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Some results concerning cyclical contractive mappings [1]

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Abstract

The purpose of this paper is to estabilish metrical fixed point theorems for some contractive orbital mappings involving a cyclical condition. Our results extend most of the fundamental metrical fixed point theorems in literature (Chatterjea, Bianchini, Reich, Hardy-Rogers, Ćirić).Examples of fixed point structures follows.

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1 Introduction

Let $p > 1, p \in \mathbb{N}$. We consider T a selfmap of a metric space (X, d) and $\{A_i\}_{i=1}^{p}$ nonempty closed subsets of X. For any given $x \in X$, we define $T^n x$

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inductively by $T^0x = x$ and $T^{n+1}x = T(T^nx)$. The mapping T is said to be *cyclical* if

$$(1) \qquad T(A_i) \subset A_{i+1}, \forall i \in \{1, 2, \ldots, p\} (where A_{p+1} = A_1).$$

If there exist a constant $k \in (0, 1)$ such that

$$(2) \qquad d(Tx, T^2x) \leq kd(x, Tx),$$

for $\forall x \in X$ than we say that T satisfies an *orbital condition.*

It is well known and easy to prove that if X is a complete metric space and $T : X \to X$ is continuos and satisfies (2), $\forall x \in X$ then T has a fixed point in X.

We first prove the following basic result for the existence of a fixed point under cyclical considerations.

Lemma 1 *If* $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ *satisfies (1) and (2), for* $\forall x \in \bigcup_{i=1}^{p} A_i$*, where* $k \in (0, 1)$*, then* $\bigcap_{i=1}^{p} A_i \neq \emptyset$.

Proof. Let $x \in \bigcup_{i=1}^{p} A_i$. We put in (2) $Tx := T^2x$ and $T^2x := T^3x$. Then we have

$$d(T^2x, T^3x) \leq kd(Tx, T^2x) \leq k^2 d(x, Tx)$$

and by induction

$$d(T^nx, T^{n+1}x) \leq k^n d(x, Tx)$$

for $n = 0, 1, 2, \ldots$. Thus for any numbers $n, m \in \mathbb{N}, m > 0$ we have

$$d(T^nx, T^{n+m}x) \leq \sum_{j=n}^{n+m-1} d(T^jx, T^{j+1}x) \leq \sum_{j=n}^{n+m-1} k^j d(x, Tx) \leq \frac{k^n}{1-k} d(x, Tx).$$

Since $0 < k < 1$, it results that $k^n \to 0$ (as $n \to \infty$), which together with the above inequality shows that $\{T^nx\}$ is a Cauchy sequence. But (X, d) is a complete metric space, therefore $\{T^n x\}$ converge to some $x^* \in X$. However in view of (1) an infinite number of terms of the sequence $\{T^n x\}$ lie in each $A_i, i \in \{1, 2, \ldots, p\}$. Therefore $x^* \in \bigcap_{i=1}^{p} A_i$, so $\bigcap_{i=1}^{p} A_i \neq \emptyset$. The present paper is motivated by a paper of W.A. Kirk, P.S. Srinivasan and P. Veeramani [7]. These authors consider the folowing result.

Theorem 2 [7] *Let* $\{A_i\}_{i=1}^{p}$ *be nonempty closed subsets of a complete metric space, and suppose* $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ *satisfies the conditions (1) and there exist a constant* $a \in (0, 1)$ *such that*

(3) $$d(Tx, Ty) \leq a d(x, y), \forall x \in A_i, y \in A_{i+1}, 1 \leq i \leq p.$$

Then T *has a unique fixed point in* $\bigcap_{i=1}^{p} A_i$.

An interesting feature about the above result is that continuity of T is no longer needed. The objective of this note is to extend the above reasoning to more general classes of mappings which do not imply the continuity.

In 1972, Zamfirescu obtained a very interesting fixed point theorem which gather together three contractive conditions i.e., conditions of Banach, of Kannan and of Chatterjea, in a rather unexpected way. Note that all the contractive conditions presented in this paper are obtaind from this three ones, as was shown by Rhoades[9], [10], and Berinde [1]. The fixed point theorem for the Zamfirescu's operator involving cyclical condition is more general that the ones presented in this paper. This theorem was obtain by the author in [8].

2 Main results

In 1968 R.Kannan [6] proved a fixed point theorem which extends the well-known Banach's contraction principle that need not to be continuous(but are continuous at their fixed point), by considering the next contractive condition: there exist a constant $a \in \left[0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] \tag{4}$$

for all $x, y \in X$, where (X, d) is a metric space and $T : X \to X$ a mapping. The cyclical extension for the Kannan's theorem was obtained by I.A. Rus in [12] using fixed point structure arguments.

Theorem 1 [12] *Let (X, d) be a complete metric space, $A_1, A_2, \ldots, A_p$ be nonempy closed subsets of X and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$. We suppose that T fulfills (1) and (2) for $x \in A_i, y \in A_{i+1}, i \in \{1, 2, \ldots, p\}$ with $a \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point in $\bigcap_{i=1}^{p} A_i$.*

We proved in [8] this theorem ussing arguments similar to those used in this paper.

Following the Kannan's theorem a lot of papers were devoted to obtaining fixed point theorems for various clases of contractive type conditions that do not require the continuity of T.

One of them, actually a sort of dual of Kannan's theorem, was proved by Chatterjea in [3].

Theorem 2 [3] *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping for which there exist a real number $0 \leq a < \frac{1}{2}$ such that for each $x, y \in X$ we have*

$$d(Tx, Ty) \leq a[d(x, Ty) + d(y, Tx)] \tag{5}$$

Then T has a unique fixed point in X.

Showing that a Chatterjea type operator T defined by (5) satisfies an orbital condition and ussing Lemma 1 we extend this result as follows.

Theorem 3 *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space, and suppose $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ satisfies the conditions (1) and there exist a constant $a \in (0, 0.5)$ such that (5) is satified for each $x \in A_i, y \in A_{i+1}, i \in \{1, 2, \dots p\}$. Then T has a unique fixed point in $\bigcap_{i=1}^{p} A_i$.*

Proof. Let $i \in \{1, 2, \dots, p\}$ arbitrary and $x \in A_i$. Then $Tx \in A_{i+1}$. Thus we can take in (5) $y := Tx$, and ussing the triangle inequality we have

$$d(Tx, T^2x) \le a[d(x, T^2x) + d(Tx, Tx)] \le a[d(x, Tx) + d(Tx, T^2x)]$$

and therefore

$$d(Tx, T^2x) \le \frac{a}{1-a} d(x, Tx).$$

We denote $k := \frac{a}{1-a}$ and since $a \in (0, 0.5)$ we have that $k \in (0, 1)$. Now we can apply Lemma 1 to get that $\bigcap_{i=1}^{p} A_i \neq \emptyset$. Then applying Theorem 2 to the restrition of operator T to $\bigcap_{i=1}^{p} A_i$ we obtain that T has a unique fixed point in $\bigcap_{i=1}^{p} A_i$.

In 1972 R.M.T. Bianchini gives the following theorem as a generalization of Kannan's fixed point theorem.

Theorem 4 [2] *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping for which there exist a real number $0 \le h < 1$ such that for each $x, y \in X$ we have*

$$d(Tx, Ty) \le h \max\{d(x, Tx), d(y, Ty)\} \tag{6}$$

The T has a unique fixed point in X.

It is possible to extend the above fixed point theorem like in the previous case, by imposing an additional cyclical condition, as showes the next theorem.

Theorem 5 *Let* $\{A_i\}_{i=1}^{p}$ *be nonempty closed subsets of a complete metric space, and suppose* $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ *satisfies the conditions (1) and there exist a constant* $h \in (0,1)$ *such that (6) is satified for each* $x \in A_i, y \in A_{i+1}, i \in \{1, 2, \dots p\}$*.Then* T *has a unique fixed point in* $\bigcap_{i=1}^{p} A_i$*.*

Proof. Let $x \in \bigcup_{i=1}^{p} A_i$. Then by (6) with $y := Tx$ we get

$$d(Tx, T^2x) \leq h \max\{d(x, Tx), d(Tx, T^2x)\}.$$

If $max\{d(x, Tx), d(Tx, T^2x)\} = d(Tx, T^2x)$ then from (6) we have that $(1 - h)d(Tx, T^2x) \leq 0$, so $1 - h \leq 0 \Rightarrow h \geq 1$, a contradiction. Thus $max\{d(x, Tx), d(Tx, T^2x)\} = d(x, Tx)$ and hence (6) becomes

$$d(Tx, T^2x) \leq hd(x, Tx).$$

Now by Lemma 1 we find that $\bigcap_{i=1}^{p} A_i \neq \emptyset$. Then applying Theorem 4 to the restrition of operator T to $\bigcap_{i=1}^{p} A_i$ we obtain that T has a unique fixed point in $\bigcap_{i=1}^{p} A_i$.

We state now a result due to S. Reich.

Theorem 6 [11] *Let* X *be a complete metric space with metric* d*, and let* $T : X \to X$ *be a function with the following property*

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, y) \tag{7}$$

for each $x, y \in X$*, where* a, b, c *are nonnegative and satisfy* $a + b + c < 1$*. The* T *has a unique fixed point.*

Note that $a = b = 0$ yields Banach's fixed point theorem, while $a = b, c = 0$ yields Kannan's fixed point theorem. Of course, we may assume always that $a = b$, but this is not essential.

The same idea enables us to extend Theorem 6.

Theorem 7 *Let $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space, and suppose $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ satisfies the conditions (1) and there exist three constants a, b, c with $a + b + c < 1$ such that (7) is satified for each $x \in A_i, y \in A_{i+1}, i \in \{1, 2, \dots p\}$. Then T has a unique fixed point in $\bigcap_{i=1}^{p} A_i$.*

Proof. Let $x \in \bigcup_{i=1}^{p} A_i$. We take in (7) $y := Tx$. It follows that

$$d(Tx, T^2x) \le ad(x, Tx) + bd(Tx, T^2x) + cd(x, Tx)$$

and therefore

$$d(Tx, T^2x) \le \frac{a + c}{1 - b} d(x, Tx).$$

We denote $k := \frac{a+c}{1-b}$ and since $a + b + c < 1$ we have that $k \in (0, 1)$. Moreover from the above inequality we get

$$d(Tx, T^2x) \le kd(x, Tx).$$

Now we can apply Lemma 1 to get that $\bigcap_{i=1}^{p} A_i \neq \emptyset$. Then applying Theorem 6 to the restrition of operator T to $\bigcap_{i=1}^{p} A_i$ we obtain that T has a unique fixed point in $\bigcap_{i=1}^{p} A_i$.

Later, in 1973, G.E.Hardy and T.D. Rogers, have obtained a similar conclusion to Reich. The basic result of the paper [5] is

Theorem 8 [5] *Let (X, d) be a complete metric space and T a self-mapping of X satisfying the condition for $x, y \in X$*

(8) $$d(Tx, Ty) \le ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y)$$

where a, b, c, e, f are nonnegative and we set $\alpha = a + b + c + e + f$. If $\alpha < 1$ then T has a unique fixed point.

Reich's result has a similar conclusion to that in (8) in the case that $\alpha = a + b + f$. Also his result generalizes the fixed point Theorem of Kannan in which $\alpha = a + b$. We proceed to obtain an extension of this theorem.

Theorem 9 *Let (X, d) be a complete metric space and $\{A_i\}_{i=1}^{p}$ be nonempty closed susets of X. Let $T : X \to X$ an operator. We suppose that T satisfies (1) and (8) for all $x \in A_i, y \in A_{i+1}, i \in \{1, 2, \dots, p\}$ with $\alpha < 1$. Then T has a unique fixed point in $\bigcap\limits_{i=1}^{p} A_i$.*

Proof. Let $x \in \bigcup\limits_{i=1}^{p} A_i$. Set $y := Tx$ in (8) and simplify to obtain

$$d(Tx, T^2x) \leq \frac{a + f}{1 - b} d(x, Tx) + \frac{c}{1 - b} d(x, T^2x).$$

Now by the triangle inequality, $d(x, T^2x) \leq d(x, Tx) + d(Tx, T^2x)$, so from the above inequation we get

$$d(Tx, T^2x) \leq \frac{a + c + f}{1 - b - c} d(x, Tx),$$

and by symetry, we may exchange a with b and c with e in the above inequation to obtain

$$d(Tx, T^2x) \leq \frac{b + e + f}{1 - a - e} d(x, Tx).$$

Then $k := \min\left\{\frac{a+c+f}{1-b-c}, \frac{b+e+f}{1-a-e}\right\}$. By the Lemma 1 we have $\bigcap\limits_{i=1}^{p} A_i \neq \emptyset$. Following the Theorem 8 applicable for the restrition of operator T to $\bigcap\limits_{i=1}^{p} A_i$ we can conclude that T has a unique fixed point in $\bigcap\limits_{i=1}^{p} A_i$.

The purpose of the paper [4] was to define and investigate a class of generalized contractions which includes Banach's contraction, Kannan's condition and Chatterjea's condition (5).

Let X be a metric space and $T : X \to X$. We recall that X is said to be T-orbitally complete if every sequence $\{T^{n_i}x : i \in \mathbb{N}\}, x \in X$, which is a Cauchy sequence, has a limit point in X. It is obsious that if X is a complete space, then T is T-orbitaly complete for any mapping $T : X \to X$. The result proved by Ćirić is the following.

Theorem 10 [4] *Let T be a mapping of T-orbitally complete metric space X into itself. If for every $x, y \in X$ there exist nonnegative numbers q, r, s and t which may depend on both x and y, such that*

$$\sup\{q + r + s + 2t : x, y \in X\} < 1$$

and

$$d(Tx, Ty) \leq qd(x, y) + rd(x, Tx) + sd(y, Ty) + t[d(x, Ty) + d(y, tx)] \tag{9}$$

then T has a unique fixed point in X.

Now, in the same manner, we are in position to prove the next extension of the above result.

Theorem 11 *Let (X, d) be a complete metric space and $\{A_i\}_{i=1}^{p}$ be nonempty closed susets of X. Let $T : X \to X$ an operator. We suppose that T satisfies (1) and for every $x \in A_i, y \in A_{i+1}, i \in \{1, 2, \ldots, p\}$there exist nonnegative numbers q, r, s and t which may depend on both x and y, such that*

$$\sup\{q + r + s + 2t : x, y \in X\} := \lambda < 1$$

and (9) hold. Then T has a unique fixed point in $\bigcap\limits_{i=1}^{p} A_i$.

Proof. Let $x \in \bigcup\limits_{i=1}^{p} A_i$. Letting $y := Tx$ in (9) we obtain

$$d(Tx, T^2x) \le (q+r)d(x, Tx) + sd(Tx, T^2x) + td(x, T^2x).$$

Using the triangle inequality, is a simple task to drive from the abore inequation to the following one

$$d(Tx, T^2x) \le \frac{q+r+t}{1-s-t} d(x, Tx).$$

In view of the fact that $\lambda < 1$ we have

$$q + r + t + \lambda s + \lambda t < \lambda$$

so

$$\frac{q+r+t}{1-s-t} \le 1$$

is true for each $x \in A_i, i \in \{1, 2, \ldots, p\}$. Let us denote $k := \frac{q+r+t}{1-s-t}$. Therefore

$$d(Tx, T^2x) \le kd(x, Tx)$$

which means that T is an orbital mapping. By Lemma 1 we have that $\bigcap\limits_{i=1}^{p} A_i \neq \emptyset$. The rest of proof follows by Theorem 10 applied to the restriction of T to $\bigcap\limits_{i=1}^{p} A_i$.

3 Fixed point structures

Let X be a nonempty set and $A \subset X$ nonempty subsets of X. We set $\mathbb{M}(A) := \{T | T : A \to A\}$ and $P(X) := \{A \subset X | A \neq \emptyset\}$.

Definition 1 [12] *A triplet $(X, S(X), M)$ is a fixed point structure (f.p.s) on X if*

1. $S(X) \subset P(X)$, $S(X) \neq \emptyset$;

2. $M : P(X) \multimap \bigcup_{A \in P(X)} (A), A \multimap M(A) \subset (M)$ is a setvalued operator such that if $B \subset A, B \neq \emptyset$, then

$$M(B) \supset \{T|_B | T \in M(A) and T(B) \subset B\};$$

3. every $A \in S(X)$ has the fixed point property with respect to $M(A)$.

It is clear that for any fixed point theorem we have an example of f.p.s. Hence we are entitled to give the following examples.

Example 1 [12] Let (X, d) a complete metric space, $S(X)$ be the set of all nonempty closed subsets of X and $M(Y) := \{T : Y \to Y \; | T satisfies (3)\}$. Then $(X, S(X), M)$ is a f.p.s.

Example 2 Let (X, d) a complete metric space, $S(X)$ be the set of all nonempty closed subsets of X and $M(Y) := \{T : Y \to Y \; | T satisfies (4)\}$. Then $(X, S(X), M)$ is a f.p.s.

Example 3 Let (X, d) a complete metric space, $S(X)$ be the set of all nonempty closed subsets of X and $M(Y) := \{T : Y \to Y \; | T satisfies (5)\}$. Then $(X, S(X), M)$ is a f.p.s.

Example 4 Let (X, d) a complete metric space, $S(X)$ be the set of all nonempty closed subsets of X and $M(Y) := \{T : Y \to Y \; | T satisfies (6)\}$. Then $(X, S(X), M)$ is a f.p.s.

Example 5 Let (X, d) a complete metric space, $S(X)$ be the set of all nonempty closed subsets of X and $M(Y) := \{T : Y \to Y \; | T satisfies (7)\}$. Then $(X, S(X), M)$ is a f.p.s.

Example 6 Let (X, d) a complete metric space, $S(X)$ be the set of all nonempty closed subsets of X and $M(Y) := \{T : Y \to Y \; | T satisfies (8)\}$. Then $(X, S(X), M)$ is a f.p.s.

Example 7 Let (X, d) a complete metric space, $S(X)$ be the set of all nonempty closed subsets of X and $M(Y) := \{T : Y \to Y \; | T satisfies (9)\}$. Then $(X, S(X), M)$ is a f.p.s.

We remark that using and the definition of a fixed point structure from the above examples we also can prove the new results from this paper and therefore those theorems can be considered as applications of the fixed point structure theory.

Consequently we have:

Example 8 If we take in Lemma 1 the f.p.s. in Example 1 we obtain the Theorem 2.

Example 9 If we take in Lemma 1 the f.p.s. in Example 2 we obtain the Theorem 1.

Example 10 If we take in Lemma 1 the f.p.s. in Example 3 we obtain the Theorem 3.

Example 11 If we take in Lemma 1 the f.p.s. in Example 4 we obtain the Theorem 5.

Example 12 If we take in Lemma 1 the f.p.s. in Example 5 we obtain the Theorem 7.

Example 13 If we take in Lemma 1 the f.p.s. in Example 6 we obtain the Theorem 9.

Example 14 If we take in Lemma 1 the f.p.s. in Example 7 we obtain the Theorem 11.

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