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Quasiconformality and Compatibility for direct product of bi-Lipschitz homeomorphisms[1]

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Abstract

We continue the research of the consequences of the linear liminf-dilatation used instead of the limsup-dilatation for bi-Lipschitz homeomorphisms. We prove that a direct product $F = f \times g$ of two homeomorphisms is bi-Lipschitz if and only if f and g are bi-Lipschitz . Another result of the paper is that the direct product $F = f \times g$ is quasiconformal homeomorphism if $F = f \times g$ is bi-Lipschitz homeomorphism.The converse is true.

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0 Introduction

In this note we shall extend the foundations of the theory of quasiconformal maps on direct products spaces with respect to linear limsup-dilatation,linear liminf-dilatation and bi-Lipschitz homeomorphisms. P. Caraman ([2],pp 127,149,286) have established equivalence the Gehring's

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metric definition and Markushevich-Pesin's definition in the theory of n-dimensional quasiconformal mappings.We remark that for more details about the evolution of the quasiconformality we shall refer by C.Andreian Cazacu [1].

In [6] and [7] is introduced and developed the quasiconformality by the basis of Markushevich-Pesin's definition in connection with linear limsup-dilatation.

Let D and D' be a domains in $\mathbb{R}^n$, $F : D \to D'$ is a homeomorphism and

$$d(z, z_0) = |z - z_0| = (|x - x_0|^2 + |y - y_0|^2)^{1/2} \tag{0.1}$$

the Euclidean distance. For any point $z_0 \in D$ and $t > 0$ such that the ball $\bar{B}(z_0, t) = \{z : d(z, z_0) = |z - z_0| \leq t\}$ be included in D, denote

$$L(z_0, F, t) = \max_{d(z,z_0)=t} d(F(z), F(z_0)) \tag{0.2}$$

and

$$l(z_0, F, t) = \min_{d(z,z_0)=t} d(F(z), F(z_0)). \tag{0.3}$$

J. Väisälä [8] gives that if the *linear limsup-dilatation* of F at z_0

$$H(z_0, F) = \limsup_{t\to 0} \frac{L(z_0, F, t)}{l(z_0, F, t)} \tag{0.4}$$

is bounded in D, i.e. there exists a constant $H < \infty$ such that $H(z_0, F) \leq H$ for every $z_0 \in D$, F is a quasiconformal mapping after the metric definition. In [4], M. Cristea used also in the study of the quasiconformal mappings the *linear liminf-dilatation*

$$h(z_0, F) = \liminf_{t\to 0} \frac{L(z_0, F, t)}{l(z_0, F, t)} \tag{0.5.}$$

In [5], J. Heinonen and P. Koskela proved that $h(z_0, F) \leq H$ for every $z_0 \in D$ implies that F is quasiconformal and $H(z_0, F) = h(z_0, F)$ a.e., what

increased the importance of $h(z_0, F)$.
Let U and V be domains in $\mathbb{R}^k$ and $\mathbb{R}^m$; x and y arbitrary points in U and V, respectively; $f : U \to U' \subset \mathbb{R}^k$, $g : V \to V' \subset \mathbb{R}^m$ be a homeomorphisms; $F = f \times g : U \times V \to U' \times V'$ the direct product of f and g; $U \times V$ and $U' \times V'$ being domains in $R^k \times R^m$ identified with R^n, $n = k+m$; $z = (x, y)$ is a point in $U \times V$ and $F(z) = (f(x), g(y)) \in U' \times V'$.

Starting from Karmazin's limsup compatibility condition and Theorem 2[6],(for bi-Lipschitz homeomorphisms). We say that f and g are compatible if there is a constant C respectively c, such that:

Condition 1: $\limsup_{t\to 0} \dfrac{L(x_0, f, t)}{l(y_0, g, t)} \leq C$ and $\limsup_{t\to 0} \dfrac{L(y_0, g, t)}{l(x_0, f, t)} \leq C$, for any $x_0 \in U$ and $y_0 \in V$, and

Condition 2:[3] there exists a sequence $t_p \to 0$,$p \in \mathbb{N}$ such that

$$\frac{L(x_0, f, t_p)}{l(y_0, g, t_p)} \leq c \text{ and } \frac{L(y_0, g, t_p)}{l(x_0, f, t_p)} \leq c, \text{ for any } x_0 \in U, y_0 \in V.$$

Remark.[3] Condition 2 is fulfilled e.g. if

$$\liminf_{t\to 0} \frac{L(x_0, f, t)}{l(y_0, g, t)} \leq c \text{ and } \limsup_{t\to 0} \frac{L(y_0, g, t)}{l(x_0, f, t)} \leq C$$

or vice versa, and for case when

$$\liminf_{t\to 0} \frac{L(x_0, f, t)}{l(y_0, g, t)} \leq c \text{ and } \liminf_{t\to 0} \frac{L(y_0, g, t)}{l(x_0, f, t)} \leq c$$

but there exists a sequence t_p as above.

With these compatibility conditions we succeeded to characterize the quasiconformality and bi-Lipschitz condition of F extending Karmazin's Theorem 2[6]. The main issue is the problem of definition, which follow to be described.

1 Direct products of bi-Lipschitz homeomorpfisms

Definition 1.1. *A homeomorphism* $F : D \to D^{'}(D, D^{'} \subset \mathbb{R}^n)$ *is called limsup-Lipschitz in domain* D *if there exists a constant* $L, 0 < L < \infty$, *such that for almost every points* $z_0 \in D$ *is satisfied following limsup-Lipschitz condition*

$$\limsup_{t\to 0} \frac{\mid F(z) - F(z_0) \mid}{\mid z - z_0 \mid} \leq L.$$

The limsup-Lipschitz condition is a classical concept of Lipschitz condition.
Definition 1.2. *Let* U *be an open subset of* $\mathbb{R}^n$. *A homeomorphism* $F : U \to \mathbb{R}^n$ *is said to be locally limsup-Lipschitz if for every compact set* $A \subset U$ *there exists a constant* $L_A < \infty$ *such that*

$$\limsup_{z\to z_0} \frac{\mid F(z) - F(z_0) \mid}{\mid z - z_0 \mid} \leq L_A$$

for almost every $z_0 \in A$.
Definition 1.3. *A homeomorphism* $F : D \to D^{'}(D, D^{'} \subset \mathbb{R}^n)$ *is called limsup bi-Lipschitz homeomorphism, if* F *and* F^{-1} *are limpsup-Lipschitz.*

As a variant of Definition 3 with beautiful application is following.
Definition 1.4.

Let $D, D^{'}$ *be domains of* $\mathbb{R}^n$. *A homeomorphism* $F : D \to D^{'}$ *is said to be bi-Lipschitz if there exists a constant* $L, 0 < L < \infty$, *such that* F *satisfies the inequalities*

$$\frac{1}{L} \mid z - z_0 \mid \leq \mid F(z) - F(z_0) \mid \leq L \mid z - z_0 \mid$$

for every $z_0 \in D$ *and for* $\mid z - z_0 \mid$ *sufficiently small.*

The result was first proved by Karmazin , here we shell it by a different method.
Theorem 1.5. ([6])*A homeomorphism* $F = f \times g$ *is limsup bi-Lipschitz in domain* $U \times V$ *if and only if its homeomorphisms* f *and* g *are also limsup bi-Lipschitz in domains* U *and* V, *respectively.*

Proof. Necessity. Let $F : U \times V \to U' \times V'$ be a limsup bi-Lipschitz homeomorphism at the point $z_0 \in U \times V$ if satisfies the condition

$$\frac{| z - z_0 |}{L} \leq | F(z) - F(z_0) | \leq L \mid z - z_0 \mid, \text{where} L < \infty.$$

Consider the point $x_0 \in U$ such that for some $y_0 \in V, F(z)$ satisfies limsup-Lipschitz condition in point $z_0 = (x_0, y_0)$ with a constant $L < \infty$.

Let U_0 be the set of all x_0, then $mesU_0 = mesU$:

Indeed,

$$U \setminus U_0 = \{x_0 \in U : \text{ for which does not exists } y_0 \in V \text{ such that } z_0 = (x_0, y_0)$$
$$\text{satisfies Definition 1.1}\}.$$

Then

$$(U \setminus U_0) \times V = \{(x_0, y_0) : x_0 \in U \setminus U_0, y_0 \in V \text{ such that at } \; z_0 = (x_0, y_0)$$
$$\text{does not satisfy Definition 1.1 }\} \subset \{(x, y) \in U \times V : (x, y)$$
$$\text{does not satisfy Definition 1.1}\},$$

has measure zero. By inclusion n-mes $((U \setminus U_0) \times V) = 0$.

By Theorem I ([89], p.153), $((U \setminus U_0) \times V)$ is n-measurable.

Thus

$$n\text{-mes}((U \setminus U_0) \times V) = (k\text{-mes } (U \setminus U_0)) \cdot (m\text{-mes } V) = 0.$$

Hence

$$k\text{-mes } (U \setminus U_0) = 0 \Rightarrow k\text{-mes } U_0 = k\text{-mes } U.$$

Let $x \in U_0$ and $\varepsilon > 0$, arbitrary. Then, for each point $z = (x, y_0)$ we have $|z - z_0| = |x - x_0|$

$$|f(x) - f(x_0)| \leq |F(z) - F(z_0)| \leq (L + \varepsilon)|z - z_0| = (L + \varepsilon)|x - x_0|,$$

$$|f(x) - f(x_0)| \leq (L + \varepsilon)|x - x_0|.$$

Then clearly the homeomorphism $f(x)$ satisfies the limsup-Lipschitz condition at point x_0. Similarly, we can prove for the homeomorphism $g(y)$. Let $y \in V$ and $\epsilon > 0$, arbitrary.

Then, for each point $z = (x_0, y)$ we have $|z - z_0| = |y - y_0|$, therefore $|g(y) - g(y_0)| \leq (L + \varepsilon)|y - y_0|$.

Hence, $g(y)$ is limsup-Lipschitz homeomorphism. We also, f^{-1} and g^{-1} satisfies the limsup-Lipschitz condition. Consequently,the homeomorphisms f and g are the limsup bi-Lipschitz.

Sufficiency: Let f and g be a limsup bi-Lipschitz homeomorphisms, then we show that F is limsup bi-Lipschitz homeomorphism. With this aim we first suppose that $f(x)$ at the point $x_0 \in U$ satisfies the limsup-Lipschitz condition with a constant L and $g(y)$ at the point $y_0 \in V$ satisfies the limsup-Lipschitz condition with constant L.

If $\epsilon > 0$ is a positive number arbitrary and for $\mid x - x_0 \mid, \mid y - y_0 \mid$ sufficiently small, we have

$$|f(x) - f(x_0)| \leq (L + \varepsilon)|x - x_0|, \quad |g(y) - g(y_0)| \leq (L + \varepsilon)|y - y_0|.$$

Hence

$$|F(z) - F(z_0)| = |(f(x), g(y)) - (f(x_0), g(y_0))| \leq$$

$$|f(x) - f(x_0)| + |g(y) - g(y_0)| \leq$$

$$(L + \varepsilon)|x - x_0| + (L + \varepsilon)|y - y_0| \leq 2(L + \varepsilon)|z - z_0|.$$

These proves that $F(z)$ satisfies the limsup-Lipschitz condition with constant $(2L)$. Obviously, $F^{-1}(z) = f^{-1}(x) \times g^{-1}(y)$ satisfies the limsup-Lipschitz condition. Hence, $F(z)$ is the bi-Lipschitz homeomorphism.

Theorem 1.6 ([6], p.31) *Let $U, U' \subset \mathbb{R}^k$ and $V, V' \subset \mathbb{R}^m$ be a domains. If $f : U \to U'$ and $g : V \to V'$ be a limsup-compatible homeomorphisms in domain $D = U \times V$, then f and g are limsup bi-Lipschitz homeomorphisms in domains U and V, respectively.*

Proof. Suppose that f and g be a limsup-compatible homeomorphisms in the domain D. Then,there exist a constant $C < \infty$,such that for almost every points $z_0 \in U \times V$ we have

$$\limsup_{t\to 0} \frac{L(x_0,f,t)}{l(y_0,g,t)} \leq C \text{ and } \limsup_{t\to 0} \frac{L(y_0,g,t)}{l(x_0,f,t)} \leq C,$$

where $L(x_0,f,t) = \max_{x\in\partial B^k(x_0,t)} |f(x)-f(x_0)|$, $l(x_0,f,t) = \min_{x\in\partial B^k(x_0,t)} |f(x)-f(x_0)|$, then for $t > 0$ sufficient to small $B^k(x_0,t) \subset U$; $L(y_0,g,t) = \max_{y\in\partial B^m(y_o,t)} |g(y)-g(y_0)|$, $l(y_0,g,t) = \min_{y\in\partial B^m(y_0,t)} |g(y)-g(y_0)|$, then for $t > 0$ sufficient small $B^m(y_0,t) \subset V$.

The role of neighbourhoods $U_t and V_t$ is taken the balls $B^k(x_0,t)$ and $B^m(y_0,t)$, respectively.

We shown that there exist a constant L such that for almost every point x_0 in U and respectively y_0 in V we have

$$\limsup_{t\to 0} \frac{L(x_0,f,t)}{t} \leq L \text{ şi } \limsup_{t\to 0} \frac{L(y_0,g,t)}{t} \leq L,$$

is it implies that f and g are limsup bi-Lipschitz.

By Corollary 13[3] the f and g be quasiconformal homeomorphisms.Using Theorem 1.5[3] are differentiable almost every in domains U and V, respectively

Then,we to choose $i \in \mathbb{N}$, such that for

$$A_i = \left\{ x_0 \in U : \limsup_{x\to x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|} \leq i \right\},$$

mes $A_i > 0$.

Let us show that there exist the $i < \infty$.

Indeed,if there exist a x_0 be a point of differentiable and is satisfying inequality:

$$\limsup_{x\to x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|} \leq i(x_0).$$

For each $i \in \mathbb{N}$ to find $A_i \subset A_{i+1}$, therefore $\bigcup\limits_{i \in \mathbb{N}} A_i \subset U$

and $\bigcup\limits_{i \in \mathbb{N}} A_i$ included the set of points in U for where f is differentiable, we have measure equal with measure of U. Hence mes$\left(\bigcup\limits_{i \in \mathbb{N}} A_i\right) =$ mes$U$, which implies that there exist $i \in \mathbb{N}$, with mes$A_i > 0$.

Similarly, we choose $j < \infty$, such that for

$$B_j = \left\{ y_0 \in V : \limsup_{y \to y_0} \frac{|g(y) - g(y_0)|}{|y - y_0|} \leq j \right\},$$

mes$B_j > 0$.

Because y_0 is a point of differentiability, this establishes that

$$\limsup_{y \to y_0} \frac{|g(y) - g(y_0)|}{|y - y_0|} \leq j(y_0).$$

For all $j \in \mathbb{N}$ we find $B_j \subset B_{j+1}$, therefore $\bigcup\limits_{j \in \mathbb{N}} B_j \subset V$ and $\bigcup\limits_{j \in \mathbb{N}} B_j$ included all the points by V which g is differentiable, we have measure equal with measure of V. Hence mes $\bigcup\limits_{j \in \mathbb{N}} B_j =$ mes V, this implies that there exist $j \in \mathbb{N}$ with mes $V_j > 0$.

This implies that exist the point $y_0 \in V$ such that for almost every points $x \in U$ is satisfies limsup-compatibility condition in (x, y_0).

We show that exist a point y_0. For every point $y \in V$, we consider

$$U_y = \{x \in U : \text{ Condition 1 is satisfies in } (x, y)\}$$

şi

$$D_0 = \{z \in D : \text{ Condition 1 is satisfies in } (x, y)\}.$$

Evidently, $D_0 = \bigcup\limits_{y \in V} (U_y \times \{y\}) \subset D$. By hypotheses, mes $D_0 =$ mes D and from Fubini's Theorem ([89], p.156-159) we have

$$\text{mes } D_0 = \int\limits_V \text{ mes } U_y dy \leq \text{ mes } U \cdot \text{mes } V = \text{ mes } D,$$

where $\int\limits_V$ mes $U_y dy =$ mes $U\cdot$ mes V.

We denote by $A = \{y \in V:\ \text{mes}\, U_y < \ \text{mes}\, U\}$ and mes $A =$ $=$ mes $\{y \in V:\ \text{mes}\, U_y < \ \text{mes}\, U\} = 0$. Hence mes $A = 0$ the implies for almost every points $y \in V$, mes $U_y =$ mes U, therefore

$$\text{mes}V = \ \text{mes}(V \setminus A) \text{ where } V \setminus A = \{y \in V:\ \text{with mes}U_y = \text{mes}U\}\,.$$

There exist a point $y_0 \in V \setminus A$ with mes $U_{y_0} =$ mes U, hence we have that for almost every points $x \in U$, Condition 1 is satisfies in point (x, y_0).

We show that, if $x_0 \in A_i \bigcap U_{y_0}$ and for any $x \in U$ we have

$$\limsup_{t\to 0} \frac{L(x,f,t)}{L(x_0,f,t)} \le C^2$$

$$\frac{L(x,f,t)}{L(x_0,f,t)} = \frac{L(x,f,t)}{l(y_o,g,t)} \cdot \frac{l(y_0,g,t)}{L(x_0,f,t)} \le \frac{L(x,f,t)}{l(y_0,g,t)} \cdot \frac{L(y_0,g,t)}{l(x_0,f,t)} \le C^2.$$

By $\dfrac{L(x,f,t)}{L(x_0,f,t)} \le C^2$ and $\limsup\limits_{t\to 0} \dfrac{L(x_0,f,t)}{t} \le i$, we obtain

$$L(x,f,t) \le C^2 L(x_0,f,t)$$

,

$$\frac{L(x,f,t)}{t} \le C^2 \frac{L(x_0,f,t)}{t},$$

$$\limsup_{t\to 0} \frac{L(x,f,t)}{t} \le C^2 \ \limsup_{t\to 0} \frac{L(x_0,f,t)}{t} \le C^2 i,$$

$$\limsup_{t\to 0} \frac{L(x,f,t)}{t} \le C^2 i.$$

Therefore, $f(x)$ for almost every points in U satisfies limsup-Lipschitz condition with a constant $L = C^2 i$.

Similarly, we show for almost every point y in V, such that the homeomorphism $g(y)$ satisfies limsup-Lipschitz condition with constant $L = C^2 j$. Mappings f^{-1} and g^{-1} are limsup-compatibly in domain $U' \times V'$ and with abouve reasoning, also limsup-Lipschitz condition is satisfies. Therefore , $f(x)$ and $g(y)$ are bi-Lipschitz in domains U, V respectively.

Proposition 1.7 *Let $f : U \to \mathbb{R}^k (U \subset \mathbb{R}^k)$ be a homeomorphism.If f is L-bi-Lipschitz homeomorphism,then f is L^2-quasiconformal.*

Proof. By $f : U \to U'$ bi-Lipschitz for almost all point x_0 from U,we have

$$\frac{1}{L} \leq \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq L.$$

Denoted $|z - z_0| = t$, we can may write

$$\frac{t}{L} \leq l(x_0, f, t) \leq L(x_0, f, t) \leq Lt,$$

for almost every in U, which implies

$$\frac{L(x_0, f, t)}{l(x_0, f, t)} \leq \frac{Lt}{\frac{t}{L}} = L^2.$$

By limsup-quasiconformal Condition, we have f is L^2-quasiconformal homeomorphism. In particular, this result is applied and for $F = f \times g$ with the above condition.

Theorem 1.8 *If f and g are bi-Lipschitz homeomorphisms, then f and g are compatible homeomorphisms*

Proof. Suppose that f and g are bi-Lipschitz, by Theorem 4.1.5 we have that F is bi-Lipschitz, from Propozition 4.1.7 we have that F is quasiconformal, using Theorem 3.3.5 f and g are compatibles.

Corollary 1.9 *In the above conditions for homeomorphism $F = f \times g$, we have*

$F = f \times g$	$\Leftrightarrow$ f and g	$\Leftrightarrow$ f and g	$F = f \times g$
homeomorphism	limsup$-$	bi $-$ Lipschitz	bi $-$ Lipschitz
limsup$-$	compatible		
quasiconformal			

2 The relation between liminf-compatible condition and bi-Lipschitz condition of direct products of homeomorphisms

Theorem 2.1 *Let $f : U \to \mathbb{R}^k (U \subset \mathbb{R}^k)$ and $g : V \to \mathbb{R}^m (V \subset \mathbb{R}^m)$ be a liminf-compatible homeomorphisms in domain $D = U \times V \subset \mathbb{R}^k \times \mathbb{R}^m$. Suppose that there is $x_1, x_2 \in U$ and $y_1, y_2 \in V$ such that*

i)
$$\liminf_{t\to 0} \frac{L(y_1, g, t)}{t} > 0 \text{ and } \limsup_{t\to 0} \frac{l(y_2, g, t)}{t} < \infty,$$

ii)
$$\liminf_{t\to 0} \frac{L(x_1, f, t)}{t} > 0 \text{ and } \limsup_{t\to 0} \frac{l(x_2, f, t)}{t} < \infty.$$

Then f and g are bi-Lipschitz homeomorphisms in domains U and V, respectively.

Proof. Let f and g be a liminf-compatible at the point $z = (x, y) \in D$. By compatibility, Condition 2, there is a constant $c < \infty$ and a sequence common $t_p \to 0, p \in \mathbb{N}$, such that conditions

$$\frac{L(x, f, t_p)}{l(y, g, t_p)} \le c \text{ and } \frac{L(y, g, t_p)}{l(x, f, t_p)} \le c$$

are fulfilled.

Step 1.

a) Suppose that there is a constant $l_2 > 0$ such that

$$\liminf_{t\to 0} \frac{L(y_1, g, t)}{t} > l_2 > 0;$$

then for t sufficient to small, we have $L(y, g, t) > l_2 t$.

b) Suppose that there is a constant $l_1 > 0$ such that

$$\liminf_{t\to 0} \frac{l(x, f, t)}{t} \ge l_1 > 0$$

for all $x \in U$.

Indeed, if assertion is false, then there is $x_k \in U$, for every $k \in \mathbb{N}$ with

$$\liminf_{t\to 0} \frac{l(x_k, f, t)}{t} < \frac{1}{k}.$$

We can take a sequence $\tilde{t}_p(x_k) \to 0, p \in \mathbb{N}$, such that

$$l(x_k, f, \tilde{t}_p(x_k)) < \frac{\tilde{t}_p(x_k)}{k}$$

and

$$L(y_1, g, \tilde{t}_p(x_k)) > l_2\tilde{t}_p(x_k).$$

By compatibility, Condition 2, we obtain

$$kl_2 = \frac{l_2\tilde{t}_p(x_k)}{\frac{\tilde{t}_p(x_k)}{k}} < \frac{L(y_1, g, \tilde{t}_p(x_k))}{l(x_k, f, \tilde{t}_p(x_k))} \leq c.$$

Because $k \to \infty$ and $0 < c < \infty$, we obtain a contradiction. Let now $x \in U$. There exists a constant $l_1 > 0$, such that inequality is true

$$\liminf_{t\to 0} \frac{l(x, f, t)}{t} \geq l_1 > 0$$

for every $x \in U$.

Suppose first that $x_0 \in U$ for which we have

$$\liminf_{t\to 0} \frac{l(x_0, f, t)}{t} \geq l_1 \text{ implies that}$$

$$l(x_0, f, t) \geq l_1 t, \text{for } 0 < t < t'_{x_0}.$$

Let $x \in B(x_0, t'_0), \mid x - x_0 \mid = t \leq t'_{x_0}$ and $l(x_0, f, t) \geq l_1 t = l_1 \mid x - x_0 \mid$. Because

$$l(x_0, f, t) \leq \min_{|x-x_0|=t} \mid f(x) - f(x_0) \mid \leq \mid f(x) - f(x_0) \mid,$$

to obtain

$$l_1 \mid x - x_0 \mid \leq \mid f(x) - f(x_0) \mid, \text{for all } x \in B(x_0, t'_{x_0}).$$

We proved (1) for every $x_0 \in U$.

Step 2 .

a) Suppose that is satisfied a condition

$$\limsup_{t\to 0} \frac{l(y_2, g, t)}{t} < \infty,$$

for t sufficiently small and a constant $0 < l_2 < \infty$, such that we have

$$l(y_2, g, t) \leq tl_2.$$

b) We assume that exists a constant $L_1 > 0$, such that

$$\limsup_{t\to 0} \frac{L(x, f, t)}{t} < L_1, \text{for all } x \in U.$$

Indeed, if assertion is false therefore does not exist a constant $L_1 > 0$, then for every $k \in \mathbb{N}$, exists $x_k^* \in U$, such that

$$\limsup_{t\to 0} \frac{L(x_k^*, f, t)}{t} \geq k.$$

For every $k \in \mathbb{N}$ we can find a sequence $t_p(x_k^*)$ with

$$L(x_k^*, f, t_p(x_k^*)) \geq kt_p(x_k^*)$$

and

$$l(y_2, g, t_p(x_k^*)) \leq l_2 t_p(x_k^*).$$

By compatibility, Condition 2, we have

$$\frac{k}{l_2} \leq \frac{L(x_k^*, f, t_p(x_k^*))}{l(y_2, g, t_p(x_k^*))} < c,$$

because $k \to \infty$, $\frac{k}{l_2} \to \infty$, we reach a contradiction.

Hence, there is a constant $L_1 > 0$ such that $L(x, f, t) < L_1 t$, for all $x \in U$.

Let now $x_0 \in U$, for which we have

$$L(x_0, f, t) \leq L_1 t, \text{for } 0 < t < t''_{x_0}.$$

Let $x \in B(x_0, t''_{x_0}), \mid x - x_0 \mid = t \leq t''_{x_0}$, then

$$L(x_0, f, t) = \max_{|x-x_0|=t} \mid f(x) - f(x_0) \mid \leq L_1 \mid x - x_0 \mid,$$

therefore

$$\mid f(x) - f(x_0) \mid \leq L_1 \mid x - x_0 \mid .$$

Suppose that $t_{x_0} = \min\{t'_{x_0}, t''_{x_0}\}$. By (1) and (2) we obtain

$$l_1 \mid x - x_0 \mid \leq \mid f(x) - f(x_0) \mid \leq L_1 \mid x - x_0 \mid$$

for all $x \in B(x_0, t_{x_0})$ and for all $x_0 \in U$.

By the condition $i)$ and compatibility Condition 2 we proved that f is bi-Lipschitz homeomorphism. Similarly, by $ii)$ and compatibility condition we showed that g is a bi-Lipschitz. The theorem is proved.

Remark. This theorem is valid and in case limsup-compatibility Condition 1, without not even a supplementary hypothesis.

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