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On some analytic functions with negative coefficients

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Abstract

We will study some classes of analytic functions with negative coefficients introduced by using a modified Sălăgean operator.

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1 Introduction and preliminaries

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U,

$$A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$$

and $S = \{f \in A : f \text{ is univalent in } U\}$.

In [7] the subfamily T of S consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0, j = 2, 3, ...,\ z \in U \tag{1}$$

was introduced.

Let D^n be the Sălăgean differential operator (see [6]) $D^n : A \to A$, $n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$

$$D^n f(z) = D(D^{n-1} f(z))$$

Let $n \in \mathbb{N}$ and $\lambda \geq 0$. Let denote with D_λ^n the Al-Oboudi operator (see [4]) defined by

$$D_\lambda^n : A \to A\,,$$

$$D_\lambda^0 f(z) = f(z)\ ,\ D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z)\,,$$

$$D_\lambda^n f(z) = D_\lambda \left(D_\lambda^{n-1} f(z) \right)\ .$$

Definition 1. *[3] Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. We denote by D_λ^β the linear operator defined by*

$$D_\lambda^\beta : A \to A\,,$$

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^\beta \, a_j z^j\,.$$

Theorem 1. *[6] If $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...$, $z \in U$ then the next assertions are equivalent:*

(i) $\sum_{j=2}^{\infty} j a_j \leq 1$

(ii) $f \in T$

(iii) $f \in T^*$, *where $T^* = T \bigcap S^*$ and S^* is the well-known class of starlike functions.*

Definition 2. *[6] Let $\alpha \in [0,1)$ and $n \in \mathbb{N}$, then*

$$S_n(\alpha) = \left\{ f \in A : Re\frac{D^{n+1}f(z)}{D^n f(z)} > \alpha, z \in U \right\}$$

is the set of n-starlike functions of order α.

Definition 3. *[5] Let $\alpha \in [0,1), \beta \in (0,1]$ and let $n \in \mathbb{N}$; we define the class $T_n(\alpha, \beta)$ of n-starlike functions of order α and type β with negative coefficients by*

$$T_n(\alpha, \beta) = \{ f \in A : |J_n(f, \alpha; z)| < \beta, z \in U \},$$

where

$$J_n(f, \alpha; z) = \frac{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1}{\dfrac{D^{n+1}f(z)}{D^n f(z)} + 1 - 2\alpha}, \ z \in U$$

Remark 1. *The class $T_n(\alpha, 1)$ is the class of n-starlike functions of order α with negative coefficients i.e. $T_n(\alpha, 1) = T \bigcap S_n(\alpha)$.*

Theorem 2. *[5] Let $\alpha \in [0,1), \beta \in (0,1]$ and $n \in \mathbb{N}$. The function f of the form (1) is in $T_n(\alpha, \beta)$ if and only if*

$$\sum_{j=2}^{\infty} j^n [j - 1 + \beta(j + 1 - 2\alpha)] a_j \le 2\beta(1 - \alpha)$$

Remark 2. *From Remark 1 and Theorem 2, for $f(z)$ of the form (1), we have $f \in T_n(\alpha, 1) = T_n(\alpha)$ iff*

$$\sum_{j=2}^{\infty} j^n (j - \alpha) a_j \le 1 - \alpha, \ \text{where } \alpha \in [0,1)$$

We denote by $f * g$ the modified Hadamard product of two functions $f(z)$, $g(z) \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j, (a_j \geq 0,\ j = 2, 3, ...)$ and $g(z) = z - \sum_{j=2}^{\infty} b_j z^j$, $(b_j \geq 0,$ j=2,3,...), is defined by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j.$$

We say that an analytic function f is subordinate to an analytic function g if $f(z) = g(w(z))$, $z \in U$, for some analytic function w with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$. We denote the subordination relation by $f \prec g$.

2 Main results

Definition 4. *[1] , [2] Let $f \in T$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...,$ $z \in U$.*

We say that f is in the class $TL_\beta(\alpha)$ if:

$$Re\frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^{\beta} f(z)} > \alpha, \quad \alpha \in [0, 1), \quad \lambda \geq 0, \quad \beta \geq 0, \quad z \in U.$$

We say that f is in the class $T^c L_\beta(\alpha)$ if:

$$Re\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} > \alpha, \quad \alpha \in [0, 1), \quad \lambda \geq 0, \quad \beta \geq 0, \quad z \in U.$$

Remark 3. *We observe that both classes may also be defined, by using the subordination relation, such that:*

$$TL_\beta(\alpha) = \left\{ f \in T \,:\, \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^{\beta} f(z)} - \alpha \prec \frac{1+z}{1-z} \;, \alpha \in [0, 1),\, \lambda \geq 0,\, \beta \geq 0,\, z \in U \right\}$$

and

$$T^cL_\beta(\alpha)=\left\{f\in T\,:\,\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}-\alpha\prec\frac{1+z}{1-z}\ ,\ \alpha\in[0,1),\ \lambda\geq 0,\ \beta\geq 0,\ z\in U\right\}.$$

Theorem 3. *[1] , [2] Let $\alpha\in[0,1)$, $\lambda\geq 0$ and $\beta\geq 0$.*

The function $f\in T$ of the form (1) is in the class $TL_\beta(\alpha)$ iff

$$\sum_{j=2}^{\infty}[(1+(j-1)\lambda)^\beta(1+(j-1)\lambda-\alpha)]a_j<1-\alpha. \tag{2}$$

The function $f\in T$ of the form (1) is in the class $T^cL_\beta(\alpha)$ iff

$$\sum_{j=2}^{\infty}[(1+(j-1)\lambda)^{\beta+1}(1+(j-1)\lambda-\alpha)]a_j<1-\alpha. \tag{3}$$

Proof. Let $f\in TL_\beta(\alpha)$, with $\alpha\in[0,1)$, $\lambda\geq 0$ and $\beta\geq 0$. We have

$$Re\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)}>\alpha.$$

If we take $z\in[0,1)$, $\beta\geq 0$, $\lambda\geq 0$, we have (see Definition 1.1):

$$\frac{1-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}a_jz^{j-1}}{1-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}a_jz^{j-1}}>\alpha. \tag{4}$$

From the above we obtain:

$$1-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}a_jz^{j-1}>\alpha-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}\alpha a_jz^{j-1},$$

$$\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)a_jz^{j-1}<1-\alpha.$$

Letting $z \to 1^-$ along the real axis we have:

$$\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)a_j < 1-\alpha.$$

Conversely, let take $f \in T$ for which the relation (2) hold.

The condition $Re\dfrac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}f(z)} > \alpha$ is equivalent with

$$\alpha - Re\left(\frac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}f(z)} - 1\right) < 1\,. \tag{5}$$

We have

$$\alpha - Re\left(\frac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}f(z)} - 1\right) \le \alpha + \left|\frac{D_{\lambda}^{\beta+1}f(z)}{D_{\lambda}^{\beta}f(z)} - 1\right|$$

$$= \alpha + \left|\frac{D_{\lambda}^{\beta+1}f(z) - D_{\lambda}^{\beta}f(z)}{D_{\lambda}^{\beta}f(z)}\right| = \alpha + \left|\frac{\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}a_j[(j-1)\lambda]z^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}a_j z^{j-1}}\right|$$

$$\le \alpha + \frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}a_j|1-j|\lambda|z|^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}a_j|z|^{j-1}} = \alpha + \frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}a_j(j-1)\lambda|z|^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}a_j|z|^{j-1}}$$

$$< \alpha + \frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}a_j(j-1)\lambda}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}a_j} = \frac{\alpha + \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}a_j[(j-1)\lambda-\alpha]}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}a_j} < 1.$$

Thus

$$\alpha + \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}a_j[(j-1)\lambda+1-\alpha] < 1,$$

which is the condition (2).

The proof of the second part of the theorem is similarly with the above one, so it is omitted.

Remark 4. *Using the conditions (2) and (3) it is easy to prove that*

$TL_{\beta+1}(\alpha) \subseteq TL_\beta(\alpha)$

and

$T^cL_{\beta+1}(\alpha) \subseteq T^cL_\beta(\alpha)$,

where $\beta \geq 0$, $\alpha \in [0,1)$ *and* $\lambda \geq 0$.

Theorem 4. *[1] , [2] If* $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j \in TL_\beta(\alpha), (a_j \geq 0,\ j = 2,3,...)$, $g(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j \in TL_\beta(\alpha)$, $(b_j \geq 0,\ j = 2,3,...)$, $\alpha \in [0,1)$, $\lambda \geq 0$, $\beta \geq 0$, *then* $f(z) * g(z) \in TL_\beta(\alpha)$.

If $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j \in T^cL_\beta(\alpha), (a_j \geq 0,\ j = 2,3,...)$, $g(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j \in T^cL_\beta(\alpha)$, $(b_j \geq 0,\ j = 2,3,...)$, $\alpha \in [0,1)$, $\lambda \geq 0$, $\beta \geq 0$, *then* $f(z) * g(z) \in T^cL_\beta(\alpha)$.

Proof. We have

$$\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^\beta [(j-1)\lambda + 1 - \alpha] a_j < 1 - \alpha$$

and

$$\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^\beta [(j-1)\lambda + 1 - \alpha] b_j < 1 - \alpha.$$

We know that $f(z) * g(z) = z - \sum\limits_{j=2}^{\infty} a_j b_j z^j$. From $g(z) \in T$, by using Theorem 1 , we have $\sum\limits_{j=2}^{\infty} j b_j \leq 1$. We notice that $b_j < 1,\quad j = 2,3,\ldots$.

Thus,

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(j-1)\lambda+1-\alpha]a_j b_j < \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(j-1)\lambda+1-\alpha]a_j < 1-\alpha.$$

This means that $f(z) * g(z) \in TL_{\beta}(\alpha)$, $\beta \geq 0$, $\alpha \in [0,1)$ and $\lambda \geq 0$.

The proof of the second part of the theorem is similarly with the above one, so it is omitted.

Theorem 5. *[1] , [2] Let $f_1(z) = z$ and*

$$f_j(z) = z - \frac{1-\alpha}{(1+(j-1)\lambda)^{\beta}(1-\alpha+(j-1)\lambda)} z^j,\ j = 2, 3, ...$$

Then $f \in TL_{\beta}(\alpha)$ iff it can be expressed in the form $f(z) = \sum\limits_{j=1}^{\infty} \lambda_j f_j(z)$, where $\lambda_j \geq 0$ and $\sum\limits_{j=1}^{\infty} \lambda_j = 1$.

Let $f_1(z) = z$ and

$$f_j(z) = z - \frac{1-\alpha}{(1+(j-1)\lambda)^{\beta+1}(1-\alpha+(j-1)\lambda)} z^j,\ j = 2, 3, ...$$

Then $f \in T^c L_{\beta}(\alpha)$ iff it can be expressed in the form $f(z) = \sum\limits_{j=1}^{\infty} \lambda_j f_j(z)$, where $\lambda_j \geq 0$ and $\sum\limits_{j=1}^{\infty} \lambda_j = 1$.

Proof. Let $f(z) = \sum\limits_{j=1}^{\infty} \lambda_j f_j(z)$, $\lambda_j \geq 0$, j=1,2, ... , with $\sum\limits_{j=1}^{\infty} \lambda_j = 1$. We obtain

$$f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z) = \lambda_1 z + \sum_{j=2}^{\infty} \lambda_j \left(z - \frac{1-\alpha}{[1+(j-1)\lambda]^{\beta}[1-\alpha+(j-1)\lambda]} z^j \right)$$

$$= \sum_{j=1}^{\infty} \lambda_j z - \sum_{j=2}^{\infty} \lambda_j \frac{1-\alpha}{[1+(j-1)\lambda]^{\beta}[1-\alpha+(j-1)\lambda]} z^j$$

$$= z - \sum_{j=2}^{\infty} \lambda_j \frac{1-\alpha}{[1+(j-1)\lambda]^{\beta}[1-\alpha+(j-1)\lambda]} z^j.$$

We have

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[1-\alpha+(j-1)\lambda]\lambda_j \frac{1-\alpha}{[1+(j-1)\lambda]^{\beta}[1-\alpha+(j-1)\lambda]}$$

$$= (1-\alpha)\sum_{j=2}^{\infty}\lambda_j = (1-\alpha)(\sum_{j=1}^{\infty}\lambda_j - \lambda_1) < 1-\alpha$$

which is the condition (2) for $f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z)$. Thus $f(z) \in TL_{\beta}(\alpha)$.

Conversely, we suppose that $f(z) \in TL_{\beta}(\alpha), f(z) = z - \sum_{j=2}^{\infty} a_j z^j, a_j \geq 0$ and we take $\lambda_j = \dfrac{[1+(j-1)\lambda]^{\beta}[1-\alpha+(j-1)\lambda]}{1-\alpha} a_j \geq 0$, j=2,3, ... , with $\lambda_1 = 1 - \sum_{j=2}^{\infty} \lambda_j$.

Using the condition (2), we obtain

$$\sum_{j=2}^{\infty}\lambda_j = \frac{1}{1-\alpha}\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[1-\alpha+(j-1)\lambda]a_j < \frac{1}{1-\alpha}(1-\alpha) = 1.$$

Then $f(z) = \sum_{j=1}^{\infty} \lambda_j f_j$, where $\lambda_j \geq 0$, j=1,2, ... and $\sum_{j=1}^{\infty} \lambda_j = 1$. This completes our proof.

The proof of the second part of the theorem is similarly with the above one, so it is omitted.

Corollary 1. *[1] , [2] The extreme points of $TL_{\beta}(\alpha)$ are $f_1(z) = z$ and*

$$f_j(z) = z - \frac{1-\alpha}{(1+(j-1)\lambda)^{\beta}(1-\alpha+(j-1)\lambda)} z^j,\ j = 2, 3, \ldots\ .$$

The extreme points of $T^c L_{\beta}(\alpha)$ are $f_1(z) = z$ and

$$f_j(z) = z - \frac{1-\alpha}{(1+(j-1)\lambda)^{\beta+1}(1-\alpha+(j-1)\lambda)} z^j,\ j = 2, 3, \ldots\ .$$

Remark 5. *We notice that in the particulary case, obtained for* $\lambda = 1$ *and* $\beta \in \mathbb{N}$, *we find similarly results for the class* $T_n(\alpha)$ *of the n-starlike functions of order* α *with negative coefficients (inclusive the necessary and sufficiently condition presented in Remark 2) and for the class* $T_n^c(\alpha)$ *of the n-convex functions of order* α *with negative coefficients .*

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