General Mathematics Vol. 12, No. 4 (2004), 29–38

From Finsler Geometry to Noncommutative Geometry

Laurian Pişcoran

Abstract

One link between Finsler geometry and noncommutative geometry is given by a differential operator, which is called the Dirac operator. In this paper, we construct such an operator and we analyze some of its properties. Also, in this paper is presented an elocvent example. After that we continued the construction of this operator for the case of Randers spaces, which are some particular example of Finsler spaces.

AMS Subject Classification: 41A60, 41A15

Keywords and phrases: Dirac-Finsler operator

1 Preliminaries

Definition A Finsler space is a real differential manifold endowed with a norm bundle with some properties, so for any $p \in M$, we have a norm over

T_pM :

$$||...||_p : T_pM \to \mathbb{R}$$

$$\xi_p \to ||\xi_p||_p$$

Next, we will note with ξ_p the elements of T_pM and with α_p or ξ_p^* the elements of the cotangent space.

We construct the map

$$G_p : T_pM \times T_pM \to \mathbb{R}$$

$$(\xi_p, \eta_p) \to \ G_p(\xi_p, \eta_p)$$

with:

$$G_p(\xi_p, \eta_p) = \frac{1}{4}(||\xi_p + \eta_p||_p^2 - ||\xi_p - \eta_p||_p^2)$$

This is a nondegenerate application in a Finsler space, in the sense that this application define a Finsler duality:

$$\Gamma_p : T_pM \to T_p^*M$$

defined by: $\Gamma_p(\xi_p) = G_p(\xi_p, \eta_p)$, $\xi_p \to \Gamma_p(\xi_p)$.

In this way, we define a norm in the cotangent space:

$$||...||_p^* : T_pM \to \mathbb{R}$$

$$\xi_p^* \to ||\xi_p^*||_p^*$$

where: $||\xi_p^*||_p^* = ||\xi_p||_p$, so we have: $||\xi_p^*||_p^*$=$||\Gamma_p^{-1}(\xi_p^*)||_p$.

All of this, can be extended for maps, vector fields and forms, $C^\infty(M)$, $\chi(M)$, $\Omega^1(M)$:

$$||...||_p : \chi(M) \to \ C^\infty(M)$$

$$X \to \chi(X)$$

where $||X||_p(p) = ||X_{|_p}||_p$. We also have

$$G : \chi(M) \times \chi(M) \to \ C^{\infty}(M)$$

$$(X, Y) \to G(X, Y)$$

where $G(X, Y)(p) = G_p(X_{|_p}, Y_{|_p})$.

Also,we have Finsler duality:

$$\Gamma : \chi(M) \to \ \Omega^1(M)$$

$$X \to \Gamma(X)$$

where: $\Gamma(X)(Y) = G(X, Y)$, so, we have: $\Gamma(X)_{|_p}(\eta_p) = G_p(X_{|_p}, \eta_p)$. Finaly, we have:

$$||...||_p^* : \Omega^1(M) \to \ C^{\infty}(M)$$

$$\alpha \to ||\alpha||^*$$

where $||\alpha||^*(p) = ||\alpha_{|_p}||_p^*$ and $||\alpha||^* = ||\Gamma^{-1}(\alpha)||$.

Next, we will remember some well known notions about differentials operators:

Let be given two bundles (E, M) and (F, M) over the same manifold M , and we consider thecoresponding section of this bundles $\Gamma(M, E)$ and $\Gamma(M, F)$.

Next, we consider:

$$Hom_{\mathbb{R}}(\Gamma(M, E), \Gamma(M, F)) \equiv L_{\mathbb{R}}(E, F)$$

and

$$Hom_{\ C^\infty(M)}(\Gamma(M,E),\Gamma(M,F)) \equiv L_{\ C^\infty(M)}(E,F)$$

Let $f \in C^\infty(M)$ and we define:

$$\underline{f}\ :\ \Gamma(M,E) \to \Gamma(M,F)$$

$$u \to uf$$

and we consider: $ad(f)(D) = \underline{f}\ D - D\underline{f}$, so we have:

$$ad(f)(D) : \Gamma(M,E) \to \Gamma(M,F)$$

$$u \to (\underline{f}\ D - D\underline{f}\ )(u)$$

and also:

$$ad(f)(D) : L_{\mathbb{R}}(E,F) \to L_{\mathbb{R}}(E,F)$$

$$D \to ad(f)(D)$$

and

$$ad(f)(D)_{|_p} : E_p \to F_{|_p}$$

$$u_p \to [(Du)f]_{|_p} - [D(uf)]_{|_p}$$

. **Properties:**

1) $ad(f_1) \circ ad(f_2) = ad(f_2) \circ ad(f_1)$

2) for $f, g \in C^\infty(M)$ with $df = dg$ we have: $ad(f) = ad(g)$

3) for $f_1,...,f_k, g_1, ..., g_k \in C_R^\infty(M)$ with $df_i = dg_i$, we have:

$ad(f_1) \circ ... \circ ad(f_k) = ad(g_1) \circ ... \circ ad(g_k)$

Let consider now the set: $d(\ C^{\infty}(M)) \subseteq \Omega^1(M)$ of exact forms and $P \in Diff^{(k)}(E,F)$. We define:

$$\sigma_k(P) : d(\ C^{\infty}(M)) \to L_{\mathbb{R}}(E,F)$$

$$df \to \frac{(-1)^k}{k!}\zeta(df,...,df)$$

In particular we start with: $D \in Diff^k(E,F) \subset L_{\mathbb{R}}(E,F)$, where D: $\Gamma(M,E) \to \Gamma(M,F)$, $u \to Du$ and if $D \in Diff^k(E,F)$, we have $\sigma_k(P) \neq 0$, we say that the operator D has order k.

If $P \in Diff^{(k)}(E,F)$ and $Q \in Diff^{(k')}(F,G)$, we have $P{\circ}Q \in Diff^{k+k'}(E,G)$ and also $\sigma_{k+k'}(Q \circ P) = \sigma_k(Q) \circ \sigma_{k'}(P)$.

Definition We say that the operator $P \in Diff^{(2)}(E,F)$ is a generalized Laplacian, if $\sigma_2(P)(f) = (||df||^*)^2 \cdot 1_{\Gamma(M,E)}$

2 Dirac Finsler Operator

Let consider now some notions about C*-algebras

Definition: A right module over a C*-algebra is a set endowed with two aplications: the sum and an external operation over E , the right multiplication with scalars, such that:

1) $(E,+)$ is an Abelian group

2) $(x+y)\lambda = x\lambda + y\lambda$, $x(\lambda+\mu) = x\lambda + x\mu$, $(x\mu)\lambda = x(\mu\lambda)$ with $x, y \in E$

Definition: Let A be a C*-algebra and A_+ be the positive elements of the algebra. An $A-$Finsler module, is an module E, endowed with an aplication $\rho : E \to A_+$, with the following properties:

1) $||.||_E, x \to \rho(x)$, is a Banach norm over E

2) $\rho^2(ax) = a\rho^2(x)a^*$, where $a \in A$, $x \in E$

We have: $||ax||_E^2 = ||\rho^2(ax)||_E^2 = ||a\rho^2(x)a^*||_E^2 \leq ||a||_E^2||x||_E^2$

Let (E, M, π) be an Banach module over an $A-$Finsler module E.

We define now a map $c : \Gamma(M,E) \otimes \Omega^1(A) \to \Gamma(M,E)$ and also we define a connection :

$$\bigtriangledown : \Gamma(M,E) \to \Gamma(M,E) \otimes \Omega^1(A)$$

$$\bigtriangledown(as) = a \bigtriangledown s + s \otimes da$$

With this notations we can define the Dirac Finsler operator:

$$D : \Gamma(M,E) \to \Gamma(M,E),$$

$$D = c \circ \bigtriangledown,$$

where $\Gamma(M,E)$ represent the section of the Banach bundle.

We say that D is a Dirac operator if D^2 is a generalized Laplacian.

If we consider now, two bundle $(E,M), (F,M)$ over the Finsler manifold M, we say that $P \in Diff^2(E,F)$ is a generalized laplacian, if $\sigma_2(P)(df) = (|||df|||^*)^2 \cdot 1_{\Gamma(M,E)}$.

If we have a covariant derivative $\overline{\bigtriangledown}$ over the bundle (E,M), with

$$\overline{\bigtriangledown} : \Gamma(M,E) \to \Gamma(M,E) \otimes_{\ C^\infty(M)} \Omega^1(A)$$

and if we have a Clifford Finsler action, i.e. a map: $\gamma : \Gamma(M,E) \otimes \Omega^1(A) \to \Gamma(M,E)$. Then, we can define Dirac Finsler operator $D : \Gamma(M,E) \to \Gamma(M,E)$ with $D = \gamma \circ \overline{\bigtriangledown}$

Example:

On $\mathbb{R}$ we define the following euclidian-Finsler norm:

$$|||\xi|||_x = |\xi| + sgn(x\xi)x\xi = |\xi|(1 + sgn(x)x) = |\xi|(1 + |x|) = |\xi|m(x),$$

where we make the notation $m(x) = 1 + |x|$.

We consider now the following identifications: $\chi(\mathbb{R}) \equiv C^\infty(\mathbb{R})$, $\Omega^1(\mathbb{R}) \equiv C^\infty(\mathbb{R})$.

We have: $G_x(\xi, \eta) = m^2(x)\xi\eta$, i.e. $G(X, Y) = m^2XY$ with $X, Y \in C^\infty(\mathbb{R})$

The following Finsler duality holds

$$\Gamma : \chi(\mathbb{R}) \to \Omega^1(\mathbb{R}),$$

$$X \to \Gamma(X)$$

and if we consider $\Gamma(X)$ as an element from $C^\infty(\mathbb{R})$ one obtain: $\Gamma(g)(X) = G_x(g(x), 1) = m^2(x)g(x)$.

Also, we have:

$$\Gamma^{-1} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$$

$$g \to m^{-2}g$$

Observe that: $||g||^* = ||\Gamma^{-1}(g)|| = ||m^{-2}g|| = m|m^{-2}g| = m^{-1}|g|$, so one obtain $(||g||^*)^2 = m^{-2}g^2$.

We can define a cvasi-Clifford action by:

$$\chi(\mathbb{R}) \otimes_{C^\infty(M)} \Omega^1(\mathbb{R}) \simeq C^\infty(\mathbb{R}) \to \chi(\mathbb{R}) \simeq C^\infty(M)$$

with $\gamma : C^\infty(M) \to C^\infty(M)$, $g \to m^{-1}$and it follows that: $(\gamma \circ \gamma)(g) = (m^{-1}g)^2 = m^{-2}g^2 =(||g||^*)^2$.

Let us consider the covariant derivative:

$$\overline{\bigtriangledown}: \ C^{\infty}(\mathbb{R}) \to \ C^{\infty}(\mathbb{R})$$

$$g \to g'.$$

In this way we obtain the Dirac Finsler operator $D = \gamma \circ \overline{\bigtriangledown}$ where:

$$D: \ C^{\infty}(M) \to \ C^{\infty}(M)$$

$$g \to m^{-1}g'$$

and we have $\sigma_1(D)(df) = m^{-1}f'1_{\ C^{\infty}(\mathbb{R})}$, and after computation one obtain:

$$\sigma_2(D^2)(df) = m^{-2}(f')^2 1_{\ C^{\infty}(\mathbb{R})}$$

Also we have: $(||df||^*)^2 = m^{-2}(f')^2$ and after that we obtain: $\sigma_2(D^2)(df) = (||df||^*)^2$ which means that D is a generalized Laplacian.

Proposition. *The Dirac Finsler operator defined above is an differential eliptic operator.*

Proof. Using the definition given by Narasimhan ([1]), we can easily verify the following property:from $D(f) = D(g)$ we obtain: $\gamma(\overline{\bigtriangledown}(f)) = \gamma(f^{\cdot}) = \gamma(g^{\cdot}) = \gamma(\overline{\bigtriangledown}(g))$ and using the fact that D is a generalized Laplacian, we obtain that D is an elliptic operator.

In the case of Randers spaces, with the norm: $F(x, \xi) = \sqrt{g_{ij}\xi_i\xi_j} + b_i(x)\xi_i$, where g_{ij} are the coeficients of a Riemann metric g and $b_i(x)$ are the coeficients of an 1-form, we can construct the Dirac Finsler operator in the following way: first, we consider $F(x, \xi) = \xi_i \left(\sqrt{g_{ij}\frac{\xi_i}{\xi_j}} + b_i \right) = \xi_i \cdot m(x, \xi)$, where $m(x, \xi) = \sqrt{g_{ij}\frac{\xi_i}{\xi_j}} + b_i$.

Second, we have: $G_x(\xi,\eta) = m^2(x)\xi\eta$, i.e. $G(X,Y) = m^2XY$ with $X,Y \in \mathcal{F}(M)$.

Then :

$$\Gamma : \chi(\mathbb{R}) \to \Omega^1(\mathbb{R})$$

$$X \to \Gamma(X)$$

with: $\Gamma(g)(X) = G_x(g(x),1) = m^2(x)g(x)$.

Also, we obtain:

$$\Gamma^{-1} : \ C^\infty(M) \to \ C^\infty(M)$$

$$g \to m^{-2}g$$

Then: $||g||^* = ||\Gamma^{-1}(g)|| = ||m^{-2}g|| = m|m^{-2}g| = m^{-1}|g|$, so one obtain $(||g||^*)^2 = m^{-2}g^2$.

We can define:

$$\gamma : \ C^\infty(M) \to \ C^\infty(M)$$

$$g \to m^{-1}g$$

and one obtain: $(\gamma \circ \gamma)(g) = (m^{-1}g)^2 = m^{-2}g^2 =(||g||^*)^2$.

Let us consider the covariant derivative:

$$\overline{\bigtriangledown} : \ C^\infty(M) \to \ C^\infty(M)$$

$$f \to \overline{\bigtriangledown}(f).$$

We obtain the Dirac Finsler operator $D = \gamma \circ \overline{\bigtriangledown}$ and we say that he is a Dirac type operator if it is a generalized Laplacian. For the case of Randers spaces, in local coordinate we obtain:

$$\gamma = \left(\frac{1}{\sqrt{g_{ij}\frac{\xi_i}{\xi_j}} + b_i} \right) g$$

$$D = \left(\frac{1}{\sqrt{g_{ij} \frac{\xi_i}{\xi_j}} + b_i} \right) \overline{\nabla}(g)$$

.

References

[1] Almeida, P., *Introduction to noncommutative geometry*, Lisabona 2001

[2] Narasimhan, R., *Analiză pe varietăţi reale şi complexe*, Ed. Theta 2001

[3] Shen, Z., *An introduction to Riemann Finsler geometry*, Ed. Springer Verlag, 2000

[4] Varilly, J., *Elements of noncommutative geometry*, Ed. Birkhauser 2001