General Mathematics, Vol. 16, Nr. 2 (2008), 95–107

General common fixed point theorems for weakly compatible maps [1]

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Abstract

The aim of this paper is to prove a common fixed point theorem for four weakly compatible maps satisfying an implicit relation without need of continuity. This theorem generalizes, improves and extends some results on compatible continuous maps of [1], [2], [10], [12] and others.

2000 Mathematical Subject Classification: 47H10, 54H25.

Key words and phrases: Weakly compatible maps, compatible and compatible maps of type (A), (B), (C) and (P), implicit relations, common fixed point theorems

1 Introduction and preliminaries

Generalizing the concept of commuting mappings, Sessa [15] introduces the concept of weakly commuting mappings. He defines $\mathcal{S}$ and $\mathcal{T}$ to be weakly commuting if

$$d(\mathcal{S}\mathcal{T}x, \mathcal{T}\mathcal{S}x) \leq d(\mathcal{T}x, \mathcal{S}x)$$

for all $x \in \mathcal{X}$, where $\mathcal{S}$ and $\mathcal{T}$ are two self maps of a metric space $(\mathcal{X}, d)$.

[1] *Received 11 Octomber, 2007*

Accepted for publication (in revised form) 20 December, 2007

And in 1986, Jungck [3], introduced more generalized commuting mappings, called compatible mappings, which are more general that commuting and weakly commuting maps. $\mathcal{S}$ and $\mathcal{T}$ above are said to be compatible if

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}\mathcal{S}x_n) = 0 \tag{1}$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{S}x_n = \lim_{n\to\infty} \mathcal{T}x_n = t$ for some $t \in \mathcal{X}$. This concept has been useful as a tool for obtaining more comprehensive fixed point theorems. Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible, but neither implication is reversible (see [14], [3]).

Further, G. Jungck, P. P. Murthy and Y. J. Cho [4] gave the notion of compatible mappings of type (A) as follows, $\mathcal{S}$ and $\mathcal{T}$ above are said to be compatible of type (A) if, in place of (1), we have the two conditions

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) = 0.$$

Clearly, weakly commuting maps are compatible of type (A). From [4], it follows that the implication is not reversible. But this definition is equivalent to the concept of compatible mappings under some conditions and examples are given to show that the two notions are independent.

Afterwards, H. K. Pathak and M. S. Khan [9] extended type (A) mappings by introducing the concept of compatible maps of type (B) and compared these mappings with compatible and compatible mappings of type (A) in normed spaces. To be compatible of type (B), $\mathcal{S}$ and $\mathcal{T}$ above have to satisfy, in lieu of condition (1), the inequalities

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n)\right],$$

$$\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{T}t) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{T}^2x_n)\right].$$

It is clear to see that compatible maps of type (A) are compatible of type (B), to show that the converse is not true (see [9]).

In [6] the concept of compatible maps of type (P) was introduced and compared with compatible and compatible maps of type (A). $\mathcal{S}$ and $\mathcal{T}$ above are compatible of type (P) if, instead of (1) we have,

$$\lim_{n\to\infty} d(\mathcal{S}^2x_n, \mathcal{T}^2x_n) = 0.$$

Some fixed points theorems for compatible mappings of type (P) are proved in [7] and [13].

In 1998, H. K. Pathak, Y. J. Cho, S. M. Kang, B. Madharia [8] introduced an other new extension of compatible maps of type (A) called compatible maps of type (C). They define $\mathcal{S}$ and $\mathcal{T}$ above to be compatible of type (C) if, we replace (1) by the inequalities

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{T}^2x_n)$$

$$\leq \frac{1}{3}\left[\lim_{n\to\infty} d(\mathcal{S}\mathcal{T}x_n, \mathcal{S}t) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{T}^2x_n) + \lim_{n\to\infty} d(\mathcal{S}t, \mathcal{S}^2x_n)\right],$$

$$\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}^2x_n)$$

$$\leq \frac{1}{3}\left[\lim_{n\to\infty} d(\mathcal{T}\mathcal{S}x_n, \mathcal{T}t) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{S}^2x_n) + \lim_{n\to\infty} d(\mathcal{T}t, \mathcal{T}^2x_n)\right].$$

The same authors gave some examples to show that compatible maps of type (C) need not be neither compatible nor compatible of type (A) (resp. compatible of type (B)) in normed spaces.

Recently, Jungck and Rhoades [5] defined weakly compatible maps and showed that compatible maps are weakly compatible but the converse in not true. They defined $\mathcal{S}$ and $\mathcal{T}$ above to be weakly compatible if $\mathcal{S}t = \mathcal{T}t, t \in \mathcal{X}$ implies $\mathcal{S}\mathcal{T}t = \mathcal{T}\mathcal{S}t$. By Lemma 1 in ([3], [4], [6]) it follows that $\mathcal{S}$ and $\mathcal{T}$ are compatible (resp. compatible of type (A), compatible of type (P)) then, $\mathcal{S}$ and $\mathcal{T}$ are weakly compatible. It is known that all of the above compatibility notions imply weakly compatible notion. However, as we shall show in the example below, there exists weakly compatible maps which are neither compatible nor compatible of type (A) (resp. compatible of type (B), type (C), type (P)).

Example 1.1. *Let* $\mathcal{X} = [0, 20]$ *with the usual metric. Define* $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ *by* $\mathcal{S}x = \begin{cases} 0 & if \quad x = 0 \\ x + 16 & if \quad 0 < x \leq 4 \\ x - 4 & if \quad 4 < x \leq 20 \end{cases}$;$\mathcal{T}x = \begin{cases} 0 & if \quad x \in \{0\} \cup (4, 20] \\ 3 & if \quad 0 < x \leq 4. \end{cases}$

Let $\{x_n\}$ *be the sequence defined by* $x_n = 4 + \frac{1}{n}, n \in \mathbb{N}^*$. *Then*

$$\mathcal{S}x_n = x_n - 4 \to 0; \mathcal{T}x_n = 0 \to 0 \ as \ n \to \infty,$$

$$S(0) = 0 = \mathcal{T}(0); \qquad \mathcal{S}\mathcal{T}(0) = 0 = \mathcal{T}\mathcal{S}(0).$$

Clearly, $\mathcal{S}$ and $\mathcal{T}$ are weakly compatible maps, since they commute at their coincidence point $t = 0$. On the other hand , we have

$$\mathcal{S}\mathcal{T}x_n = \mathcal{S}(0) = 0; \qquad \mathcal{S}^2 x_n = \mathcal{S}(x_n - 4) = x_n + 12,$$

$$\mathcal{T}\mathcal{S}x_n = \mathcal{T}(x_n - 4) = 3; \qquad \mathcal{T}^2 x_n = \mathcal{T}(0) = 0.$$

Consequently, $\lim_{n\to\infty} |\mathcal{S}\mathcal{T}x_n - \mathcal{T}\mathcal{S}x_n| = 3 \neq 0$ that is, $\mathcal{S}$ and $\mathcal{T}$ are not compatible. Moreover, we have

$$\lim_{n\to\infty} |\mathcal{T}\mathcal{S}x_n - \mathcal{S}^2 x_n| = \lim_{n\to\infty} |3 - x_n - 12| = 13 \neq 0$$

thus, $\mathcal{S}$ and $\mathcal{T}$ are not compatible of type (A). Furthermore,

$$13 = \lim_{n\to\infty} |\mathcal{T}\mathcal{S}x_n - \mathcal{S}^2 x_n| \nleqq \frac{1}{2}\left[\lim_{n\to\infty} |\mathcal{T}\mathcal{S}x_n - \mathcal{T}t| + \lim_{n\to\infty} |\mathcal{T}t - \mathcal{T}^2 x_n|\right] = \frac{3}{2}$$

which tells us that $\mathcal{S}$ and $\mathcal{T}$ are not compatible of type (B). Again, one have $13 = \lim_{n\to\infty} |\mathcal{T}\mathcal{S}x_n - \mathcal{S}^2 x_n| \nleqq \frac{1}{3}\Big[\lim_{n\to\infty} |\mathcal{T}\mathcal{S}x_n - \mathcal{T}t| + \lim_{n\to\infty} |\mathcal{T}t - \mathcal{S}^2 x_n|$ $+ \lim_{n\to\infty} |\mathcal{T}t - \mathcal{T}^2 x_n|\Big] = = \frac{19}{3}$ hence, the maps $\mathcal{S}$ and $\mathcal{T}$ are not compatible of type (C). Also, we have

$$\lim_{n\to\infty} |\mathcal{S}^2 x_n - \mathcal{T}^2 x_n| = 16 \neq 0$$

therefore, $\mathcal{S}$ and $\mathcal{T}$ are not compatible of type (P).

2 Implicit relations

Like in [10], we denote by $\mathcal{F}$ the set of all real continuous functions $F:\mathbb{R}_+^6\to\mathbb{R}$ satisfying the following conditions:

(F_1) : F is non-increasing in variables t_5 and t_6,

(F_2) : there exists $h\in(0,1)$ such that for every $u,v\geq 0$ with:

(F_a) : $F(u,v,v,u,u+v,0)\leq 0$ or

(F_b) : $F(u,v,u,v,0,u+v)\leq 0$

we have $u\leq hv$,

(F_3) : $F(u,u,0,0,u,u)>0$, for all $u>0$.

Example 2.1. $F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-k\max\left\{t_2,t_3,t_4,\frac{1}{2}(t_5+t_6)\right\}$ *where* $k\in(0,1)$.

Example 2.2. $F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^2-c_1\max\{t_2^2,t_3^2,t_4^2\}-c_2\max\{t_3t_5,t_4t_6\}$ $-c_3t_5t_6$, *where* $c_1>0,c_2,c_3\geq 0,c_1+2c_2<1$ *and* $c_1+c_3<1$.

Example 2.3. $F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^2-t_1(at_2+bt_3+ct_4)-dt_5t_6$, *where* $a>0,b,c,d\geq 0,a+b+c<1$ *and* $a+d<1$.

Example 2.4. $F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^p-at_1^{p-1}t_2-bt_1^{p-2}t_3t_4-ct_5^{p-1}t_6-dt_5t_6^{p-1}$, *where* $a>0$, $b,c,d\geq 0,a+b<1$ *and* $a+c+d<1$ *and* p *an integer such as* $p\geq 3$.

Example 2.5. $F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^3-c\dfrac{t_3^2t_4^2+t_5^2t_5^2}{1+t_2+t_3+t_4}$, *where* $c\in(0,1)$.

Example 2.6. $F(t_1,t_2,t_3,t_4,t_5,t_6)=at_1^2-bt_2^2-\dfrac{ct_5t_6}{dt_3^2+et_4^2+1}$, *where* $c,d,e\geq 0,0<b<a$ *and* $b+c<a$.

Example 2.7. $F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1-t_1[at_2^p+bt_3^p+ct_4^p]^{\frac{1}{p}}-d\sqrt{t_5t_6}$, *where* $0<a<(1-d)^p,b,c,d\geq 0,a+b+c<1$ *and* $d<1,p\in\mathbb{N}^*$.

Example 2.8. $F(t_1,t_2,t_3,t_4,t_5,t_6)=t_1^2-k\max\{t_2^2,t_3t_4,t_5t_6\}$, *where* $k\in(0,1)$.

The subject of the preset paper is to obtain common fixed point theorems by using a minimal commutativity condition. Our results extend the recent results due to Bouhadjera, Popa and others.

3 Main results

Now we state our main theorems

Theorem 3.1. *Let $\mathcal{S}, \mathcal{T}, \mathcal{I}$ and $\mathcal{J}$ be mappings from a complete metric space $(\mathcal{X}, d)$ into itself satisfying the conditions:*

(a) $\mathcal{S}\mathcal{X} \subset \mathcal{J}\mathcal{X}$ and $\mathcal{T}\mathcal{X} \subset \mathcal{I}\mathcal{X}$,

(b) one of $\mathcal{S}\mathcal{X}$ or $\mathcal{T}\mathcal{X}$ is closed,

(c) $\mathcal{S}$ and $\mathcal{I}$ as well as $\mathcal{T}$ and $\mathcal{J}$ are weakly compatible,

(d) inequality

(2) $F(d(\mathcal{S}x, \mathcal{T}y), d(\mathcal{I}x, \mathcal{J}y), d(\mathcal{I}x, \mathcal{S}x), d(\mathcal{J}y, \mathcal{T}y), d(\mathcal{I}x, \mathcal{T}y), d(\mathcal{J}y, \mathcal{S}x)) \leq 0$

holds for all $x, y \in \mathcal{X}$, where $F \in \mathcal{F}$. Then $\mathcal{S}, \mathcal{T}, \mathcal{I}$ and $\mathcal{J}$ have a unique common fixed point.

Proof. Suppose x_0 is an arbitrary point in $\mathcal{X}$. Then, since (a) holds, we can define inductively a sequence $\{y_n\}$ as follows

$$\{\mathcal{S}x_0, \mathcal{T}x_1, \mathcal{S}x_2, \mathcal{T}x_3, \ldots, \mathcal{S}x_{2n}, \mathcal{T}x_{2n+1}, \ldots\} \tag{3}$$

such that
$y_{2n} = \mathcal{S}x_{2n} = \mathcal{J}x_{2n+1}$ and $y_{2n+1} = \mathcal{T}x_{2n+1} = \mathcal{J}x_{2n+2}$ for $n \in \mathbb{N}$.

Using the inequality (2), we have successively
$F(d(\mathcal{S}x_{2n}, \mathcal{T}x_{2n+1}), d(\mathcal{I}x_{2n}, \mathcal{J}x_{2n+1}), d(\mathcal{I}x_{2n}, \mathcal{S}x_{2n}), d(\mathcal{J}x_{2n+1}, \mathcal{T}x_{2n+1}),$
$d(\mathcal{I}x_{2n}, \mathcal{T}x_{2n+1}), d(\mathcal{J}x_{2n+1}, \mathcal{S}x_{2n})) = F(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}),$
$d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n+1}), 0) \leq 0.$

By (F_1), we have $F(d(\mathcal{I}_{2n}, \mathcal{I}_{2n+1}), d(\mathcal{I}_{2n-1}, \mathcal{I}_{2n}), d(\mathcal{I}_{2n-1}, \mathcal{I}_{2n}), d(\mathcal{I}_{2n}, \mathcal{I}_{2n+1}),$
$d(\mathcal{I}_{2n-1}, \mathcal{I}_{2n}) + d(\mathcal{I}_{2n}, \mathcal{I}_{2n+1}), 0) \leq 0.$

So, we obtain by (F_a)

$$d(y_{2n}, y_{2n+1}) \leq hd(y_{2n-1}, y_{2n}).$$

Similarly, by (F_1) and (F_b), one may get

$$d(y_{2n-1}, y_{2n}) \leq hd(y_{2n-2}, y_{2n-1})$$

and so,

$$d(y_{2n}, y_{2n+1}) \leq h^{2n} d(y_0, y_1)$$

for $n \in \mathbb{N}$. An easy calculation shows that the sequence $\{y_n\}$ defined by (3) is a Cauchy one. Since $\mathcal{X}$ is complete, the sequence $\{y_n\}$ converges to a point z in $\mathcal{X}$. Hence, z is also the limit of its subsequences $\{\mathcal{S}x_{2n}\} = \{\mathcal{J}x_{2n+1}\}, \{\mathcal{T}x_{2n-1}\} = \{\mathcal{I}x_{2n}\}$ and $\{\mathcal{T}x_{2n+1}\} = \{\mathcal{I}x_{2n+2}\}$.

Suppose that $\mathcal{S}\mathcal{X}$ is closed, since $\mathcal{S}\mathcal{X} \subset \mathcal{J}\mathcal{X}$, then there exists a point u in $\mathcal{X}$ such that $z = \mathcal{J}u$. Using estimation (2), we obtain $F(d(\mathcal{S}x_{2n}, \mathcal{T}u), d(\mathcal{I}x_{2n}, \mathcal{J}u), d(\mathcal{I}x_{2n}, \mathcal{S}x_{2n}), d(\mathcal{J}u, \mathcal{T}u), d(\mathcal{I}x_{2n}, \mathcal{T}u), d(\mathcal{J}u, \mathcal{S}x_{2n})) \leq 0$

By letting $n \to \infty$, we have by the continuity of F

$$F(d(z, \mathcal{T}u), 0, 0, d(z, \mathcal{T}u), d(z, \mathcal{T}u), 0) \leq 0$$

which implies by (F_a), that $z = \mathcal{T}u$. Therefore, $\mathcal{J}u = z = \mathcal{T}u$. But $\mathcal{J}$ and $\mathcal{T}$ are weakly compatible, then $\mathcal{J}\mathcal{T}u = \mathcal{T}\mathcal{J}u$ and so, $\mathcal{J}z = \mathcal{J}\mathcal{T}u = \mathcal{T}\mathcal{J}u = \mathcal{T}z$. Again, from inequality (2), we have $F(d(\mathcal{S}x_{2n}, \mathcal{T}z), d(\mathcal{I}x_{2n}, \mathcal{J}z), d(\mathcal{I}x_{2n}, \mathcal{S}x_{2n}), d(\mathcal{J}z, \mathcal{T}z), d(\mathcal{I}x_{2n}, \mathcal{T}z), d(\mathcal{J}z, \mathcal{S}x_{2n})) \leq 0$.

Taking the limit as $n \to \infty$, we get

$$F(d(z, \mathcal{T}z), d(z, \mathcal{T}z), 0, 0, d(z, \mathcal{T}z), d(\mathcal{T}z, z)) \leq 0$$

contradicting (F_3), then, we deduce that, $z = \mathcal{T}z = \mathcal{J}z$. This means that z is in the range of $\mathcal{T}$ and, since $\mathcal{T}\mathcal{X} \subset \mathcal{I}\mathcal{X}$, there exists an element v in $\mathcal{X}$ such that $z = \mathcal{T}z = \mathcal{I}v$. The use of condition (2) gives

$$F(d(\mathcal{S}v, \mathcal{T}z), d(\mathcal{I}v, \mathcal{J}z), d(\mathcal{I}v, \mathcal{S}v), d(\mathcal{J}z, \mathcal{T}z), d(\mathcal{I}v, \mathcal{T}z), d(\mathcal{J}z, \mathcal{S}v))$$

$$= F(d(\mathcal{S}v, z), 0, d(\mathcal{S}v, z), 0, 0, d(\mathcal{S}v, z)) \leq 0.$$

which implies by (F_b), that $\mathcal{S}v = z = \mathcal{I}v$.But the mappings $\mathcal{S}$ and $\mathcal{I}$ are weakly compatible, hence, $\mathcal{S}\mathcal{I}v = \mathcal{I}\mathcal{S}v$ i.e $\mathcal{S}z = \mathcal{S}\mathcal{I}v = \mathcal{I}\mathcal{S}v = \mathcal{I}z$.Moreover, by (2), we can estimate

$$F(d(\mathcal{S}z, \mathcal{T}z), d(\mathcal{I}z, \mathcal{J}z), d(\mathcal{I}z, \mathcal{S}z), d(\mathcal{J}z, \mathcal{T}z), d(\mathcal{I}z, \mathcal{T}z), d(\mathcal{J}z, \mathcal{S}z))$$

$$= F(d(\mathcal{S}z, z), d(\mathcal{S}z, z), 0, 0, d(\mathcal{S}z, z), d(z, \mathcal{S}z)) \leq 0$$

which contradicts (F_3) if $\mathcal{S}z \neq z,$. We conclude that, $z = \mathcal{S}z = \mathcal{I}z$. Consequently, $\mathcal{T}z = \mathcal{J}z = \mathcal{S}z = z$, this means that the point z is a common fixed point for both $\mathcal{S}, \mathcal{T}\mathcal{I}$ and $\mathcal{J}$. The uniqueness follows immediately from proceding inequality (2) and the proof is complete. Similarly, one can obtain this conclusion by supposing $\mathcal{T}\mathcal{X}$ is closed.

Truly Theorem 3.1 generalizes the results of [1],[2],[10],[11],[12] and others, since no continuity assumption is assumed here and the weak compatibility is least condition for mapping to have fixed point.

Corollary 3.1. *If in the hypotheses of Theorem 3.1, he have in the lieu of (2) the condition*

$$d(\mathcal{S}x, \mathcal{T}y) \leq k \max\{d(\mathcal{I}x, \mathcal{J}y), d(\mathcal{I}x, \mathcal{S}x), d(\mathcal{I}y, \mathcal{T}y),$$

$$\frac{1}{2}(d(\mathcal{I}x\mathcal{T}y) + d(\mathcal{I}y, \mathcal{S}x))\ \}$$

for all $x, y \in \mathcal{X}$, were $k \in (0, 1)$. Then, the mappings $\mathcal{S}, \mathcal{T}, \mathcal{I}$ and $\mathcal{J}$ have a unique common fixed point.

Proof. Use Theorem 3.1 and Example 3.1.

In a similar way as in Corollary 3.1, one can obtain additional corollaries using the Examples given above.

Remarks.

(1) If we put in Theorem 3.1 and its Corollaries $\mathcal{I} = \mathcal{J} = \mathcal{I}_{\mathcal{X}}$ (: the identity mapping on $\mathcal{X}$) and also $\mathcal{S} = \mathcal{T}$ and $\mathcal{I} = \mathcal{J} = \mathcal{I}_{\mathcal{X}}$, then we can get much more corollaries.

(2) Theorem 3.1 remains valid if we have $\mathcal{I}\mathcal{X}$ or $\mathcal{J}\mathcal{X}$ is closed (resp. $\mathcal{I}$ or $\mathcal{J}$ is surjective) instead of $\mathcal{S}\mathcal{X}$ or $\mathcal{T}\mathcal{X}$ is closed.

Now, we give an example to illustrate our result.

Example 3.1. *Let $\mathcal{X} = [0, \infty)$ be endowed with the usual metric d. Define*

$$\mathcal{I}x = \begin{cases} 0 \; if \; x \in [0,1) \\ x \; if \; x \in [1, \infty) \end{cases} ; \; \mathcal{S}x = \begin{cases} 2 \; if \; x \in [0,1) \\ \frac{1}{\sqrt{x}} \; if \; x \in [1, \infty) \end{cases}$$

$$\mathcal{J}x = \begin{cases} 0 \ if \ x \in [0,1) \\ x^2 \ if \ x \in [1,\infty) \end{cases} ; \mathcal{T}x = \begin{cases} 2 \ if \ x \in [0,1) \\ \dfrac{1}{x} \ if x \in [1,\infty) \end{cases} .$$

Clearly, $\mathcal{S}\mathcal{X} = (0,1] \cup \{2\} \subset \mathcal{J}\mathcal{X} = [0,\infty) = \mathcal{X}$ and $\mathcal{T}\mathcal{X} = (0,1] \cup \{2\} \subset \mathcal{I}\mathcal{X} = [0,\infty)$ and $\mathcal{J}\mathcal{X}, \mathcal{I}\mathcal{X}$ are closed. Further, we see that $\mathcal{S}$ and $\mathcal{I}$ as well as $\mathcal{J}$ and $\mathcal{T}$ are weakly compatible since they commute as their coincidence point $x = 1$. Now, define $f : \mathbb{R}_+^6 \to \mathbb{R}$ by $F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - \frac{1}{4}t_2^2$. It is clear to see that $F \in \mathcal{F}$. Moreover, we have

$$F(d(\mathcal{S}x,\mathcal{T}y), d(\mathcal{I}x,\mathcal{J}y), d(\mathcal{I}x,\mathcal{S}x), d(\mathcal{J}y,\mathcal{T}y), d(\mathcal{I}x,\mathcal{T}y), d(\mathcal{J}y,\mathcal{S}x))$$
$$= \left| \frac{1}{\sqrt{x}} - \frac{1}{y} \right|^2 - \frac{1}{4}|x - y^2|^2 = \frac{\sqrt{x} - y|^2}{xy^2} - \frac{1}{4}|\sqrt{x} - y|^2|\sqrt{x} + y|^2$$
$$= |\sqrt{x} - y|^2 \left[\frac{1}{xy^2} - \frac{1}{4}(\sqrt{x} + y)^2 \right] \leq 0$$

for all $x, y \geq 1$. Then, F satisfies condition (2). So, all assumptions of Theorem 3.1 are satisfied and 1 is the unique common fixed point of the above maps. Now, we show that Theorems in [1], [2],[10],[11] and [12] are not applicable. Indeed, let us consider a sequence $\{x_n\}$ in $\mathcal{X}$ defined by $x_n = 1 + \dfrac{1}{n}$ for $n \in \mathbb{N}^*$.
Clearly, we have as $n \to \infty$,

$$\mathcal{S}x_n = \frac{1}{\sqrt{x_n}} \to 1 = t; \mathcal{J}x_n = x_n^2 \to 1$$

$$\mathcal{I}x_n = x_n \to; \mathcal{T}x_n = \frac{1}{x_n} \to 1.$$

Further, one have

$$\mathcal{I}\mathcal{S}x_n = \mathcal{I}\left(\frac{1}{\sqrt{x_n}}\right) = 0; \mathcal{S}\mathcal{I}x_n = \mathcal{S}(x_n) = \frac{1}{\sqrt{x_n}} \to 1$$
$$\mathcal{J}\mathcal{T}x_n = \mathcal{J}\left(\frac{1}{x_n}\right) = 0; \mathcal{T}\mathcal{J}x_n = \mathcal{T}(x_n^2) = \frac{1}{x_n^2} \to 1$$
$$\mathcal{I}\mathcal{I}x_n = \mathcal{I}(x_n) = x_n \to 1; \mathcal{S}\mathcal{S}x_n = \mathcal{S}\left(\frac{1}{\sqrt{x_n}}\right) = 2$$

$$\mathcal{J}\mathcal{J}x_n = \mathcal{J}(x_n^2) = x_n^4 \to 1; \mathcal{T}\mathcal{T}x_n = \mathcal{T}\left(\frac{1}{x_n}\right) = 2.$$

But,

$$\lim_{n\to\infty} d(\mathcal{S}\mathcal{I}x_n, \mathcal{I}\mathcal{S}x_n) = \lim_{n\to\infty}\left|0 - \frac{1}{\sqrt{x_n}}\right| = 1 \neq 0$$

$$\lim_{n\to} d(\mathcal{J}\mathcal{T}x_n, \mathcal{T}\mathcal{J}x_n) = \lim_{n\infty}\left|0 - \frac{1}{x_n^2}\right| = 1 \neq 0,$$

so, $\mathcal{S}$ and $\mathcal{I}$ as well as $\mathcal{J}$ and $\mathcal{T}$ are not compatible. Again, we have

$$\lim_{n\to\infty} d(\mathcal{I}\mathcal{S}x_n, \mathcal{S}^2x_n) = \lim_{n\to\infty} |0-2|2 \neq 0,$$

$$\lim_{n\to\infty} d(\mathcal{J}\mathcal{T}x_n, \mathcal{T}^2x_n) = \lim_{n\to\infty} |0-2| = 2 \neq 0,$$

thus, the pairs $(\mathcal{S},\mathcal{I})$ and $(\mathcal{J},\mathcal{T})$are not compatible of type (A). Now, one have

$$\lim_{n\to\infty} d(\mathcal{S}^2x_n, \mathcal{I}^2x_n) = \lim_{n\to\infty} |2 - x_n| = 1 \neq 0,$$

$$\lim_{n\to\infty} d(\mathcal{J}^2x_n, \mathcal{T}^2x_n) = \lim_{n\to\infty} |x_n^4 - 2| = 2 \neq 0,$$

this tells us that the maps $\mathcal{S}$ and $\mathcal{I}$ and $\mathcal{J}$ and $\mathcal{T}$ are not compatible of type (P). Also we have

$$2 = \lim_{n\to\infty} d(\mathcal{I}\mathcal{S}x_n, \mathcal{S}^2x_n) \nleq \frac{1}{2}[\lim_{n\to\infty} d(\mathcal{I}\mathcal{S}x_n, \mathcal{I}1) + \lim_{n\to} d(\mathcal{I}1, \mathcal{I}^2x_n)] = \frac{1}{2},$$

$$2 = \lim_{n\to\infty} d(\mathcal{J}\mathcal{T}x_n, \mathcal{T}^2x_n) \nleq \frac{1}{2}[\lim_{n\to\infty} d(\mathcal{J}\mathcal{T}x_n, \mathcal{J}1) + \lim_{n\to\infty} d(\mathcal{J}1, \mathcal{J}^2x_n)] = \frac{1}{2},$$

that is $(\mathcal{S},\mathcal{I})$ and $(\mathcal{J},\mathcal{T})$ are not compatible of type (B). Finally,

$$2 = \lim_{n\to\infty} d(\mathcal{I}\mathcal{S}x_n, \mathcal{S}^2x_n) \nleq \frac{1}{3}[\lim_{n\to\infty} d(\mathcal{I}\mathcal{S}x_n\mathcal{I}1) + \lim_{n\to\infty} d(\mathcal{I}1, \mathcal{S}^2x_n)$$

$$+ \lim_{n\to\infty} d(\mathcal{I}1, \mathcal{I}^2x_n)] = \frac{2}{3}$$

and

$$2 = \lim_{n\to\infty} d(\mathcal{J}\mathcal{T}x_n, \mathcal{T}^2x_n) \nleq \frac{1}{3}[\lim_{n\to\infty} d(\mathcal{J}\mathcal{T}x_n, \mathcal{J}1) + \lim_{n\to\infty} d(\mathcal{J}1, \mathcal{T}^2x_n)$$

$$+\lim_{n\to\infty} d(\mathcal{J}1, \mathcal{J}^2 x_n)] = \frac{2}{3},$$

therefore, neither $\mathcal{S}$ and $\mathcal{I}$ nor $\mathcal{J}$ and $\mathcal{T}$ are compatible of type (C).
Now we give a generalization to the above result.

Theorem 3.2. *let $\mathcal{I}, \mathcal{J}$ and $\{\mathcal{T}_i\}_{i\in\mathbb{N}^*}$ be mapping from a complete metric space $(\mathcal{X}, d)$ into itself such that*

(i) $\mathcal{T}_i\mathcal{X} \subset \mathcal{J}\mathcal{X}$ *and* $\mathcal{T}_{i+1}\mathcal{X} \subset \mathcal{I}\mathcal{X}$,

(ii) one of $\mathcal{T}_i\mathcal{X}$ *or* $\mathcal{T}_{i+1}\mathcal{X}$ *is closed,*

(iii) the pairs $\{\mathcal{T}_i, \mathcal{I}\}$ *and* $\{\mathcal{T}_{i+1}, \mathcal{I}\}$ *are weakly compatible,*

(iv) the inequality

$$F(d(\mathcal{T}_i x, \mathcal{T}_{i+1y}), d(\mathcal{I}x, \mathcal{J}y), d(\mathcal{I}x, \mathcal{T}_i x),$$
$$d(\mathcal{J}y, \mathcal{T}_{i+1}y), d(\mathcal{I}x, \mathcal{T}_{i+1}y), d(\mathcal{J}y, \mathcal{T}_i x)) \leq 0$$

holds for all $x, y \in \mathcal{X}$, for all $i \in \mathbb{N}^*$ and $F \in \mathcal{F}$.Then, $\mathcal{I}, \mathcal{J}$ and $\{\mathcal{T}_i\}_{i\in\mathbb{N}^*}$ have a unique common fixed point in $\mathcal{X}$.

Proof. Letting $i = 1$, we get the hypotheses of Theorem 3.1 for the maps $\mathcal{I}, \mathcal{J}, \mathcal{T}_1$ and $\mathcal{T}_2$ with the unique common fixed point z. Now, z is a unique common fixed point of $\mathcal{I}, \mathcal{J}, \mathcal{T}_1$ and of $\mathcal{I}, \mathcal{J}, \mathcal{T}_2$. Otherwise, if z' is a second distinct fixed point of $\mathcal{I}, \mathcal{J}$ and $\mathcal{T}_1$, then by inequality (2), we get

$$F(d(\mathcal{T}_1 z, \mathcal{T}_2 z'), d(\mathcal{I}z, \mathcal{J}z'), d(\mathcal{I}z, \mathcal{T}_1 z),$$
$$d(\mathcal{J}z', \mathcal{T}_2 z'), d(\mathcal{I}z, \mathcal{T}_2 z'), d(\mathcal{J}z', \mathcal{T}_1 z))$$
$$= F(d(z, z'), d(z, z'), 0, 0, d(z, z'), d(z', z)) \leq 0$$

contradicts (F_3), hence $z' = z$.

By the same method, we prove that z is the unique common fixed point of the mappings $\mathcal{I}, \mathcal{J}$ and $\mathcal{T}_2$.

Now by letting $i = 2$, we get the hypotheses of the same theorem for the maps $\mathcal{I}, \mathcal{J}$ and $\mathcal{T}_3$ and consequently they have a unique common fixed point z'. Analougously, z' is the unique common fixed point of $\mathcal{I}, \mathcal{J}, \mathcal{T}_2$ and of $\mathcal{I}, \mathcal{J}, \mathcal{T}_3$. Thus $z' = z$. Continuing in this way, we clearly see that z is the required point.

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