General Mathematics Vol. 13, No. 1 (2005), 55–66

Boundness of Cesàro Means Operators

Amelia Bucur

Dedicated to Professor Emil C. Popa for his sixtieth birthday

Abstract

The aim of this paper is presenting the evolution of the results regarding the boundeness of Cesàro operators.

2000 Mathematics Subject Classification: 05A40

1 Introduction

In this section we will be concerned with Jacobi series and Pollard′s result on uniform boundness of the partial sum operators of the Fourier expansion in Jacobi series.

Let α and β be two values with $\alpha, \beta > -1$. The Jacobi weight $w^{(\alpha,\beta)}$ is the function defined by $w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ for $x \in [-1, 1]$.

The Jacobi polynomials

$$p_j(x) = p_j^{(\alpha,\beta)}(x) = \gamma_j^{(\alpha,\beta)}(x) + ... + \delta_j^{(\alpha,\beta)} x^0, \; j \in \mathbb{N}$$

are the unique polynomials of precise degree j, with leading coefficients $\gamma_j^{(\alpha,\beta)} > 0$, fullfilling the orthonormal condition

$$\int_{-1}^{1} p_j(x)pk(x)w^{(\alpha,\beta)}(x)dx = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases}, \ \jmath, k \in \mathbb{N}$$

Let $f : [-1,1] \to \mathbb{R}$ be a function such that the Fourier coefficients

$$c_j(f) := c_j^{(\alpha,\beta)}(f) := \int_{-1}^{1} f(x)p_j(x)w^{(\alpha,\beta)}(x)dx, \quad j \in \mathbb{N} \tag{1}$$

exist.

The Jacobi series

$$\sum c_j(f)p_j \tag{2}$$

is the formal Fourier expansion of f in Jacobi polynomials.

The main concern is the convergence of the Jacobi series (2). To investigate the convergence, we define the partial sums of the Fourier expansion of f:

$$s_k^{(\alpha,\beta)}(f) := \sum_{j=0}^{k} c_j(f)p_j, \ k \in \mathbb{N}.$$

We are interested in Banach spaces B of functions $f : [-1,1] \to \mathbb{R}$, for which $s_k^{(\alpha,\beta)}f$ convergences to f, i.e.

$$||f - s_k^{(\alpha,\beta)}f||_B \to 0 \ \text{ as } \ k \to \infty \tag{3}$$

for all $f \in B$, where $||\cdot||_B$ denotes the norm of B. The convergence (3) is ensured if $\overline{\Pi} = B$ and the partial sum operators as $s_k^{(\alpha,\beta)}, k \in \mathbb{N}$, are uniformly bounded in B, namely

$$||s_k^{(\alpha,\beta)}f||_B \leq C||f||_B$$

for all $k \in \mathbb{N}$ and $f \in B$ (Π is space of algebraic polynomials).

One of the first results on uniform boundness was found in 1947 by Pollard [7]. Pollard determined a simple condition under which the partial sum operators of the Legendre series, which is the Jacobi series in the case $\alpha = \beta = 0$, are uniformly bounded in $B = L^p[-1,1]$. The precise results is stated in the following theorem.

Theorem 1. *(Pollard, 1947). If* $\frac{4}{3} < p < 4$*, then the Legendre Fourier operators* $s_n := s_n^{(\alpha,\beta)}$ *are uniformly bounded* $L^p[-1,1]$ *i.e.,*

$$||s_n f||_p \leq C_p ||f||_p$$

for all $n \in \mathbb{N}$ *and* $f \in L^p[-1,1]$ *with a positive constant* C_p *being independent of* f *and* n*.*

Then, in 1949, Pollard [8] generalized his result to include the Jacobi spaces $L^p_{w^{(\alpha,\beta)}}[-1,1]$ with $\alpha, \beta \geq -\frac{1}{2}$. Here $L^p_{w^{(\alpha,\beta)}}[-1,1], 1 \leq p \leq \infty$ denotes the space of all measurable functions $f : [-1,1] \to \mathbb{R}$ for which the weighted norm

$$||f||L^p_{w^{(\alpha,\beta)}}[-1,1] := \left(\int_{-1}^{1} |f(x)|^p w^{(\alpha,\beta)}(x)dx \right)^{1/p} \tag{4}$$

is finite.

Theorem 2. *(Pollard 1949) Let*

$$\widetilde{M}(\alpha,\beta) := 2\max\left\{ \frac{\alpha+1}{\alpha+\frac{3}{2}}, \frac{\beta+1}{\beta+\frac{3}{2}} \right\}$$

and

$$\widetilde{m}(\alpha,\beta) := 2\min\left\{ \frac{\alpha+1}{\alpha+\frac{1}{2}}, \frac{\beta+1}{\beta+\frac{1}{2}} \right\}$$

Suppose $\alpha, \beta \geq -\frac{1}{2}$, *then for values* p *with* $\widetilde{M}(\alpha,\beta) < p < \widetilde{m}(\alpha,\beta)$ *the Fourier projection operators* $s_n^{(\alpha,\beta)}$ *are uniformly bounded in* $L^p_{w^{(\alpha,\beta)}}[-1,1]$ *i.e.*

$$||s_n^{(\alpha,\beta)} f||_{L^p_w(\alpha,\beta)[-1,1]} \leq C||f||_{L^p_w(\alpha,\beta)[-1,1]}$$

holds for all $f \in L^p_{w^{(\alpha,\beta)}}[-1,1]$ *and* $n \in \mathbb{N}$ *with a positive constant* $C = C(\alpha,\beta,p)$ *being independent of* f *and* n.

Twenty years after Pollard′s results, Muckenhoupt [6] published in 1969 a theorem in which Pollard′s results is included. Muckenhoupt gave a comprehensive answer to the question as to when the Fourier projection operators $s_n^{(\alpha,\beta)}$ are uniformly bounded in $B = \{f | w^{a,b}(f) \in L^p[-1,1]\}$ Muckenhoupt′s results reads as follows.

Theorem 3. *(Muckenhoupt, 1969) Assume that* $\alpha, \beta > -1$, $1 < p < \infty$ *and* $a, b \in \mathbb{R}$ *such that*

$$\left|\frac{\alpha}{2} + \frac{1}{2} - \frac{1}{p} - a\right| < \min\left\{\frac{1}{4}, \frac{\alpha}{2} + \frac{1}{2}\right\} \tag{5}$$

$$\left|\frac{\beta}{2} + \frac{1}{2} - \frac{1}{p} - b\right| < \min\left\{\frac{1}{4}, \frac{\beta}{2} + \frac{1}{2}\right\} \tag{6}$$

Then

$$||w^{(a,b)} s_n^{(\alpha,\beta)} f||_p \leq C||w^{(a,b)} f||_p$$

for all $n \in \mathbb{N}$ *and* f *with* $w^{(a,b)} f \in L^p[-1,1]$, *where* $C = C(\alpha,\beta,a,b,p)$ *is a positive constant being independent of* f *and* n.

2 Boundness of Cesàro means operators

If the expansion of a function f in a Jacobi series fails to converge, we are then led to consider the Cesàro means (of first order)

$$\sigma_n^{(\alpha,\beta)}(f) := \frac{1}{n}\sum_{k=1}^{n} s_n^{(\alpha,\beta)} f,\ \ n\in\mathbb{N},$$

where $f : [-1,1] \to\to \mathbb{R}$ is assumed to be a function such that the Fourier coefficients (1) exists.

We main concern refers to Banach spaces B, consisting of functions $f : [-1,1] \to \mathbb{R}$, such that the Cesàro operators $\sigma_n^{(\alpha,\beta)}$, $n \in \mathbb{N}$, are uniformly bounded in B, i.e.,

$$||\sigma_n^{(\alpha,\beta)} f||_B \le C||f||_B$$

for all $n \in \mathbb{N}$ and $f \in B$.

One of the first results in this sense was found in 1963 by Askey and Hirshmann [1]. They proved, in the case $\alpha = \beta = 0$, the uniform boundness of the Cesàro operators in $B = L^p[-1,1]$.

Theorem 4. *(Askey & Hirschmann, 1963) If $1 \le p \le \infty$, then the Legendre Cesàro operators $\sigma_n := \sigma_n^{(0,0)}$ are uniformly bounded in $L^p[-1,1]$, i.e.,*

$$||\sigma_n f||_p \le C_p||f||_p$$

for all $n \in \mathbb{N}$ and $f \in L^p[-1,1]$ with a positive constant C_p being independent of f and n.

In 1994 Lubinsky and Totik [5] observed that for $\alpha, \beta > 0$ the Cesàro operators $\sigma_n^{(\alpha,\beta)}$ are uniformly bounded in $B = \left\{ f | w^{\left(\frac{\alpha}{2},\frac{\beta}{2}\right)} f \in L^p[-1,1] \right\}$ where the weight of B has halved indices α and β.

Theorem 5. *(Lubinsky & Totik, 1994) Let $\alpha, \beta > 0$ and $1 \le p \le \infty$. Then*

$$||w^{\left(\frac{\alpha}{2},\frac{\beta}{2}\right)} \sigma_n^{(\alpha,\beta)} f||_p \le C||w^{\left(\frac{\alpha}{2},\frac{\beta}{2}\right)} f||_p \tag{7}$$

holds for all $n \in \mathbb{N}$ and f with $w^{\left(\frac{\alpha}{2},\frac{\beta}{2}\right)} f \in L^p[-1,1]$, where $C = C(\alpha,\beta)$ is a positive constant being independent of f and n.

For proving Theorem 5, Lubinsky and Totik modified a method which goes back to G. Freud [4]. The method is based on a decomposition of the Cesàro operator $\sigma_n^{(\alpha,\beta)}$. They essentially considered the case $p = \infty$. For this purpose, Lubinsky and Totik introduced the following modified Jacobi weight

$$w_n^{(\alpha,\beta)}(x) := \left(\sqrt{1-x} + \frac{1}{n}\right)^{2\alpha} \left(\sqrt{1+x} + \frac{1}{n}\right)^{2\beta}$$

with $x \in [-1,1]$ and $n \in \mathbb{N}$. They proved

$$||w_n^{\left(\frac{\alpha}{2},\frac{\beta}{2}\right)} \sigma_n^{(\alpha,\beta)} f||_\infty \le C||w^{\left(\frac{\alpha}{2},\frac{\beta}{2}\right)} f||_\infty \tag{8}$$

for all $n \in \mathbb{N}$ and f with $w^{\left(\frac{\alpha}{2},\frac{\beta}{2}\right)} f \in L^p[-1,1]$. It should be noted that $w^{\left(\frac{\alpha}{2},\frac{\beta}{2}\right)}(x) \le w_n^{\left(\frac{\alpha}{2},\frac{\beta}{2}\right)}(x)$ for $x \in [-1,1]$, since $\alpha, \beta > 0$. Thus (8) is sharped than (7) with $p = \infty$. Then Lubinsky and Totik obtained the case $p = 1$ from (8) by the duality principle. Finally, by simple application of Riesz and Thorin′s interpolation principle, they proved the estimate

$$||w_n^{\left(\frac{\alpha}{2},\frac{\beta}{2}\right)} \sigma_n^{(\alpha,\beta)} f||_p \le C||w^{\left(\frac{\alpha}{2},\frac{\beta}{2}\right)} f||_p,$$

from which (7) follows, since $\alpha, \beta > 0$.

Lubinsky and Totik′s result was the starting point for investigation of M. Felton [3]. He determine conditions under which the Cesàro operators $\sigma_n^{(\alpha,\beta)}$ are uniformly bounded in $B = \left\{ f | w^{(\alpha,\beta)} f \in L^p[-1,1] \right\}$. He determine conditions for a and b such that the uniform estimate

$$||w^{(a,b)}\sigma_n^{(\alpha,\beta)} f||_p \leq C||w^{(a,b)} f||_p$$

holds true. This will be Lubinsky and Totik′s results if $a = \frac{\alpha}{2}$, $b = \frac{\beta}{2}$ and $\alpha, \beta > 0$. In 2004 M. Felton obtain a results which is similar to Muckenhount′s Theorem 3.

Theorem 6. *(see [3]). Let $\alpha, \beta \geq -\frac{1}{2}$, $1 \leq p \leq \infty$ and let $a, b \in \mathbb{R}$ such that $\sigma_n^{(\alpha,\beta)} : B \to B$ with $B = \{f | w^{(a,b)} f \in L^p[-1,1]\}$ and*

$$\left| \frac{\alpha}{2} + \frac{1}{4} - \frac{1}{2p} - a \right| < \frac{1}{2} \text{ and } \left| \frac{\beta}{2} + \frac{1}{4} - \frac{1}{2p} - b \right| < \frac{1}{2}$$

Then

$$||w^{(a,b)}\sigma_n^{(\alpha,\beta)} f||_p \leq C||w^{(a,b)} f||_p$$

is valid for all $f \in B$ and $n \in \mathbb{N}$, where $C = C(\alpha, \beta, a, b, p)$ is a positive constant being independent of f and n.

3 Cesàro Means and Riesz Means

In this section we introduce Riesz means as they as are defined in [3]. Riesz means are closely related to Cesàro means. The section ends with the result that Riesz means are uniformly bounded for appropiate choices of parameters.

Let $B := L^p_{w^{(a,b)}}[-1,1]$ be a fixed Jacobi space with $a, b > -1$ and $1 \leq p \leq \infty$. Moreover, let $\alpha, \beta > -1$ and $p_j = p_j^{(\alpha,\beta)}$, $j \in \mathbb{N}^*$, be the corresponding orthonormal Jacobi polynomials.

(Let $w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}, x \in [-1,1]$, be a Jacobi weight with $\alpha, \beta > -1$. The Jacobi polynomials

$$p_n(x) = p_n^{(\alpha,\beta)}(x) = \gamma_n^{(\alpha,\beta)} x^n + ... + \delta_n^{(\alpha,\beta)} x^0,\ n \in \mathbb{N}^*,$$

are the unique polynomials of precise degree n, with leading coefficients $\gamma_n^{(\alpha,\beta)} > 0$, fulfilling the orthonormal condition

$$\int_{-1}^{1} p_n(x) p_m(x) w^{(\alpha,\beta)}(x) dx = \begin{cases} 0, & \text{if } n \neq m \\ 1, & \text{if } n = m \end{cases},\ n, m \in \mathbb{N}^*).$$

Is known that $B = L^p_{w^{(a,b)}}[-1,1] \subset L^1_{w^{(\alpha,\beta)}}[-1,1]$ (see [3]).

Then the Fourier coefficients $c_k(f) = c_k^{(\alpha,\beta)}(f)$ and the partial sums

$$s_k f = s_k^{(\alpha,\beta)} f = \sum_{j=0}^{k} c_j(f) p_j \tag{9}$$

are defined for all $f \in B$. Let $P(D) = P^{(\alpha,\beta)}(D)$ be the Jacobi differential operator

$$P^{(\alpha,\beta)}(D) := (w^{(\alpha,\beta)})^{-1} \frac{d}{dx} w^{(\alpha+1,\beta+1)} \frac{d}{dx},$$

with both α and β are greater than -1.

Since the eigenfunction of $P(D)$ are the orthonormal Jacobi polynomials p_n, (9) can be understood as the partial sum of the expansion in the eigenfunctions of $P(D)$.

Let $\widetilde{\sigma_n} f = \widetilde{\sigma_n}^{(\alpha,\beta)} f$ be the Cesàro means of $f \in B$ defined by

$$\widetilde{\sigma_n} f := \frac{1}{n} \sum_{k=0}^{n-1} s_k f,\ n \in \mathbb{N}. \tag{10}$$

Thus, in (10), we add up from $k = 0$ to $k = n-1$. Hence $\widetilde{\sigma_n} f \in \Pi_{n-1}$ (space of algebraic polynomials of degree at most $n-1$). If we put (9) in (10), a rearrangement of the sum immediately gives

$$\widetilde{\sigma_n} f = \sum_{k=0}^{n} \left(1 - \frac{k}{n}\right) c_k(f) p_k,\ n \in \mathbb{N} \tag{11}$$

The eigenvalues of $P(D)$ are $-\lambda(n)$ with

$$\lambda(n) := \lambda^{(\alpha,\beta)}(n) := n(n+\alpha+\beta+1),\ n \in \mathbb{N}^* \tag{12}$$

that is

$$P(D)p_n = -\lambda(n)p_n \ (see\ [3]).$$

Definition 1. *Riesz means $R_n = R_n^{(\alpha,\beta)}$ are defined as*

$$R_n f := \sum_{k=0}^{n} \left(1 - \frac{\lambda(k)}{\lambda(n)}\right) c_k(f) p_k,\ n \in \mathbb{N} \tag{13}$$

for $f \in B$.

Thus Riesz means are defined in a similar way as the Cesàro means, except that the term $\left(1 - \frac{k}{n}\right)$ in (11) is replaced by $\left(1 - \frac{\lambda(k)}{\lambda(n)}\right)$ to obtain (13). Riesz means $R_n f$ are polynomials of degree at most $n-1$, i.e., $R_n f\ in \Pi_{n-1}$.

The following lemma shows that Riesz means can be represented via Cesàro means. The proof of the following lemma follows Totik′s idea.

Lemma 1. *Let $R_n = R_n^{(\alpha,\beta)}$ and $\widetilde{\sigma_n} = \widetilde{\sigma_n}^{(\alpha,\beta)}$ be the Riesz and Cesàro means (13) and (11) respectively. Moreover, let $\lambda(n) = \lambda^{(\alpha,\beta)}(n)$ as they are in (12). Then*

$$R_n = \left(1 - \frac{n(n+1)}{\lambda(n)}\right)\widetilde{\sigma_n} - \frac{2}{\lambda(n)} \sum_{k=1}^{n} k\widetilde{\sigma}_k$$

for $k \in \mathbb{N}$.

Proof. From

$$\sum_{k=0}^{n-1} (2k+2+\alpha+\beta) s_k f = \sum_{k=0}^{n-1} \sum_{j=0}^{k} (2k+2+\alpha+\beta) c_j(f) p_k =$$

$$= \sum_{j=0}^{n-1} \left\{ \sum_{k=j}^{n-1} (2k+2+\alpha+\beta) \right\} c_j(f) p_k =$$

$$= \sum_{j=0}^{n-1} \underbrace{(n+\alpha+\beta+1+j)(n-j)}_{\lambda(n)-\lambda(j)} c_j(f) p_k =$$

$$= \lambda(n) \sum_{j=0}^{n-1} \left(1 - \frac{\lambda(j)}{\lambda(n)}\right) c_j(f) p_k$$

are the definition of R_n in (13) we obtain

$$\sum_{k=0}^{n-1} (2k+2+\alpha+\beta) s_k = \lambda(n) R_n \tag{14}$$

Now $\sum\limits_{k=0}^{n-1} (k+1)\widetilde{\sigma}_{k+1} = \sum\limits_{k=0}^{n-1} \sum\limits_{j=0}^{k} s_j = \sum\limits_{j=0}^{n-1} \sum\limits_{k=j}^{n-1} s_j = \sum\limits_{j=0}^{n-1} (n-j) s_j$ yields

$$\sum_{k=0}^{n-1} (2n-2k) s_k = 2 \sum_{k=1}^{n} k \widetilde{\sigma}_k \tag{15}$$

Addition of (14) and (15) gives

$$(2n+2+\alpha+\beta) \underbrace{\sum_{k=0}^{n-1}}\ s_k \quad_{n\widetilde{\sigma_n}} = \lambda(n) R_n + 2 \sum_{k=1}^{n} k \widetilde{\sigma}_k$$

and hence

$$(\lambda(n) + n(n+1)) \widetilde{\sigma_n} = \lambda(n) R_n + 2 \sum_{k=1}^{n} k \widetilde{\sigma}_k,$$

which proves the statement of lemma 1.

Theorem 7. *Let* $\alpha, \beta \geq -\frac{1}{2}$ *and* $B = L^p_{w^{(a,b)}}[-1,1]$ *with* $a, b > -1$ *and* $1 \leq p \leq \infty$ *such that*

$$\begin{cases} \pi \subset B \subset L^1_{w^{(\alpha,\beta)}}[-1,1] \\ \left|\frac{\alpha}{2} + \frac{1}{4} - \frac{1}{2p} - \frac{a}{p}\right|, \left|\frac{\beta}{2} + \frac{1}{4} - \frac{1}{2p} - \frac{b}{p}\right| < \frac{1}{2} \ \text{if } 1 \leq p < \infty \\ \left|\frac{\alpha}{2} + \frac{1}{4} - a\right|, \left|\frac{\beta}{2} + \frac{1}{4} - b\right| < \frac{1}{2} \ \text{if } p = \infty \end{cases} \tag{16}$$

Then Riesz means $R_n^{(\alpha,\beta)}$, defined in (13), are uniformly bounded in B, i.e.,

$$||R_n^{(\alpha,\beta)} f||_B \leq C||f||_B$$

for all $f \in B$ and $n \in \mathbb{N}$ with a positive constant $C = C(\alpha, \beta, a, b, p)$ being independent of f and n.

Proof. The inclusions $\pi \subset B \subset L^1_{w^{(\alpha,\beta)}}[-1, 1]$ in (16) ensure that the rule of assignment $\widetilde{\sigma_n}^{(\alpha,\beta)} : B \to B$ is satisfied. Since $\alpha, \beta \geq -\frac{1}{2}$ and (16) is fulfilled, it follows from Theorem 5.7 ([3]) that he Cesàro operators $\widetilde{\sigma_n}^{(\alpha,\beta)}$ are uniformly bounded in B, i.e.,

$$||\widetilde{\sigma_n}^{(\alpha,\beta)} f||_B \leq C||f||_B$$

for $f \in B$ and $n \in \mathbb{N}$ with $C = C(\alpha, \beta, a, b, p) > 0$. from Lemma 1 we therefore obtain

$$||R_n^{(\alpha,\beta)} f||_B \leq C \left\{ 1 + \frac{\lambda(n+1)}{\lambda^{(\alpha,\beta)}(n)} + \frac{2}{\lambda^{(\alpha,\beta)}(n)} \sum_{k=1}^{n} k \right\} ||f||_B \leq$$

$$\leq C \left\{ 1 + 2\frac{n+1}{n+1+\alpha+\beta} \right\} ||f||_B \leq 5C||f||_B$$

for $f \in B$ and $n \in \mathbb{N}$.

Problem. Are these calcules still valid for the case in which the Cesàro means are replaced with generalized Cesàro means?

References

[1] R. Askey, I. I. Hirschmann Jr., *Mean summability for ultraspherical polynomials*, Math. Scand., 12 (1963), 167 - 177.

[2] S. Chamillo, B. Muckenhoupt, *Weak type estimates for Cesàro sums of Jacobi polynomials series*, Mem. Amer. Math. Soc., 102 (1993), no. 487, viii, 90.

[3] M. Felten, *Boundedness of First Order Cesàro Means in Jacobi Spaces and Weighted Approximation on [-1, 1]*, Seminarberichte aus dem Fachbereich Mathematik der FernUniversität in Hagen, Band 75 - 2004.

[4] G. Freud, *Orthogonale Polynome*, Birkhäuser Verlag, Basel, 1969, Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften, Mathematische Reihe, Band 33.

[5] D. S. Lubinsky, V. Totik, *Best weighted polynomial approximation via Jacobi expansions, SIAM, J. Math. Anal., 25 (1994), no. 2, 550 - 570.*

[6] B. Muckenhoupt, *Mean convergence of Jacobi series*, Proc. Amer. Math. Soc. 23 (1969), 306 - 310.

[7] H. Pollard, *The mean convergence of orthogonal series I*, Trans. Amer. Math. Soc., 62 (1947), 387 - 403.

[8] H. Pollard, *The mean convergence of orthogonal series II*, Trans. Amer. Math. Soc. 63 (1948), 355 - 367.

"Lucian Blaga" University of Sibiu
Department of Mathematics
Str. Dr. I. Raţiu, No. 5-7
550012 - Sibiu, Romania
E-mail address: amelia.bucur@ulbsibiu.ro