$, $C_2$ is the hexagon from the last theorem.

**Conjecture.** ([Ma 94]) The set $C_n \subset \mathbb{R}^n$ is a universal cover in $\mathbb{R}^n$.

Our next theorem answers this conjecture for $\mathbb{R}^3$ in the positive way:

**Theorem 4.** *The set $C_3 \subset \mathbb{R}^3$, i.e., a rhombic dodecahedron, with distance of the opposite faces being equal to* 1*, is a universal cover in $\mathbb{R}^3$.*

As a possible application, we point out the following. The so called Borsuk problem, in $\mathbb{R}^3$, askes the following. Let $X \subset \mathbb{R}^3$ be an arbitrary set of diameter at most 1. Then is it possible to cover $X$ with four sets $X_1, ..., X_4 \subset \mathbb{R}^3$, of diameters $\mathrm{diam} X_i$ strictly less than 1? This question has been answered positively, by several authors. The sharpest published result is due to [Grü], who proved that one can attain $\mathrm{diam} X_i \leq 0.989$. For his proof he applied another universal cover, namely the regular octahedron, with distance of the opposite faces being equal to 1. Then he truncated this octahedron, still obtaining a universal cover, and decomposed this new universal cover to four sets of diameters at most 0.989. Since intuitively our rhombic dodecahedron seems to be a "smaller" set than this octahedron, it might be possible that a similar procedure applied to the rhombic dodecahedron would yield a substantially sharper estimate.

## 6 Methods of proofs of the new results

An interesting question, to which our methods did not apply, is one stated in [KW]:

**Question.** ([KW], Problem 11.6) Let $K \subset \mathbb{R}^3$ be a convex body. Then does $K$ admit an inscribed regular octahedron?

Lastly we say some words about the proofs of our theorems.

Partly we have used some theorems of [Gri], who in turn has used intersection number techniques. We elucidate his techniques on an example, of inscribing squares, to Jordan curves $C$ in $\mathbb{R}^2$, that are $C^1$-submanifolds of $\mathbb{R}^2$. One has to find four points, that satisfy two conditions: (1) they are the vertices of some square, and (2) all lie on $C$. In other words, one has to prove that two 4-submanifolds $M_1$, $M_2 \subset \mathbb{R}^8$ have a non-empty intersection, where $M_1$ is the submanifold of all ordered quadruples of points in $\mathbb{R}^2$, forming the vertices of some square (in positive orientation, say), and $M_2 = C^4$. Here $M_1$ depends on the configuration looked for, while $M_2$ depends on the surface (curve $C$) we look the configuration on.

Moreover, $M_1$ and $M_2$ are of complementary dimensions, and both $M_1$ and $M_2$ are orientable manifolds, that makes possible to define their (signed) intersection number, provided they are in generic position. For a suitably chosen "test" surface one can readily determine this intersection number, which should be non-zero, and then one still has to prove that this intersection number is invariant under deformations of the surface.

Also we have used techniques from algebraic topology. This we elucidate on the example already mentioned in §1, that of circumscribing cubes to a convex body $K \subset \mathbb{R}^3$ (although we actually use this technique for another theorem). There we have associated to a 3-frame at the origin an ordered triple of real numbers, namely the edge-lengths of the rectangular parallelepiped circumscribed to $K$, with edges parallel to the vectors of the 3-frame.

Thus we have a map from the special orthogonal group $SO(3)$ to $\mathbb{R}^3$. We have to show that its image intersects the diagonal of $\mathbb{R}^3$. Of course, this is not true for an arbitrary continuous map $SO(3) \to \mathbb{R}^3$. However, we have a rather large group acting on the set of our 3-frames, and correspondingly we can define an action of the same group on $\mathbb{R}^3$, in such a way that our map becomes an equivariant map $SO(3) \to \mathbb{R}^3$.

Then the question of nonexistence of equivariant maps with image avoiding the diagonal can be reduced to the question that a certain vector bundle does not have a nowhere 0 section, which question in turn can be settled by calculating top Stiefel-Whitney classes.

Lastly, with non-inscribability of parallelotopes to generic convex bodies in $\mathbb{R}^n$, for $n$ sufficiently large, we have proceeded as follows. Simple calculations by polynomials show that already a generic cubic surface (a set of the form $p^{-1}(0)$ where $p : \mathbb{R}^n \to \mathbb{R}$ is a generic polynomial of third degree, that of course depends only on finitely many parameters) does not admit an inscribed parallelepiped. More exactly, the set of cubic surfaces, admitting an inscribed parallelepiped, is a set of "smaller dimension" than the set of all cubic surfaces. (Therefore was it necessary to bound the degree, since for all convex bodies, both dimensions are infinite.) Then, choosing a generic cubic polynomial $p$ sufficiently close to $\sum_{i=1}^n x_i^2 - 1$, we find a convex body $K$ close to the unit ball, with $\partial K \subset p^{-1}(0)$, hence $K$ does not admit an inscribed parallelepiped.

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