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Hyperstructures associated to $\mathcal{E}$-lattices[1]

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Abstract

The goal of this paper is to present some basic properties of $\mathcal{E}$-lattices and their connections with hyperstructure theory.

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1 Introduction

The starting point for our discussion is given by the paper [8], where there is introduced the category of $\mathcal{E}$–lattices and there are made some elementary constructions in this category. Given a nonvoid set L and a map $\varepsilon : L \to L$,

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we denote by $\mathrm{Ker}\,\varepsilon$ the kernel of ε (i.e. $\mathrm{Ker}\,\varepsilon = \{(a,b) \in L \times L \mid \varepsilon(a) = \varepsilon(b)\}$), by $\mathrm{Im}\,\varepsilon$ the image of ε (i.e. $\mathrm{Im}\,\varepsilon = \{\varepsilon(a) \mid a \in L\}$) and by $\mathrm{Fix}\,\varepsilon$ the set consisting of all fixed points of ε (i.e. $\mathrm{Fix}\,\varepsilon = \{a \in L \mid \varepsilon(a) = a\}$). We say that L is an $\mathcal{E}$*–lattice* (relative to ε) if there exist two binary operations $\wedge_\varepsilon, \vee_\varepsilon$ on L which satisfy the following properties:

a) $a \wedge_\varepsilon (b \wedge_\varepsilon c) = (a \wedge_\varepsilon b) \wedge_\varepsilon c,\ a \vee_\varepsilon (b \vee_\varepsilon c) = (a \vee_\varepsilon b) \vee_\varepsilon c$, for all $a, b, c \in L$;

b) $a \wedge_\varepsilon b = b \wedge_\varepsilon a,\ a \vee_\varepsilon b = b \vee_\varepsilon a$, for all $a, b \in L$;

c) $a \wedge_\varepsilon a = a \vee_\varepsilon a = \varepsilon(a)$, for any $a \in L$;

d) $a \wedge_\varepsilon (a \vee_\varepsilon b) = a \vee_\varepsilon (a \wedge_\varepsilon b) = \varepsilon(a)$, for all $a, b \in L$.

Clearly, in an $\mathcal{E}$–lattice L (relative to ε) the map ε is idempotent and $\mathrm{Im}\,\varepsilon = \mathrm{Fix}\,\varepsilon$. Moreover, for any $a, b \in L$, we have:

$$a \wedge_\varepsilon \varepsilon(a) = a \vee_\varepsilon \varepsilon(a) = \varepsilon(a),$$
$$a \wedge_\varepsilon \varepsilon(b) = \varepsilon(a) \wedge_\varepsilon b = \varepsilon(a) \wedge_\varepsilon \varepsilon(b) = \varepsilon(a \wedge_\varepsilon b),$$
$$a \vee_\varepsilon \varepsilon(b) = \varepsilon(a) \vee_\varepsilon b = \varepsilon(a) \vee_\varepsilon \varepsilon(b) = \varepsilon(a \vee_\varepsilon b).$$

Also, note that the set $\mathrm{Fix}\,\varepsilon$ is closed under the binary operations $\wedge_\varepsilon, \vee_\varepsilon$ and, denoting by $\wedge, \vee$ the restrictions of $\wedge_\varepsilon, \vee_\varepsilon$ to $\mathrm{Fix}\,\varepsilon$, we have that $(\mathrm{Fix}\,\varepsilon, \wedge, \vee)$ is a lattice. The connection between the $\mathcal{E}$–lattice concept and the lattice concept is very powerful. So, if $(L, \wedge_\varepsilon, \vee_\varepsilon)$ is an $\mathcal{E}$–lattice and $\sim$ is an equivalence relation on L such that $\sim\ \subseteq \mathrm{Ker}\,\varepsilon$, then the factor set $L/\sim$ is a lattice isomorphic to the lattice $\mathrm{Fix}\,\varepsilon$. Conversely, if L is a nonvoid set and $\sim$ is an equivalence relation on L having the property that the factor set

$L/\sim$ is a lattice, then the set L can be endowed with a $\mathcal{E}$–lattice structure (relative to a map $\varepsilon : L \to L$) such that $\sim\ \subseteq$ Ker ε and $L/\!\sim\ \cong \mathrm{Fix}\,\varepsilon$.

If $(L, \wedge_\varepsilon, \vee_\varepsilon)$ is an $\mathcal{E}$–lattice and for every $x \in L$ we denote by $[x]$ the equivalence class of x modulo $\mathrm{Ker}\,\varepsilon$ (i.e. $[x] = \{y \in L \mid \varepsilon(x) = \varepsilon(y)\}$), then we have $a \wedge_\varepsilon b \in [\varepsilon(a) \wedge \varepsilon(b)]$ and $a \vee_\varepsilon b \in [\varepsilon(a) \vee \varepsilon(b)]$, for all $a, b \in L$. We say that L is a *canonical $\mathcal{E}$–lattice* if $a \wedge_\varepsilon b, a \vee_\varepsilon b \in \mathrm{Fix}\,\varepsilon$, for all $a, b \in L$. Three fundamental types of canonical $\mathcal{E}$–lattices have been identified, as follows:

– let $(L, \wedge, \vee)$ be a lattice, ε be an idempotent endomorphism of L and define $a \wedge_\varepsilon b = \varepsilon(a \wedge b), a \vee_\varepsilon b = \varepsilon(a \vee b)$, for every $a, b \in L$; then $(L, \wedge_\varepsilon, \vee_\varepsilon)$ is a canonical $\mathcal{E}$–lattice, called a *canonical $\mathcal{E}$–lattice of type* 1;

– let $(L, \wedge, \vee)$ be a lattice, $\varepsilon : L \to L$ be an idempotent map such that $\mathrm{Fix}\,\varepsilon$ is a sublattice of L and define $a \wedge_\varepsilon b = \varepsilon(a) \wedge (b)$, $a \vee_\varepsilon b = \varepsilon(a) \vee (b)$, for every $a, b \in L$; then $(L, \wedge_\varepsilon, \vee_\varepsilon)$ is a canonical $\mathcal{E}$–lattice, called a *canonical $\mathcal{E}$–lattice of type* 2;

– let L be a set, $\varepsilon : L \to L$ be an idempotent map such that $\mathrm{Fix}\,\varepsilon$ is a lattice (we denote by $\wedge, \vee$ its binary operations) and define $a \wedge_\varepsilon b = \varepsilon(a) \wedge \varepsilon(b)$, $a \vee_\varepsilon b = \varepsilon(a) \vee \varepsilon(b)$, for every $a, b \in L$; then $(L, \wedge_\varepsilon, \vee_\varepsilon)$ is a canonical $\mathcal{E}$–lattice, called a *canonical $\mathcal{E}$–lattice of type* 3.

The above constructions furnish us many examples of canonical $\mathcal{E}$–lattices. Mention also that any canonical $\mathcal{E}$–lattice is isomorphic to a canonical $\mathcal{E}$–lattice of type 3 (see [8], Section 2, Proposition 2).

Let $(L_1, \wedge_{\varepsilon_1}, \vee_{\varepsilon_1})$ and $(L_2, \wedge_{\varepsilon_2}, \vee_{\varepsilon_2})$ be two $\mathcal{E}$–lattices. According to [8], a map $f : L_1 \to L_2$ is called an *$\mathcal{E}$–lattice homomorphism* if:

a) $f \circ \varepsilon_1 = \varepsilon_2 \circ f$;

b) for all $a, b \in L_1$, we have:

i) $f(a \wedge_{\varepsilon_1} b) = f(a) \wedge_{\varepsilon_2} f(b)$;

ii) $f(a \vee_{\varepsilon_1} b) = f(a) \vee_{\varepsilon_2} f(b)$.

Moreover, if f is one-to-one and onto, then we say that it is an *$\mathcal{E}$–lattice isomorphism*. $\mathcal{E}$–lattice homomorphisms (respectively $\mathcal{E}$–lattice isomorphisms) of an $\mathcal{E}$–lattice into itself are called *$\mathcal{E}$–lattice endomorphisms* (respectively *$\mathcal{E}$–lattice automorphisms*). The most significant results concerning to $\mathcal{E}$–lattice homomorphisms / isomorphisms have been obtained in the particular case of subgroup $\mathcal{E}$–lattices (see [9]).

Most of our notation is standard and will usually not be repeated here. Basic definitions and results on lattices can be found in [1] and [4]. For hyperstructure theory notions we refer the reader to [3].

2 Basic properties of $\mathcal{E}$–lattices

In this section we investigate some properties of $\mathcal{E}$–lattices, as modularity, distributivity or complementation. We shall prove that they are strongly connected to the similar properties of lattices.

In order to introduce the modularity for $\mathcal{E}$–lattices, we need to extend at this situation the notion of ordering relation. Let $(L, \wedge_\varepsilon, \vee_\varepsilon)$ be an $\mathcal{E}$–lattice. A binary relation $\leq_\varepsilon$ on L is called an *$\mathcal{E}$-ordering relation* (relative to ε) if, for all $a, b \in L$, we have:

a) $\varepsilon(a) \leq_\varepsilon \varepsilon(a)$;

b) $a \leq_\varepsilon b$ and $b \leq_\varepsilon a$ imply that $a = b$;

c) $a \leq_\varepsilon b$ and $b \leq_\varepsilon c$ imply that $a \leq_\varepsilon c$.

In a natural way, on L we define the following two $\mathcal{E}$-ordering relations:

– $a \leq'_\varepsilon b$ iff $a \wedge_\varepsilon b = a$;

– $a \leq''_\varepsilon b$ iff $a \vee_\varepsilon b = b$.

These are not equivalent ($a \leq'_\varepsilon b$ implies that $a \vee_\varepsilon b = \varepsilon(b)$ and $a \leq''_\varepsilon b$ implies that $a \wedge_\varepsilon b = \varepsilon(a)$). Moreover, we have

$$a \leq'_\varepsilon b \;\text{ iff }\; a \in \operatorname{Fix}\varepsilon \;\text{ and }\; a \leq \varepsilon(b)$$

and

$$a \leq''_\varepsilon b \;\text{ iff }\; b \in \operatorname{Fix}\varepsilon \;\text{ and }\; \varepsilon(a) \leq b,$$

where $\leq$ is the ordering relation associated to the lattice $\operatorname{Fix}\varepsilon$.

Definition 1 *We say that an $\mathcal{E}$–lattice $(L, \wedge_\varepsilon, \vee_\varepsilon)$ is $\wedge_\varepsilon$-modular if $a \leq'_\varepsilon b$ implies that $a \vee_\varepsilon (b \wedge_\varepsilon c) = b \wedge_\varepsilon (a \vee_\varepsilon c)$, and $\vee_\varepsilon$-modular if $a \leq''_\varepsilon b$ implies that $a \vee_\varepsilon (b \wedge_\varepsilon c) = b \wedge_\varepsilon (a \vee_\varepsilon c)$.*

The following result shows that the above two concepts are equivalent and, moreover, the modularity of an $\mathcal{E}$–lattice $(L, \wedge_\varepsilon, \vee_\varepsilon)$ can be reduced to the modularity of the lattice $\operatorname{Fix}\varepsilon$.

Proposition 1 *For an $\mathcal{E}$–lattice $(L, \wedge_\varepsilon, \vee_\varepsilon)$, the next conditions are equivalent:*

a) *L is $\wedge_\varepsilon$-modular.*

b) *L is $\vee_\varepsilon$-modular.*

c) *The lattice* $\mathrm{Fix}\,\varepsilon$ *is modular.*

Proof. a) $\Longleftrightarrow$ c) Suppose that L is $\wedge_\varepsilon$-modular and let $a, b, c \in \mathrm{Fix}\,\varepsilon$ such that $a \le b$. Then $a \wedge_\varepsilon b = \varepsilon(a) \wedge_\varepsilon \varepsilon(b) = \varepsilon(a) \wedge \varepsilon(b) = \varepsilon(a) = a$ and so $a \le'_\varepsilon b$. It obtains that $a \vee_\varepsilon (b \wedge_\varepsilon c) = b \wedge_\varepsilon (a \vee_\varepsilon c)$, which means $a \vee (b \wedge c) = b \wedge (a \vee c)$ in the lattice $\mathrm{Fix}\,\varepsilon$.

Conversely, assume that $\mathrm{Fix}\,\varepsilon$ is modular and let a, b, c be three elements of L satisfying $a \le'_\varepsilon b$. Then $a \in \mathrm{Fix}\,\varepsilon$ and $a \le \varepsilon(b)$. This last relation implies that $a \vee (\varepsilon(b) \wedge \varepsilon(c)) = \varepsilon(b) \wedge (a \vee \varepsilon(c))$. Since a is a fixed point, the previous equality is equivalent to $a \vee_\varepsilon (b \wedge_\varepsilon c) = b \wedge_\varepsilon (a \vee_\varepsilon c)$ and hence L is $\wedge_\varepsilon$-modular.

b) $\Longleftrightarrow$ c) Similarly with a) $\Longleftrightarrow$ c).

Note that each of the following well-known conditions (which for lattices are equivalent to the modularity – see, for example, Chapter IV of [4]):

(1) $a \wedge_\varepsilon (b \vee_\varepsilon c) = a \wedge_\varepsilon \{[b \wedge_\varepsilon (a \vee_\varepsilon c)] \vee_\varepsilon c\}$, for all $a, b, c \in L$,

(2) $a \wedge_\varepsilon [b \vee_\varepsilon (a \wedge_\varepsilon c)] = (a \wedge_\varepsilon b) \vee_\varepsilon (a \wedge_\varepsilon c)$, for all $a, b, c \in L$,

(3) $x \wedge_\varepsilon a = x \wedge_\varepsilon b$, $x \vee_\varepsilon a = x \vee_\varepsilon b$ and $a \le'_\varepsilon b$ (or $a \le''_\varepsilon b$) imply that $a = b$,

(4) L does not contain five distinct elements x, a, b, c, y satisfying $a \wedge_\varepsilon c = b \wedge_\varepsilon c = x$, $a \vee_\varepsilon c = b \vee_\varepsilon c = y$ and $a \le'_\varepsilon b$ (or $a \le''_\varepsilon b$)

assures the modularity of the $\mathcal{E}$–lattice $(L, \wedge_\varepsilon, \vee_\varepsilon)$, but not conversely.

Our next aim is to study the concept of distributivity for $\mathcal{E}$–lattice.

Definition 2 *We say that an $\mathcal{E}$–lattice $(L, \wedge_\varepsilon, \vee_\varepsilon)$ is $\wedge_\varepsilon$-distributive if $a \wedge_\varepsilon (b \vee_\varepsilon c) = (a \wedge_\varepsilon b) \vee_\varepsilon (a \wedge_\varepsilon c)$, for all $a, b, c \in L$, and $\vee_\varepsilon$-distributive if $a \vee_\varepsilon (b \wedge_\varepsilon c) = (a \vee_\varepsilon b) \wedge_\varepsilon (a \vee_\varepsilon c)$, for all $a, b, c \in L$.*

The above two types of distributivity of an $\mathcal{E}$–lattice are not equivalent, as shows the following example.

Example 1 *Let L be the set consisting of all natural divisors of 72 and $\varepsilon : L \to L$ be the map defined by* $\varepsilon(1) = 1$, $\varepsilon(2) = \varepsilon(4) = \varepsilon(8) = 2$, $\varepsilon(3) = \varepsilon(9) = 3$, $\varepsilon(6) = \varepsilon(12) = \varepsilon(18) = \varepsilon(24) = \varepsilon(36) = \varepsilon(72) = 6$. *On L we introduce an $\mathcal{E}$–lattice structure, by defining two binary operations $\wedge_\varepsilon, \vee_\varepsilon$ in the next manner:*

- *if two elements a, b of L are contained in distinct classes of equivalence modulo $\operatorname{Ker}\varepsilon$, put $a \wedge_\varepsilon b = \varepsilon(a) \wedge \varepsilon(b)$ and $a \vee_\varepsilon b = \varepsilon(a) \vee \varepsilon(b)$ (note that in this case the binary operations $\wedge$ and $\vee$ on $\operatorname{Fix}\varepsilon$ are G.C.D. and L.C.M., respectively);*
- $4 \wedge_\varepsilon 8 = 4 \vee_\varepsilon 8 = 2$;
- $12 \wedge_\varepsilon 18 = 12 \wedge_\varepsilon 24 = 12 \wedge_\varepsilon 72 = 36,\ 12 \wedge_\varepsilon 36 = 6$
  $18 \wedge_\varepsilon 24 = 18 \wedge_\varepsilon 36 = 18 \wedge_\varepsilon 72 = 6$
  $24 \wedge_\varepsilon 36 = 24 \wedge_\varepsilon 72 = 6$
  $36 \wedge_\varepsilon 72 = 6$
  $12 \vee_\varepsilon 18 = 12 \vee_\varepsilon 24 = 36,\ 12 \vee_\varepsilon 36 = 12 \vee_\varepsilon 72 = 6$

$18 \vee_\varepsilon 24 = 72,\ 18 \vee_\varepsilon 36 = 18 \vee_\varepsilon 72 = 6$

$24 \vee_\varepsilon 36 = 24 \vee_\varepsilon 72 = 6$

$36 \vee_\varepsilon 72 = 6.$

By a direct calculation, it is easy to see that L *is* $\vee_\varepsilon$*-distributive. On the other hand, we have* $12 \wedge_\varepsilon (18 \vee_\varepsilon 24) \neq (12 \wedge_\varepsilon 18) \vee_\varepsilon (12 \wedge_\varepsilon 24)$ *and therefore* L *is not* $\wedge_\varepsilon$*-distributive.*

In the previous example, remark that the lattice $\mathrm{Fix}\,\varepsilon = \{1, 2, 3, 6\}$ is distributive and so the distributivity of the lattice $\mathrm{Fix}\,\varepsilon$ does not imply that of the $\mathcal{E}$–lattice L. Clearly, the converse implication holds, i.e. any $\wedge_\varepsilon$-distributive (or $\vee_\varepsilon$-distributive) $\mathcal{E}$–lattice has a distributive lattice of fixed points. Mention also that these two properties are equivalent for canonical $\mathcal{E}$–lattices and that both $\wedge_\varepsilon$-distributivity and $\vee_\varepsilon$-distributivity of an $\mathcal{E}$–lattice imply its modularity.

Into an $\mathcal{E}$–lattice $(L, \wedge_\varepsilon, \vee_\varepsilon)$, each of the following well-known conditions (which for lattices are equivalent to the distributivity – see, for example, Chapter II of [4]):

(1) $(a \wedge_\varepsilon b) \vee_\varepsilon (b \wedge_\varepsilon c) \vee_\varepsilon (c \wedge_\varepsilon a) = (a \vee_\varepsilon b) \wedge_\varepsilon (b \vee_\varepsilon c) \wedge_\varepsilon (c \vee_\varepsilon a)$, for all $a, b, c \in L$,

(2) $x \wedge_\varepsilon a = x \wedge_\varepsilon b$ and $x \vee_\varepsilon a = x \vee_\varepsilon b$ imply that $a = b$,

(3) L is modular and it does not contain five distinct elements x, a, b, c, y satisfying $a \wedge_\varepsilon b = b \wedge_\varepsilon c = c \wedge_\varepsilon a = x$ and $a \vee_\varepsilon b = b \vee_\varepsilon c = c \vee_\varepsilon a = y$

assures the distributivity of the lattice $\mathrm{Fix}\,\varepsilon$, but not the $\wedge_\varepsilon$-distributivity or the $\vee_\varepsilon$-distributivity of L.

Next we shall indicate some sufficient condition for an $\mathcal{E}$–lattice in order to have its distributivity. Let $(L, \wedge_\varepsilon, \vee_\varepsilon)$ be an $\mathcal{E}$–lattice and $\{a_i \mid i \in I\}$ be a set of representatives for the equivalence classes modulo $\operatorname{Ker}\varepsilon$. A nonvoid subset L' of L is called an *$\mathcal{E}$-sublattice* of L if $\varepsilon(L') \subseteq L'$ and L' is closed under the binary operations $\wedge_\varepsilon, \vee_\varepsilon$ (note that $\operatorname{Fix}\varepsilon$, as well as every equivalence class modulo $\operatorname{Ker}\varepsilon$ are $\mathcal{E}$-sublattice of L). For any two distinct elements $x, y \in [a_i] \setminus \{a_i\}$ $(i \in I)$, the $\mathcal{E}$-sublattice $\langle x, y\rangle$ of L generated by x and y can have one of the following forms:

$$\langle x, y\rangle = L'_0 = \{a_i, x, y\}, \text{ where } x \wedge_\varepsilon y = x \vee_\varepsilon y = a_i,$$

$$\langle x, y\rangle = L'_1 = \{a_i, x, y, x \wedge_\varepsilon y\}, \text{ where } x \vee_\varepsilon y = a_i,$$

$$\langle x, y\rangle = L'_2 = \{a_i, x, y, x \vee_\varepsilon y\}, \text{ where } x \wedge_\varepsilon y = a_i,$$

$$\langle x, y\rangle = L'_3 = \{a_i, x, y, x \wedge_\varepsilon y, x \vee_\varepsilon y\}.$$

Obviously, all $\mathcal{E}$–lattices L'_i, $i = \overline{0,3}$, are included in the class $[a_i]$ and each of them possesses an $\mathcal{E}$-sublattice of type L'_0. Then the following two conditions are equivalent:

i) $[a_i]$ does not contain an $\mathcal{E}$-sublattice of type L'_0, for any $i \in I$.

ii) $|[a_i]| \leq 2$, for any $i \in I$.

Assume now that the $\mathcal{E}$–lattice L satisfies the above conditions and it has a fully ordered lattice of fixed points. Since $\operatorname{Fix}\varepsilon$ is distributive, for any $a, b, c \in L$, both $a \wedge_\varepsilon (b \vee_\varepsilon c)$ and $(a \wedge_\varepsilon b) \vee_\varepsilon (a \wedge_\varepsilon c)$ (respectively $a \vee_\varepsilon (b \wedge_\varepsilon c)$ and $(a \vee_\varepsilon b) \wedge_\varepsilon (a \vee_\varepsilon c)$) are contained in the same equivalence class modulo $\operatorname{Ker}\varepsilon$. Clearly, if one of the elements a, b, c is a fixed point, then the equalities $a \wedge_\varepsilon (b \vee_\varepsilon c) = (a \wedge_\varepsilon b) \vee_\varepsilon (a \wedge_\varepsilon c)$ and $a \vee_\varepsilon (b \wedge_\varepsilon$

$c) = (a \vee_\varepsilon b) \wedge_\varepsilon (a \vee_\varepsilon c)$ hold. Let us consider that $a, b, c \notin \mathrm{Fix}\,\varepsilon$ and $a \wedge_\varepsilon (b \vee_\varepsilon c) \neq (a \wedge_\varepsilon b) \vee_\varepsilon (a \wedge_\varepsilon c)$ (the other situation can be treated in a similar way). Put $a \in [a_i] = \{a_i, a\}$, $b \in [a_j] = \{a_j, b\}$, $c \in [a_k] = \{a_k, c\}$ and suppose $a_i \leq a_j \leq a_k$. Then $a \wedge_\varepsilon (b \vee_\varepsilon c)$, $(a \wedge_\varepsilon b) \vee_\varepsilon (a \wedge_\varepsilon c) \in [a_i]$ and so we have the next two cases:

Case 1. $a \wedge_\varepsilon (b \vee_\varepsilon c) = a_i$ and $(a \wedge_\varepsilon b) \vee_\varepsilon (a \wedge_\varepsilon c) = a$

Since a is not a fixed point, the same property is verified by $a \wedge_\varepsilon b$ and $a \wedge_\varepsilon c$. But $a \wedge_\varepsilon b, a \wedge_\varepsilon c \in [a_i]$ and therefore $a \wedge_\varepsilon b = a \wedge_\varepsilon c = a$. This implies that $a \in \mathrm{Fix}\,\varepsilon$, a contradiction.

Case 2. $a \wedge_\varepsilon (b \vee_\varepsilon c) = a$ and $(a \wedge_\varepsilon b) \vee_\varepsilon (a \wedge_\varepsilon c) = a_i$

Because $a \notin \mathrm{Fix}\,\varepsilon$, we have $b \vee_\varepsilon c \notin \mathrm{Fix}\,\varepsilon$ and thus $b \vee_\varepsilon c = c$. This equality shows that $b \leq''_\varepsilon c$. Hence $c \in \mathrm{Fix}\,\varepsilon$, a contradiction.

Mention that the study of the other five situations of ordering between a_i, a_j and a_k is analogous to the above. Therefore we have proved the next proposition.

Proposition 2 *Let $(L, \wedge_\varepsilon, \vee_\varepsilon)$ be an $\mathcal{E}$–lattice satisfying the previous equivalent conditions* i), ii) *and having a fully ordered lattice of fixed points. Then L is both $\wedge_\varepsilon$-distributive and $\vee_\varepsilon$-distributive.*

Finally, we present some results concerning to the concept of complementation for $\mathcal{E}$–lattices. Since on an $\mathcal{E}$–lattice we have two $\mathcal{E}$-ordering relations, it is natural to introduce two different types of initial (respectively final) elements. Let $(L, \wedge_\varepsilon, \vee_\varepsilon)$ be an $\mathcal{E}$–lattice. An element $a_0 \in L$

is called $\wedge_\varepsilon$-*initial* if $a_0 \leq'_\varepsilon a$, for all $a \in L$, and $\vee_\varepsilon$-*initial* if $a_0 \leq''_\varepsilon a$, for all $a \in L$. By duality, an element $a_1 \in L$ is called $\wedge_\varepsilon$-*final* if $a \leq'_\varepsilon a_1$, for all $a \in L$, and $\vee_\varepsilon$-*final* if $a \leq''_\varepsilon a_1$, for all $a \in L$. The notions of $\vee_\varepsilon$-initial element or $\wedge_\varepsilon$-final element of an $\mathcal{E}$–lattice $(L, \wedge_\varepsilon, \vee_\varepsilon)$ lead to the trivial case $L = \operatorname{Fix}\varepsilon$ and so we shall consider only the other situations. For two elements $a_0, a_1 \in L$, we have that

$$a_0 \text{ is } \wedge_\varepsilon\text{-initial in } L \text{ iff } a_0 \text{ is an initial element of } \operatorname{Fix}\varepsilon$$

and

$$a_1 \text{ is } \vee_\varepsilon\text{-final in } L \text{ iff } a_1 \text{ is a final element of } \operatorname{Fix}\varepsilon.$$

Remark also that, under the hypothesis of their existence, we have the uniqueness of a $\wedge_\varepsilon$-initial element or of a $\vee_\varepsilon$-final element of an $\mathcal{E}$–lattice. In the following, by a *bounded $\mathcal{E}$–lattice* we shall understand an $\mathcal{E}$–lattice having both a $\wedge_\varepsilon$-initial element (denoted usually by a_0) and a $\vee_\varepsilon$-final element (denoted usually by a_1).

Definition 3 *Let $(L, \wedge_\varepsilon, \vee_\varepsilon)$ be a bounded $\mathcal{E}$–lattice and $a \in L$. An element $\bar{a} \in L$ is called an $\mathcal{E}$-complement of a if $a \wedge_\varepsilon \bar{a} = a_0$ and $a \vee_\varepsilon \bar{a} = a_1$. We say that L is $\mathcal{E}$-complemented if every element of L has an $\mathcal{E}$-complement.*

First of all, we show that the $\mathcal{E}$-complementation of an $\mathcal{E}$–lattice is equivalent to the complementation of its lattice of fixed points.

Proposition 3 *Let $(L, \wedge_\varepsilon, \vee_\varepsilon)$ be a bounded $\mathcal{E}$–lattice. Then L is $\mathcal{E}$-complemented if and only if* $\operatorname{Fix}\varepsilon$ *is a complemented lattice.*

Proof. Suppose that L is $\mathcal{E}$-complemented. If $\bar{a}$ is an $\mathcal{E}$-complement of $a \in L$, then, by applying ε to the equalities $a \wedge_\varepsilon \bar{a} = a_0$ and $a \vee_\varepsilon \bar{a} = a_1$, it obtains that $\varepsilon(\bar{a})$ is a complement of $\varepsilon(a)$ in $\mathrm{Fix}\,\varepsilon$ (and an $\mathcal{E}$-complement of a in L, too). Since $\mathrm{Fix}\,\varepsilon = \mathrm{Im}\,\varepsilon$, it results that $\mathrm{Fix}\,\varepsilon$ is complemented.

Conversely, assume that $\mathrm{Fix}\,\varepsilon$ is a complemented lattice and let $a \in L$. Then a complement of $\varepsilon(a)$ in $\mathrm{Fix}\,\varepsilon$ is also an $\mathcal{E}$-complement of a in L. Hence L is $\mathcal{E}$-complemented.

Corollary 1 *A bounded $\mathcal{E}$–lattice $(L, \wedge_\varepsilon, \vee_\varepsilon)$ is uniquely $\mathcal{E}$-complemented if and only if $L = \mathrm{Fix}\,\varepsilon$ and $\mathrm{Fix}\,\varepsilon$ is a uniquely complemented lattice.*

Proof. Suppose that L is uniquely $\mathcal{E}$-complemented, that is, every element of L possesses a unique $\mathcal{E}$-complement. Let a be an element of L and $\bar{a} \in L$ such that $a \wedge_\varepsilon \bar{a} = a_0$ and $a \vee_\varepsilon \bar{a} = a_1$. Then $\varepsilon(a) \wedge_\varepsilon \bar{a} = a_0$ and $\varepsilon(a) \vee_\varepsilon \bar{a} = a_1$.
Since $\bar{a}$ has a unique $\mathcal{E}$-complement, it follows that $\varepsilon(a) = a$ and so $L = \mathrm{Fix}\,\varepsilon$. In this case the concepts of $\mathcal{E}$-complement and complement coincide, therefore $\mathrm{Fix}\,\varepsilon$ is a uniquely complemented lattice. The converse implication is obvious.

As we have already seen, if an element a of a bounded $\mathcal{E}$–lattice L possesses an $\mathcal{E}$-complement, this is not unique in general. Let C_a be the set of all $\mathcal{E}$-complements of a. Then we can easily verify that the following relations hold: $C_a \subseteq C_{\varepsilon(a)}$, and $\varepsilon(C_{\varepsilon(a)}) = \varepsilon(C_a) \subseteq C_a$. With the supplementary assumption that L is $\wedge_\varepsilon$-distributive (respectively $\vee_\varepsilon$-distributive), it obtains that C_a is closed under the binary operation $\vee_\varepsilon$ (respectively

$\wedge_\varepsilon$). Thus, for a bounded $\mathcal{E}$–lattice L which is both $\wedge_\varepsilon$-distributive and $\vee_\varepsilon$-distributive, C_a is an $\mathcal{E}$–sublattice of L. Because the $\wedge_\varepsilon$-distributivity or the $\vee_\varepsilon$-distributivity of L implies the distributivity of the lattice $\mathrm{Fix}\,\varepsilon$, we also have

(1) $C_a \subseteq [\bar{a}] \subseteq C_{\varepsilon(a)}$,

where $\bar{a}$ is an arbitrary $\mathcal{E}$-complement of a. Note that if a is a fixed point, then $C_a = C_{\varepsilon(a)}$ and hence

(2) $C_a = [\bar{a}]$.

It is well-known that an element of a distributive lattice can have only one complement. This uniqueness fails for $\mathcal{E}$-complements, as shows the equality (2).

3 Links to hyperstructure theory

There are well-known the connections between the lattice theory and the hyperstructure theory (for example, see Chapter 4 of [3]). In this way, many properties of lattices (as modularity, distributivity, ... and so on) can be characterized by properties of some hyperstructures associated to them. Since $\mathcal{E}$–lattices constitute generalizations of lattices, it is natural to study their links with hyperstructures. The first steps of this study represent the main purpose of the present section.

Let $(L, \wedge_\varepsilon, \vee_\varepsilon)$ be an $\mathcal{E}$–lattice, $\mathcal{P}^*(L)$ be the set of all nonempty subsets of L and, for every $a \in L$, denote by $[a]$ the equivalence class of a modulo

$\operatorname{Ker}\varepsilon$. The simplest hyperoperations which can be defined on L are the following:

$$\overline{\wedge}_\varepsilon, \overline{\vee}_\varepsilon : L \times L \longrightarrow \mathcal{P}^*(L)$$

$$a \,\overline{\wedge}_\varepsilon\, b = [a \wedge_\varepsilon b],\ \ a \,\overline{\vee}_\varepsilon\, b = [a \vee_\varepsilon b],\quad \text{for all } a, b \in L.$$

These are associative and commutative, therefore $(L, \overline{\wedge}_\varepsilon)$ and $(L, \overline{\vee}_\varepsilon)$ are commutative semihypergroups. Note also that if $(L_1, \wedge_{\varepsilon_1}, \vee_{\varepsilon_1})$, $(L_2, \wedge_{\varepsilon_2}, \vee_{\varepsilon_2})$ are two $\mathcal{E}$–lattices and $f : L_1 \longrightarrow L_2$ is an $\mathcal{E}$–lattice homomorphism, then f is a semihypergroup homomorphism both from $(L_1, \overline{\wedge}_{\varepsilon_1})$ to $(L_2, \overline{\wedge}_{\varepsilon_2})$ and from $(L_1, \overline{\vee}_{\varepsilon_1})$ to $(L_2, \overline{\vee}_{\varepsilon_2})$.

On the other hand, $\overline{\wedge}_\varepsilon$ and $\overline{\vee}_\varepsilon$ verify the conditions in the definition of a new concept, which extends that of hyperlattice.

Definition 4 *Let L be a nonvoid set and $\overline{\wedge}, \overline{\vee}$ be two hyperoperations on L. We say that $(L, \overline{\wedge}, \overline{\vee})$ is a generalized hyperlattice if, for any $(a, b, c) \in L^3$, the following conditions are satisfied:*

a) $a \in (a \overline{\wedge} a) \cap (a \overline{\vee} a)$;

b) $a \overline{\wedge} b = b \overline{\wedge} a,\ a \overline{\vee} b = b \overline{\vee} a$;

c) $a \overline{\wedge} (b \overline{\wedge} c) = (a \overline{\wedge} b) \overline{\wedge} c,\ a \overline{\vee} (b \overline{\vee} c) = (a \overline{\vee} b) \overline{\vee} c$;

d) $a \in [a \overline{\wedge} (a \overline{\vee} b)] \cap [a \overline{\vee} (a \overline{\wedge} b)]$;

e) $a \in a \overline{\vee} b \Longleftrightarrow b \in a \overline{\wedge} b$.

By a direct calculation, it is easy to prove the next proposition.

Proposition 4 *Let $(L, \wedge_\varepsilon, \vee_\varepsilon)$ be an $\mathcal{E}$–lattice and $\overline{\wedge}_\varepsilon, \overline{\vee}_\varepsilon$ be the above hyperoperations on L. Then $(L, \overline{\wedge}_\varepsilon, \overline{\vee}_\varepsilon)$ is a generalized hyperlattice.*

Another remarkable hyperoperation on the $\mathcal{E}$–lattice $(L, \wedge_\varepsilon, \vee_\varepsilon)$ can be constructed by using $\overline{\wedge}_\varepsilon$ and $\overline{\vee}_\varepsilon$ in the next manner:

$$* : L \times L \to \mathcal{P}^*(L)$$

$$a * b = (a \,\overline{\wedge}_\varepsilon\, b) \cup (a \,\overline{\vee}_\varepsilon\, b), \text{ for all } a, b \in L.$$

Clearly, the hyperoperation $*$ is commutative. We also have:

$$a * a = [a], \quad \text{for every } a \in L.$$

Other usual properties of $*$ are equivalent to some properties of the $\mathcal{E}$–lattice L, as show the following results.

Proposition 5 *Let $(L, \wedge_\varepsilon, \vee_\varepsilon)$ be an $\mathcal{E}$–lattice and $*$ be the previous hyperoperation on L. Then the following conditions are equivalent:*

a) *$(L, *)$ is a semihypergroup.*

b) *$(L, *)$ is a quasihypergroup.*

c) *The lattice* $\mathrm{Fix}\,\varepsilon$ *is fully ordered.*

Proof. a) $\Longrightarrow$ c) Suppose that $*$ is associative and let a, b be two arbitrary elements of L. Then $a * (a * b) = (a * a) * b$, which means:

$$\bigcup_{x \in [a \wedge_\varepsilon b] \cup [a \vee_\varepsilon b]} ([a \wedge_\varepsilon x] \cup [a \vee_\varepsilon x]) = \bigcup_{x \in [a]} ([x \wedge_\varepsilon b] \cup [x \vee_\varepsilon b]).$$

Take $y \in \bigcup_{x \in [a \wedge_\varepsilon b]} [a \vee_\varepsilon x]$. Then $y \in [a \vee_\varepsilon x]$, for some $x \in [a \wedge_\varepsilon b]$, and so $\varepsilon(y) = \varepsilon(a) \vee \varepsilon(x) = \varepsilon(a) \vee (\varepsilon(a) \wedge \varepsilon(b)) = \varepsilon(a)$. If $y \in \bigcup_{x \in [a]} [x \wedge_\varepsilon b]$ it obtains

$\varepsilon(y) = \varepsilon(a) \wedge \varepsilon(b)$ and if $y \in \bigcup_{x \in [a]} [x \vee_\varepsilon b]$ it obtains $\varepsilon(y) = \varepsilon(a) \vee \varepsilon(b)$. Thus $\varepsilon(a) = \varepsilon(a) \wedge \varepsilon(b)$ or $\varepsilon(a) = \varepsilon(a) \vee \varepsilon(b)$, which imply that $\varepsilon(a) \leq \varepsilon(b)$ or $\varepsilon(b) \leq \varepsilon(a)$. Hence $\mathrm{Fix}\,\varepsilon$ is fully ordered.

c) $\Longrightarrow$ a) Suppose $\mathrm{Fix}\,\varepsilon$ to be fully ordered and let $a, b \in L$. We have to prove that $a * (b * c) = (a * b) * c$, i.e.:

$$(3) \quad \bigcup_{x \in [b \wedge_\varepsilon c] \cup [b \vee_\varepsilon c]} ([a \wedge_\varepsilon x] \cup [a \vee_\varepsilon x]) = \bigcup_{x \in [a \wedge_\varepsilon b] \cup [a \vee_\varepsilon b]} ([x \wedge_\varepsilon c] \cup [x \vee_\varepsilon c]).$$

It is easy to see that the next equalities hold:

$$(4) \quad \begin{cases} \bigcup_{x \in [b \wedge_\varepsilon c]} [a \wedge_\varepsilon x] = \bigcup_{x \in [a \wedge_\varepsilon b]} [x \wedge_\varepsilon c] = [a \wedge_\varepsilon b \wedge_\varepsilon c], \\ \bigcup_{x \in [b \vee_\varepsilon c]} [a \vee_\varepsilon x] = \bigcup_{x \in [a \vee_\varepsilon b]} [x \vee_\varepsilon c] = [a \vee_\varepsilon b \vee_\varepsilon c], \\ \bigcup_{x \in [b \wedge_\varepsilon c]} [a \vee_\varepsilon x] = [a \vee_\varepsilon (b \wedge_\varepsilon c)], \quad \bigcup_{x \in [b \vee_\varepsilon c]} [a \wedge_\varepsilon x] = [a \wedge_\varepsilon (b \vee_\varepsilon c)], \\ \bigcup_{x \in [a \wedge_\varepsilon b]} [x \vee_\varepsilon c] = [(a \wedge_\varepsilon b) \vee_\varepsilon c)], \quad \bigcup_{x \in [a \vee_\varepsilon b]} [x \wedge_\varepsilon c] = [(a \vee_\varepsilon b) \wedge_\varepsilon c)]. \end{cases}$$

Assume that $\varepsilon(a) \leq \varepsilon(b) \leq \varepsilon(c)$ (the other five cases of ordering between $\varepsilon(a), \varepsilon(b)$ and $\varepsilon(c)$ may be treated in a similar way). Then the equalities (4) become:

$$(4)' \quad \begin{cases} \bigcup_{x\in[b\wedge_\varepsilon c]} [a \wedge_\varepsilon x] = \bigcup_{x\in[a\wedge_\varepsilon b]} [x \wedge_\varepsilon c] = [a], \\ \bigcup_{x\in[b\vee_\varepsilon c]} [a \vee_\varepsilon x] = \bigcup_{x\in[a\vee_\varepsilon b]} [x \vee_\varepsilon c] = [c], \\ \bigcup_{x\in[b\wedge_\varepsilon c]} [a \vee_\varepsilon x] = [b], \quad \bigcup_{x\in[b\vee_\varepsilon c]} [a \wedge_\varepsilon x] = [a], \\ \bigcup_{x\in[a\wedge_\varepsilon b]} [x \vee_\varepsilon c] = [c], \quad \bigcup_{x\in[a\vee_\varepsilon b]} [x \wedge_\varepsilon c] = [b]. \end{cases}$$

These imply that the both sides of (3) are equal to $[a] \cup [b] \cup [c]$ and so (3) holds.

b) $\Longrightarrow$ c) Suppose that $(L, *)$ is a quasihypergroup, that is, it satisfies the reproductive law:

$$a * L = L * a = L, \quad \text{for every } a \in L.$$

Let $a, b \in L$. Then $b \in a * L$, therefore there exists $x \in L$ such that $b \in a * x$. It results $b \in [a \wedge_\varepsilon x]$ or $b \in [a \vee_\varepsilon x]$ and thus $\varepsilon(b) = \varepsilon(a) \wedge \varepsilon(x)$ or $\varepsilon(b) = \varepsilon(a) \vee \varepsilon(x)$. So $\varepsilon(b) \leq \varepsilon(a)$ or $\varepsilon(a) \leq \varepsilon(b)$, which show that $\mathrm{Fix}\,\varepsilon$ is fully ordered.

c) $\Longrightarrow$ b) Let a and b be two elements of L. Because $\mathrm{Fix}\,\varepsilon$ is fully ordered, we have $\varepsilon(b) \leq \varepsilon(a)$ or $\varepsilon(a) \leq \varepsilon(b)$, i.e. $\varepsilon(b) = \varepsilon(a) \wedge \varepsilon(b) = \varepsilon(a \wedge_\varepsilon b)$ or $\varepsilon(b) = \varepsilon(a) \vee \varepsilon(b) = \varepsilon(a \vee_\varepsilon b)$. It obtains $b \in [a \wedge_\varepsilon b]$ or $b \in [a \vee_\varepsilon b]$ and so $b \in a * b$. Hence $L = a * L$ and our proof is finished.

By Proposition 3.3, we get immediately the next corollary.

Corollary 2 *Under the hypothesis of Proposition* 3.3*, we have that* $(L, *)$ *is a hypergroup if and only if the lattice* $\mathrm{Fix}\,\varepsilon$ *is fully ordered.*

As we have seen in the proof of Proposition 3.3, if $\mathrm{Fix}\,\varepsilon$ is fully ordered, then $\{a, b\} \subseteq a * b$, for all $a, b \in L$. This shows that any element in L is an identity of $(L, *)$. Also, remark that $(L, *)$ contains a scalar iff $|L| = 1$.

Moreover, the assumption that $\mathrm{Fix}\,\varepsilon$ is fully ordered leads us to the conclusion that the hypergroup $(L, *)$ is of a special type.

Proposition 6 *Let $(L, \wedge_\varepsilon, \vee_\varepsilon)$ be an $\mathcal{E}$–lattice having a fully ordered lattice of fixed points. Then $(L, *)$ is a join space.*

Proof. We must prove that, for any $(a, b, c, d) \in L^4$, $a/b \cap c/d \neq \emptyset$ implies that $a * d \cap b * c \neq \emptyset$. Let $x \in a/b \cap c/d$. Then $a \in x * b$ and $c \in x * d$, which mean $a \in [x \wedge_\varepsilon b] \cup [x \vee_\varepsilon b]$ and $c \in [x \wedge_\varepsilon d] \cup [x \vee_\varepsilon d]$. We distinguish the next four cases.

Case 1. $a \in [x \wedge_\varepsilon b]$ and $c \in [x \wedge_\varepsilon d]$

It obtains $\varepsilon(a) = \varepsilon(x) \wedge \varepsilon(b)$ and $\varepsilon(c) = \varepsilon(x) \wedge \varepsilon(d)$, therefore $\varepsilon(a) \wedge \varepsilon(d) = (\varepsilon(x) \wedge \varepsilon(b)) \wedge \varepsilon(d) = \varepsilon(b) \wedge (\varepsilon(x) \wedge \varepsilon(d)) = \varepsilon(b) \wedge \varepsilon(c)$. This shows that $[a \wedge_\varepsilon d] = [b \wedge_\varepsilon c]$ and so:

$$[a \wedge_\varepsilon d] \cap [b \wedge_\varepsilon c] \neq \emptyset. \tag{5}$$

Case 2. $a \in [x \vee_\varepsilon b]$ and $c \in [x \vee_\varepsilon d]$

Dually to Case 1, we obtain:

$$[a \vee_\varepsilon d] \cap [b \vee_\varepsilon c] \neq \emptyset. \tag{6}$$

Case 3. $a \in [x \wedge_\varepsilon b]$ and $c \in [x \vee_\varepsilon d]$

We have $\varepsilon(a) = \varepsilon(x) \wedge \varepsilon(b)$ and $\varepsilon(c) = \varepsilon(x) \vee \varepsilon(d)$. Assume that $\varepsilon(d) \leq \varepsilon(b)$. Then $\varepsilon(a) \vee \varepsilon(d) = (\varepsilon(x) \wedge \varepsilon(b)) \vee \varepsilon(d) = (\varepsilon(x) \vee \varepsilon(d)) \wedge (\varepsilon(b) \vee \varepsilon(d)) = \varepsilon(c) \wedge (\varepsilon(b) \vee \varepsilon(d)) = \varepsilon(c) \wedge \varepsilon(b)$, which implies that $[a \wedge_\varepsilon d] = [b \wedge_\varepsilon c]$. Thus:

$$[a \vee_\varepsilon d] \cap [b \wedge_\varepsilon c] \neq \emptyset. \tag{7}$$

Now, let us assume that $\varepsilon(b) \leq \varepsilon(d)$. Because $\varepsilon(x)$ belongs to the interval $[\varepsilon(a), \varepsilon(c)]$ of the lattice $\mathrm{Fix}\,\varepsilon$ and $\varepsilon(a) \leq \varepsilon(b) \leq \varepsilon(d) \leq \varepsilon(c)$, we have the following three situations:

i) $\varepsilon(x) \in [\varepsilon(a), \varepsilon(b)]$

Then $\varepsilon(a) = \varepsilon(x)$ and $\varepsilon(c) = \varepsilon(d)$. It results $\varepsilon(a) \vee \varepsilon(d) = \varepsilon(c) = \varepsilon(b) \vee \varepsilon(c)$, and therefore $[a \vee_\varepsilon d] = [b \vee_\varepsilon c]$ and:

$$[a \vee_\varepsilon d] \cap [b \vee_\varepsilon c] \neq \emptyset. \tag{8}$$

ii) $\varepsilon(x) \in [\varepsilon(b), \varepsilon(d)]$

Then $\varepsilon(a) = \varepsilon(b)$ and $\varepsilon(c) = \varepsilon(d)$. Clearly, we have $[a \wedge_\varepsilon d] = [b \wedge_\varepsilon c]$ (and also $[a \vee_\varepsilon d] = [b \vee_\varepsilon c]$), which shows that:

$$[a \wedge_\varepsilon d] \cap [b \wedge_\varepsilon c] \neq \emptyset \quad (\text{and also } [a \vee_\varepsilon d] \cap [b \vee_\varepsilon c] \neq \emptyset). \tag{9}$$

iii) $\varepsilon(x) \in [\varepsilon(d), \varepsilon(c)]$

Then $\varepsilon(a) = \varepsilon(b)$ and $\varepsilon(c) = \varepsilon(x)$. It results $\varepsilon(a) \wedge \varepsilon(d) = \varepsilon(a) = \varepsilon(b) \wedge \varepsilon(c)$, and therefore $[a \wedge_\varepsilon d] = [b \wedge_\varepsilon c]$ and:

$$[a \wedge_\varepsilon d] \cap [b \wedge_\varepsilon c] \neq \emptyset. \tag{10}$$

Case 4. $a \in [x \vee_\varepsilon b]$ and $c \in [x \wedge_\varepsilon d]$

Dually to Case 3, we obtain the next relations:

$[a \wedge_\varepsilon d] \cap [b \vee_\varepsilon c] \neq \emptyset$,

$[a \vee_\varepsilon d] \cap [b \vee_\varepsilon c] \neq \emptyset$,

$[a \vee_\varepsilon d] \cap [b \vee_\varepsilon c] \neq \emptyset$ (and also $[a \wedge_\varepsilon d] \cap [b \wedge_\varepsilon c] \neq \emptyset$),

$[a \wedge_\varepsilon d] \cap [b \wedge_\varepsilon c] \neq \emptyset$.

Since $a * d \cap b * c = ([a \wedge_\varepsilon d] \cup [a \vee_\varepsilon d]) \cap ([b \wedge_\varepsilon c] \cup [b \vee_\varepsilon c]) = ([a \wedge_\varepsilon d] \cap [b \wedge_\varepsilon c]) \cup ([a \wedge_\varepsilon d] \cap [b \vee_\varepsilon c]) \cup ([a \vee_\varepsilon d] \cap [b \wedge_\varepsilon c]) \cup ([a \vee_\varepsilon d] \cap [b \vee_\varepsilon c])$, the above relations show that in all cases we have $a * d \cap b * c \neq \emptyset$. Hence, $(L, *)$ is a join space.

For every element a of the previous join space $(L, *)$, we have $a * a = [a]$ and $a/a = L$. We infer that $(L, *)$ is geometric iff $|L| = 1$.

If $(L, \wedge, \vee)$ is a lattice which possesses an initial element, then, introducing on L the hyperoperation

$$a \circ b = \{x \in L \mid a \wedge b \leq x\},$$

we obtain that $(L, \circ)$ is a commutative hypergroup. This construction can be generalized to $\mathcal{E}$–lattices. So, let $(L, \wedge_\varepsilon, \vee_\varepsilon)$ be an $\mathcal{E}$–lattice having a $\wedge_\varepsilon$-initial element and, corresponding with the two $\mathcal{E}$-ordering relations $\leq'_\varepsilon, \leq''_\varepsilon$ on L, we define

$$a \circ' b = \{x \in L \mid a \wedge_\varepsilon b \leq'_\varepsilon x\} \text{ and } a \circ'' b = \{x \in L \mid a \wedge_\varepsilon b \leq''_\varepsilon x\},$$

for all $a, b \in L$. Mention that $\circ''$ is a hyperoperation on L (we have $\varepsilon(a \wedge_\varepsilon b) \in a \circ'' b$ and so the set $a \circ'' b$ is nonempty), in contrast with $\circ'$, which is not necessarily well-defined (without some additional assumptions the set $a \circ' b$ can be empty). Under the above hypothesis, we obtain the following result.

Proposition 7 *a) If L is a canonical $\mathcal{E}$–lattice, then $(L, \circ')$ is a commutative hypergroup.*

b) *$(L, \circ'')$ is a commutative hypergroup if and only if $L = \mathrm{Fix}\,\varepsilon$.*

Proof. a) Let $a, b, c \in L$. Then

$$\begin{aligned} a \circ' (b \circ' c) &= a \circ' \{x \in L \mid b \wedge_\varepsilon c \leq'_\varepsilon x\} = \\ &= \bigcup_{\substack{x \in L \\ b \wedge_\varepsilon c \leq'_\varepsilon x}} a \circ' x = \bigcup_{\substack{x \in L \\ b \wedge_\varepsilon c \leq'_\varepsilon x}} \{y \in L \mid a \wedge_\varepsilon x \leq'_\varepsilon y\} \end{aligned}$$

and

$$\begin{aligned} (a \circ' b) \circ' c &= \{x \in L \mid a \wedge_\varepsilon b \leq'_\varepsilon x\} \circ' c = \\ &= \bigcup_{\substack{x \in L \\ a \wedge_\varepsilon b \leq'_\varepsilon x}} x \circ' c = \bigcup_{\substack{x \in L \\ a \wedge_\varepsilon b \leq'_\varepsilon x}} \{y \in L \mid x \wedge_\varepsilon c \leq'_\varepsilon y\}. \end{aligned}$$

Take $y \in a \circ' (b \circ' c)$. Then there exists an element $x \in L$ such that $b \wedge_\varepsilon c \leq'_\varepsilon x$ and $a \wedge_\varepsilon x \leq'_\varepsilon y$. It results $a \wedge_\varepsilon b \wedge_\varepsilon c \leq \varepsilon(y)$ and therefore, putting $x_1 = a \wedge_\varepsilon b$, we have $a \wedge_\varepsilon b \leq'_\varepsilon x_1$ and $x_1 \wedge_\varepsilon c \leq'_\varepsilon y$. These imply that $y \in (a \circ' b) \circ' c$, so $a \circ' (b \circ' c) \subseteq (a \circ' b) \circ' c$. The converse inclusion is analogous. Then $\circ'$ is associative. Clearly, if a_0 is a $\wedge_\varepsilon$-initial element of L, then we have $b \in a \circ' a_0$, for all $a, b \in L$, which shows that $(L, \circ')$ satisfies the reproductive law. Hence $(L, \circ')$ is a hypergroup.

b) If $(L, \circ'')$ is a hypergroup, then, for each $a \in L$, there is an element $x \in L$ such that $a \in a \circ'' x$. It follows $a \wedge_\varepsilon x \leq''_\varepsilon a$, which implies that $a \in \mathrm{Fix}\,\varepsilon$. Hence $L = \mathrm{Fix}\,\varepsilon$. The converse is obvious.

A well-known result of J.C. Varlet (see [10]) states that if $(L, \wedge, \vee)$ is a lattice and $\Box$ is the hyperoperation on L defined by:

$$a \,\Box\, b = \{x \in L \mid a \wedge b \leq x \leq a \vee b\}, \text{ for all } a, b \in L,$$

then the lattice L is distributive iff $(L, \Box)$ is a join space. This can be naturally extended to the case of canonical $\mathcal{E}$–lattices in the next manner.

Proposition 8 *Let $(L, \wedge_\varepsilon, \vee_\varepsilon)$ be a canonical $\mathcal{E}$–lattice and $\Box$ be the hyperoperation on L defined by:*

$$a \,\Box\, b = \{x \in L \mid a \wedge_\varepsilon b \leq \varepsilon(x) \leq a \vee_\varepsilon b\}, \textit{ for all } a, b \in L.$$

Then the $\mathcal{E}$–lattice L is $\wedge_\varepsilon$-distributive (or $\vee_\varepsilon$-distributive) if and only if $(L, \Box)$ is a join space.

We finish our paper by mentioning that other algebraic structures (as fuzzy sets or rough sets) can be possibly connected to $\mathcal{E}$–lattices and investigated by using this method.

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