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REMARKS ON RESTRICTION EIGENFUNCTIONS IN $\mathbf{C}^n$ *

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Dedicated to Ed Saff on the occasion of his 60th birthday

Abstract. An elementary inquiry, based on examples and counterexamples, of some qualitative properties of doubly orthogonal systems of analytic functions on domains in $\mathbf{C}^n$ leads to a better understanding of the deviation from the classical Hardy space of the disk setting. The main results relay on Hilbert space with reproducing kernel techniques.

Key words. Hilbert space with reproducing kernel, restriction operator, doubly orthogonal system, min-max principle

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1. Introduction. Let Ω be a bounded domain of $\mathbf{C}^n$ and let $\mathcal{H}(\Omega)$ be a Hilbert space of analytic functions defined on Ω, with reproducing kernel

$$K(z,w) = \langle k_w, k_z \rangle, \;\; z, w \in \Omega.$$

In other terms, the point evaluation at $z \in \Omega$ is continuous in the norm of $\mathcal{H}(\Omega)$ and:

$$f(z) = \langle f, k_z \rangle, \;\; f \in \mathcal{H}(\Omega).$$

The norm of $\mathcal{H}(\Omega)$ will simply be denoted $\|f\|$ or $\|f\|_\Omega$ when necessary.

Let μ be a positive measure, compactly supported by Ω. The restriction operator

$$R : \mathcal{H}(\Omega) \longrightarrow L^2(\mu), \;\; f \mapsto f|_{\mathrm{supp}\mu},$$

is then compact by Montel's Theorem. The modulus square operator

$$R^*R : \mathcal{H}(\Omega) \longrightarrow \mathcal{H}(\Omega)$$

is positive, self-adjoint and even has a finite trace. Its spectrum is discrete and can be arranged into decreasing order:

$$\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_k \cdots \to 0.$$

The associated eigenfunctions $f_k, k \geq 0$, form a doubly orthogonal system of functions, in the spaces $\mathcal{H}(\Omega)$ and the closed range $\mathrm{Ran}R$ of R, endowed with the norm of $L^2(\mu)$. By convention, suppμ means the closed support of the measure μ.

The eigenvalues of R^*R can be characterized by the min-max principle:

$$\lambda_k = \min_{\mathrm{codim}V=k} \max_{f \in V\setminus\{0\}} \frac{\|f\|_\mu^2}{\|f\|^2}.$$

This explains their importance in the best approximation theory (in the $L^2(\mu)$ norm with control on the $\mathcal{H}(\Omega)$ norm) and in estimating the N-widths of such spaces of analytic functions. Most of the references at the end of this note illustrate various works, old and new, related to these concepts.

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While the asymptotic estimate of the decay of the eigenvalues λ_k has received considerable attention, the qualitative properties of the eigenfunctions f_k, very similar by their definition to orthogonal polynomials, is much less understood. In this respect, only two cases stand out: restrictions from the Hardy space, and from the Bergman space, both of a simply connected planar domain with smooth boundary, see [5] respectively [7]. Based on the analysis of these two single complex variable situations we raise a few natural questions about the behavior of the eigenfunctions f_k in general, and show by simple means (reproducing kernel identities and some perturbation theory) what it is reasonable not to expect from them in $\mathbf{C}^n, n > 1$.

2. Preliminaries. We explore below a few examples, in one or several complex variables. They will provide the basis for our intuition and a starting point for our discussion. The notation is the same as in the Introduction.

The eigenvalue equation for f_k can be written as $R^*Rf_k = \lambda_k f_k$, or better, after evaluating this on a function $h \in \mathcal{H}(\Omega)$:

$$\int f_k \overline{h} d\mu = \lambda_k \langle f_k, h \rangle_\Omega.$$

In particular this implies an integral equation for f_k:

$$f_k(z) = \frac{1}{\lambda_k} \int f_k(u) K(z,u) d\mu(u), \ \ z \in \Omega. \tag{2.1}$$

In most concrete cases the reproducing kernel $K(z,u)$ extends analytically to a neighborhood of $\overline{\Omega}$ when u runs over a compact subset of Ω and thus each eigenfunction f_k will share this property. A second derivation of the eigenfunction identity is obtained by taking $h = f_k g$ and then passing to real parts:

$$\int \Re g |f_k|^2 d\mu = \lambda_k \Re \langle f_k, f_k g \rangle_\Omega. \tag{2.2}$$

We assume of course that g is a bounded multiplier of the space $\mathcal{H}(\Omega)$.

EXAMPLE. Restriction from $H^2(\mathbf{D})$.

This is the case analyzed by Fisher and Micchelli [5]. The outline of proof below is reproduced from [7].

Via the proper normalization we can assume

$$K(z,w) = \frac{1}{1 - z\overline{w}}, \ \ z, w \in \mathbf{D}.$$

The eigenfunction equation (2.1) reads then:

$$f_k(z) = \frac{1}{\lambda_k} \int \frac{f_k(u) d\mu(u)}{1 - z\overline{u}},$$

and consequently every f_k is analytically extendable to the same neighborhood of the closed disk. Equation (2.2) implies

$$\int u |f_k|^2 d\mu = \lambda_k \int_{\partial \mathbf{D}} u(\zeta) |f_k(\zeta)|^2 \frac{d\zeta}{2\pi i \zeta},$$

for every function u harmonic in the disk and continuous on the closed disk. The latter equation can be interpreted as a balayage of the measure $|f_k|^2 d\mu$ to the boundary of the disk. Thus,

$$f_k(\zeta) \neq 0, \ \ |\zeta| = 1, k \geq 0.$$

Consequently the multiplicity of each λ_k is exactly one, otherwise a linear combination $f_k + \alpha f_k'$ of two distinct eigenfunctions corresponding to λ_k would vanish on the unit circle. By deforming continuously the measure $d\mu$ to the area measure supported by a concentric disk $r\mathbf{D}$, say along the path $td\mu + (1-t)\chi_{r\mathbf{D}} d\text{Area},\ t \in [0,1]$, proves then that each f_k has exactly k zeros in the disk.

These observations have far reaching implications. For instance, an optimal subspace in the min-max computation of the k-th eigenvalue is analytically invariant, generated by f_k as an analytic module: $V_k = f_k H^2,\ \text{codim} V_k = k$ and

$$\max_{f \in H^2} \frac{\|f f_k\|_\mu^2}{\|f f_k\|_2^2} \lambda_k.$$

Thus, the zeros of f_k give the optimal configuration $a_1, \ldots, a_k$ of k points in the disk, such that there are complex coefficients $c_1, \ldots, c_k$ and the quotient

$$\frac{\|f - \sum_{j=1}^k c_j f(a_j)\|_\mu^2}{\|f\|_2^2}$$

is minimal among all other choices of k points and k weights.

In the above considerations the disk can be replaced by any simply connected bounded planar domain with a sufficiently regular boundary.

EXAMPLE. Restriction from $L_a^2(\mathbf{D})$.

This is the case of the Bergman space of the disk, that is the Hilbert space of analytic functions in the disk which are square integrable with respect to the area measure (henceforth denoted dA). The corresponding reproducing kernel is, up to normalizations: $K(z,w) = (1 - z\overline{w})^{-2}$.

Again μ is a positive measure supported by a compact subset of the open disk $\mathbf{D}$ and $R : L_a^2(\mathbf{D}) \longrightarrow L^2(\mu)$ is the restriction operator. An eigenfunction f_k of R^*R satisfies:

$$f_k(z) = \frac{1}{\lambda_k} \int \frac{f(u) d\mu(u)}{(1 - z\overline{u})^2},$$

and, exactly as before,

$$\int u |f_k|^2 d\mu = \lambda_k \int_{\mathbf{D}} u(\zeta) |f_k(\zeta)|^2 dA(\zeta),$$

for every harmonic function u.

This is no longer a classical balayage identity, and it is not true in general that f_k is free of zeros on the boundary of the disk, see the example in [7]. However, using the positivity of the Green function of the bi-Laplacian and some positivity properties of the orthogonal projection in $L^2(\mathbf{D}, dA)$ onto harmonic functions one can prove that $f_k(\zeta) \neq 0,\ |\zeta| = 1,\ k \geq 0$, whenever $\text{supp}\mu \subset (\sqrt{2} - 1)\mathbf{D}$, see [7]. Then the scenario of the Hardy space framework holds word by word.

We will see below a simple explanation why f_0 may acquire zeros in the Bergman space setting.

EXAMPLE. Restriction between two Reinhardt domains.

Let ω, Ω be a pair of bounded Reinhardt domains in $\mathbf{C}^n$ and let us assume that ω is relatively compact in Ω. The associated Bergman spaces, with respect to the $2n$-dimensional Lebesgue measure dv give rise to a compact restriction operator:

$$R : L_a^2(\Omega) \longrightarrow L_a^2(\omega),$$

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whose modulus square R^*R produces a discrete spectrum $\lambda_\alpha,\ \alpha \in \mathbf{N}^n$, accumulating to zero. It is more convenient to label the spectrum by multi-indices $\alpha \in \mathbf{N}^n$ because the eigenfunctions are obviously given by the monomials z^α:

$$R^*Rz^\alpha = \lambda_\alpha z^\alpha,\ \ \alpha \in \mathbf{N}^n.$$

A simple explanation being the double orthogonality of z^α's with respect to the two norms, see for instance [14].

Two pathologies, seen as deviations from the $H^2(\mathbf{D})$ scenario, can easily be derived from this example.

Namely, take for instance $\Omega = B$ to be the unit ball in $\mathbf{C}^n$ and $\omega = rB$ a concentric ball of radius $r < 1$. Then the eigenfunction equation yields:

$$\lambda_\alpha \int_B |z^\alpha|^2 dv = \int_{rB} |z^\alpha|^2 dv.$$

Therefore

$$\lambda_\alpha = r^{2n+2|\alpha|},\ \alpha \in \mathbf{N}^n.$$

Thus *the eigenvalues* λ_α *depend only on* $|\alpha|$ *and have multiplicities*. On the other hand, the eigenfunctions z^α, $|\alpha| = k$, corresponding to a fixed eigenvalue have finitely many common zeros, or in other terms

$$\sum_{|\alpha|=k} |z^\alpha|^2 > 0,\ \ z \neq 0.$$

In this case the optimum in the min-max criterion is achieved on an analytically invariant subspace:

$$\lambda_\alpha = \max_{f \in M^k} \frac{\|f\|^2_{rB}}{\|f\|^2_B},\ \ |\alpha| = k.$$

Above M denotes the maximal ideal of functions vanishing at the origin.

Second, let us consider the unit polydisk $\Omega = \mathbf{D}^n$ and $\omega = (r_1\mathbf{D}) \times \ldots \times (r_n\mathbf{D})$ with radii $r_1, \ldots, r_n < 1$ and such that $\log r_1, \ldots, \log r_n$ are linearly independent over the rationals. The eigenfunction equation implies:

$$\lambda_\alpha \int_{\mathbf{D}^n} |z^\alpha|^2 dv \int_\omega |z^\alpha|^2 dv,$$

whence

$$\lambda_\alpha = r_1^{2+2\alpha_1} \cdots r_n^{2+2\alpha_n},\ \alpha \in \mathbf{N}^n.$$

These numbers are all distinct due to the independence assumption. Thus *all eigenvalues are simple* and, except for $z^0 = 1$ *all eigenfunctions* $z^\alpha,\ |\alpha| > 0$*, have infinitely many zeros* in the polydisk Ω. Note also that the optimal subspaces in the min-max criterion:

$$V_\alpha = \vee\{z^\beta;\ \lambda_\beta \leq \lambda_\alpha\},\ \ \alpha \in \mathbf{N}^n,$$

are analytically invariant. This is due to the fact that multiplication of an element $f \in V_\alpha$ by any $z_j,\ 1 \leq j \leq n$, is contractive with respect to both norms, in particular:

$$\langle R^*R(z_jf), z_jf\rangle = \int_\omega |z_jz^\alpha|^2 dv \leq$$

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$$\int_\omega |z^\alpha|^2 dv = \langle R^* R(f), f\rangle.$$

Along the same lines, a simple solution to an inverse problem, which singles out the class of Reinhardt domains, is available.

THEOREM 2.1. *Let ω be a relatively compact domain in the unit ball $\mathbf{B} \subset \mathbf{C}^n$. Assume that ω is pseudoconvex and has a C^2-smooth boundary.*

If the monomials $z^\alpha,\ \alpha \in \mathbf{N}^n$, are the eigenfunctions of the restriction operator $R : L^2_a(\mathbf{B}) \longrightarrow L^2_a(\omega)$ and $\ker R^* = 0$, then ω is a complete Reinhardt domain.

Proof. The technical assumption $\ker R^* = 0$ implies that the monomials are dense in $L^2_a(\omega)$. Therefore the associated Bergman kernel has the form:

$$K_\omega(z, w) = \sum_\alpha c_\alpha z^\alpha \overline{w}^\alpha,$$

where $c_\alpha \geq 0,\ \alpha \in \mathbf{N}^n$.

On the other hand, the Bergman kernel of the unit ball is

$$K_{\mathbf{B}}(z, w) = \frac{1}{[1 - \langle z, w\rangle]^{n+1}}.$$

An osculation with inner balls, tangent at boundary points of ω, and the variational interpretation of $K_\omega(z, z)$ show that

$$K_\omega(z, z) \sim \operatorname{dist}(z, \partial\omega)^{-n-1},$$

when z tends to a point of $\partial\omega$. Hence $\rho(z) \ln K_\omega(z, z)$ is a plurisubharmonic exhausting function for ω. Since

$$\rho(e^{it_1} z_1, \dots, e^{it_n} z_n)\rho(z),$$

whenever $z = (z_1, \dots, z_n) \in \omega$ and $t_1, \dots, t_n \in \mathbf{R}$, we find that $z \in \omega$ whenever $(e^{it_1} z_1, \dots, e^{it_n} z_n) \in \omega,\ \ t_1, \dots, t_n \in \mathbf{R}$. That is, ω is a complete Reinhardt domain. ☐

Due to the regularity of the boundary assumption, condition $\ker R^* = 0$ is fulfilled for every Runge domain ω. In the case of a single complex variable, a stronger form of the above theorem holds, see [7].

3. Reproducing kernel computations. This section contains some simple derivations of the reproducing kernel formula and the existence of doubly orthogonal systems of analytic functions.

Let Ω be as before a bounded domain in $\mathbf{C}^n$ and let $\mathcal{H}(\Omega)$ be a Hilbert space of analytic functions on Ω, with reproducing kernel $K(z, w)$. We consider a positive measure μ, compactly supported by Ω, and the restriction operator $R : \mathcal{H} \longrightarrow L^2(\mu)$. The eigenfunctions of R^*R will be denoted by $f_k, k \geq 0$, and will be normalized by the condition $\|f_k\|_{\mathcal{H}} = 1$. The associated eigenvalues are denoted by λ_k. We do not exclude here the possibility of a finite atomic measure, in which case $\lambda_k = 0$ for large k.

Note first a direct application of Mercer's theorem (see for instance [20]):

$$\text{Trace } R^*R = \sum_{k=0}^{\infty} \lambda_k = \int K(z, z) d\mu(z).$$

Since

$$K(z, a) = \langle k_a, k_z\rangle = \sum_{k=0}^{\infty} f_k(z)\overline{f_k(a)},$$

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we infer

$$\langle R^*Rk_a, k_a\rangle = \sum_k \lambda_k |f_k(a)|^2,$$

whence

$$\int \frac{|K(z,a)|^2}{K(a,a)} d\mu(z) = \sum_k \lambda_k \frac{|f_k(a)|^2}{K(a,a)}. \tag{3.1}$$

We can regard the right hand side of the latter equation as a limit of convex combinations of the eigenvalues. In general, for a point $\zeta \in \partial\Omega$, one has $\lim_{a\to\zeta} K(a,a) = \infty$, due to the extremality property of the reproducing kernel. Thus the above identity will imply $\sum_{k\geq k_0} |f_k(a)|^2 > 0$ for every k_0 and a close to the boundary of Ω. In order to make this statement more precise, we will consider below the case of the unit ball $\mathbf{B}$ in $\mathbf{C}^n$.

Let $L^2_a(\mathbf{B})$ be the Bergman space of the unit ball in $\mathbf{C}^n$, with the volume measure normalized so that the associated reproducing kernel is

$$K(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1}},$$

see for instance [14]. Let μ be a probability measure, with support included in the closed ball $\rho\overline{\mathbf{B}}$, $\rho < 1$.

Let $\lambda_k > \lambda_{k+1}$ be two distinct and consecutive eigenvalues of the operator R^*R. Then

$$\int \frac{|K(z,a)|^2}{K(a,a)} d\mu(z) \leq \frac{(1-|a|^2)^{n+1}}{(1-\rho|a|)^{2n+2}},$$

converges to zero as $|a|$ tends to 1. Denote

$$\tau \frac{(1-|a|^2)^{n+1}}{(1-\rho|a|)^{2n+2}},$$

and assume that $\tau < \lambda_k$.

Equation (3.1) implies

$$\tau \geq \lambda_k \sum_{j\leq k} \frac{|f_j(a)|^2}{K(a,a)} \lambda_k (1 - \sum_{p=1}^{\infty} \frac{|f_{k+p}(a)|^2}{K(a,a)}).$$

We obtain the following result.

PROPOSITION 3.1. *Let μ be a positive measure supported by the ball $\rho\mathbf{B}$, with $\rho < 1$. For every $k > 1$, let $r = r_k$ be the positive solution of the equation:*

$$\frac{(1-r^2)^{n+1}}{(1-\rho r)^{2n+2}} = \lambda_k.$$

Then

$$(1-|a|^2)^{n+1} \sum_{p=1}^{\infty} |f_{k+p}(a)|^2 \geq 1 - \frac{(1-|a|^2)^{n+1}}{\lambda_k (1-\rho|a|)^{2n+2}},$$

for every $a \in \mathbf{B}$ satisfying $|a| > \max(r_k, \rho)$.

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For the proof we use the above estimates and a simple computation showing that the function

$$r \mapsto \frac{(1-r^2)^{n+1}}{(1-\rho r)^{2n+2}}$$

is decreasing for $r > \rho$.

This shows in particular that, *for every $k \geq 0$, the eigenfunctions f_{k+p}, $p \geq 1$ cannot have common zeros close to the boundary of the ball.* Note also that each f_m is an analytic function in the ball $\rho^{-1}\mathbf{B}$.

Known boundary estimates of the Bergman kernel (see for instance [8]) allow to extend the above proposition to any strictly pseudoconvex domain with C^2 boundary in $\mathbf{C}^n$.

4. Restriction to finite atomic measures. Except for a few situations, it is very hard to obtain explicit computation of the eigenfunctions of the modulus of restriction operator considered in this note. One of the fortunate cases is the restriction from a Hilbert space of analytic functions to the Lebesgue space of a finite, atomic measure. This framework leads to linear algebra manipulations of finite combinations of the reproducing kernel, see also [6].

To fix ideas, we consider as before a Hilbert space of analytic functions $\mathcal{H}(\Omega)$ with reproducing kernel $K(z,w)$ and a finite atomic measure μ, supported by the set supp$(\mu) = \{a_1, a_2, \ldots, a_m\}$:

$$\mu = \gamma_1 \delta_{a_1} + \cdots + \gamma_m \delta_{a_m},$$

with $\gamma_j > 0$.

According to (2.1) we find:

$$f_k(z) = c_{1k} K(z, a_1) + \cdots + c_{mk} K(z, a_m), \;\; 1 \leq k \leq m,$$

for some complex constants c_{jk}. The eigenvalue equations are then reduced to a finite linear system:

$$\lambda_k c_{jk} = \sum_{p=1}^{m} \gamma_j K(a_j, a_p) c_{pk}, \;\; 1 \leq j, k \leq m.$$

A simple observation derived from this equation is contained in the following.

PROPOSITION 4.1. *Assume that the positive measure μ has finite support $\{a_1, \ldots, a_m\}$ and that the operator R^*R has a single non-zero eigenvalue. Then, and only then,*

$$K(a_i, a_j) = 0, \;\; 1 \leq i < j \leq m.$$

Proof. Indeed, if the spectral space corresponding to the unique non-zero eigenvalue λ of R^*R has maximal dimension, then each $K(z, a_i)$ is an eigenfunction. Hence

$$\lambda K(z, a_i) = \sum_j K(z, a_j) K(a_j, a_i) \gamma_j,$$

for some constants γ_j, and the linear independence of the elements $K(z, a_i)$ yields

$$\lambda = \gamma_i K(a_i, a_i),$$

and

$$K(a_j, a_i)\gamma_j = 0, \;\; i \neq j. \qquad \square$$

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Note that the most common reproducing kernels K of standard domains Ω in $\mathbf{C}^n$ do not have zeros, in the sense $K(z,w) \neq 0$ for all $z, w \in \Omega$.

Consider next a finite group G of complex bi-holomorphic maps of Ω which leaves invariant the inner product of the space $\mathcal{H}(\Omega)$ and the measure μ. Specifically, this means:

$$K(gz, gw) = K(z,w), \;\; g \in G, \; z, w \in \Omega,$$

and $g_*\mu = \mu, g \in G$. Then the equation

$$\lambda_k f_k(gz) = \int f_k(w) K(gz, w) d\mu(w) =$$

$$\int f_k(gw) K(gz, gw) d\mu(w) = \int f_k(gw) K(z,w) d\mu(w),$$

implies that $f_k(gz)$ is also an eigenfunction for the eigenvalue λ_k. Thus we have proved

LEMMA 4.2. *Assume that a finite group G leaves invariant the reproducing kernel $K(z,w)$ and the measure μ. Then each eigenspace of the operator R^*R is invariant under the action of G.*

To give a simple example, consider the unit ball $\mathbf{B}$ in $\mathbf{C}^n$ and the Szegö kernel $S(z,w) = [1 - \langle z, w\rangle]^{-n}$. This kernel is invariant under all biholomorphic maps of the ball. Let us consider the simplest situation of a symmetric pair of points $\{a, -a\} \subset \mathbf{B}$ and the measure $\mu = \delta_a + \delta_{-a}$. The group $G = \{\pm 1\}$ leaves invariant both the kernel and the measure, hence, by the above lemma, it leaves invariant the eigenspaces of R^*R. On the other hand, $S(a, -a) \neq 0$, therefore the operator R^*R has exactly two simple non-zero eigenvalues. Let

$$f(z) = \alpha S(z,a) + \beta S(z, -a) = \alpha S(z,a) + \beta S(-z, a),$$

be one eigenfunction. Then according to the lemma $f(-z)$ is also an eigenfunction corresponding to the same eigenvalue. Thus, the two eigenfunctions of R^*R are:

$$f_\pm(z) = S(z,a) \pm S(z,-a).$$

A direct computation identifies the two eigenvalues as

$$S(a,a) \pm S(a,-a).$$

See also the more invariant argument below.

Next we restrict these computations to the case $n = 3$.

EXAMPLE. Let $R : H^2(\mathbf{B}) \longrightarrow \mathbf{L}^2(\mu)$ be the restriction operator between the Hardy space of the unit ball in $\mathbf{C}^3$ and the Lebesgue space of the two point mass measure $\mu = \delta_a + \delta_{-a}, a \in \mathbf{B}$.

Then the eigenfunctions of the rank-two operator R^*R are:

$$f_+(z) = \frac{1 + 3\langle z, a\rangle^2}{[1 - \langle z, a\rangle^2]^3},$$

and

$$f_-(z) = \frac{\langle z, a\rangle[3 + \langle z, a\rangle^2]}{[1 - \langle z, a\rangle^2]^3}.$$

In particular each function $f_\pm$ admits infinitely many zeros in the ball.

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Let n now be arbitrary and let $K(z,w)$ be a positive definite kernel in the ball which remains invariant under all holomorphic automorphisms of the ball and which is the reproducing kernel of a Hilbert space H_K of analytic functions in the ball. Let p be a prime number, and let ϵ be a p-root of unity. By taking a point $a \in \mathbf{B} \subset \mathbf{C}^n$ and the measure $\mu = \sum_{k=1}^{p} \delta_{\epsilon^k a}$ we obtain a restriction operator

$$R : H_K \longrightarrow L^2(\mu)$$

of rank p. According to the above lemma, each eigenspace V of R^*R is invariant under the group $\mathbf{Z}_p$. Hence either each element f of V is left invariant under the action of the group, that is $f(z) = \mathrm{const} f(\epsilon z)$, or V is p dimensional. The latter scenario is excluded by an adaptation of Proposition (4.1).

Therefore the eigenfunctions f of R^*R satisfy the identity:

$$f(z) = \sum_{j=1}^{p} c_j K(z, \epsilon^j a) = C \sum_{j=1}^{p} c_j K(\epsilon z, \epsilon^j a),$$

where C is a constant. Whence

$$C c_j = c_{j-1}, \quad 1 \le j \le p,$$

with the convention $c_0 = c_p$.

From here we deduce:

$$f_k(z) K(z,a) + \epsilon^k K(z, \epsilon a) + \epsilon^{2k} K(z, \epsilon^2 a) + \cdots + \epsilon^{(p-1)k} K(z, \epsilon^{p-1} a).$$

The eigenfunction equation reads:

$$\lambda_k f_k(z)$$
$$= K(z,a) f_k(a) + \epsilon^k K(z, \epsilon a) \epsilon^{(p-1)k} f_k(\epsilon a) + \cdots + \epsilon^{(p-1)k} K(z, \epsilon^{p-1} a) \epsilon^k f_k(\epsilon^{p-1} a),$$

and since

$$f_k(a) = \epsilon^{(p-1)k} f_k(\epsilon a) = \cdots \epsilon^k f_k(\epsilon^{p-1} a),$$

we infer

$$\lambda_k = f_k(a), \quad 0 \le k \le p-1.$$

With a little more algebra, these values and functions are computable. We give below a simple example.

EXAMPLE. Let a be a point of the unit ball $\mathbf{B}$ in $\mathbf{C}^2$ and let $S(z,w) = [1 - \langle z, w \rangle]^{-2}$ be the Szegö kernel of the ball. Let $\epsilon = e^{2\pi i/3}$ be the root of order 3 of unity. We consider the positive measure $\mu = \delta_a + \delta_{\epsilon a} + \delta_{\epsilon^2 a}$ and the restriction operator

$$R : H^2(\mathbf{B}) \longrightarrow L^2(\mu).$$

In view of the above computations the non-zero eigenfunctions of the operator R^*R are:

$$f_k(z) = \frac{1}{(1 - \langle z, a \rangle)^2} + \epsilon^k \frac{1}{(1 - \langle z, \epsilon a \rangle)^2} + \epsilon^{2k} \frac{1}{(1 - \langle z, \epsilon^2 a \rangle)^2}, \quad k = 0, 1, 2.$$

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An elementary computation leads to the following closed forms:

$$f_0(z) = \frac{3 + 6\langle z, a\rangle^3}{(1 - \langle z, a\rangle^3)^2},$$

$$f_1(z) = \frac{6\langle z, a\rangle + 3\langle z, a\rangle^4}{(1 - \langle z, a\rangle^3)^2},$$

$$f_2(z) = \frac{9\langle z, a\rangle^2}{(1 - \langle z, a\rangle^3)^2}.$$

The associated eigenvalues are:

$$\frac{3 + 6\|a\|^6}{(1 - \|a\|^6)^2} > \frac{6\|a\|^2 + 3\|a\|^8}{(1 - \|a\|^6)^2} > \frac{9\|a\|^4}{(1 - \|a\|^6)^2}.$$

All other eigenvalues are equal to zero.

Again, this shows that f_0 can have zeros in the ball. The kernel in the latter example can be reduced to one complex variable, in which case it becomes the Bergman kernel of the unit disk: $K(z, w) = [1 - z\overline{w}]^{-1}$. Thus, mutas mutandis, the functions f_0, f_1, f_2 are doubly orthogonal with respect to the Bergman space metric of $L^2_2(\mathbf{D})$ and $L^2(\delta_a + \delta_{\epsilon a} + \delta_{\epsilon^2 a})$. In particular f_0 can have zeros in the disk as soon as $|a| > 2^{-1/3}$.

The following proposition is derived from the same context. Henceforth we denote by X_1 the unit sphere of a normed space X.

PROPOSITION 4.3. *Let a be a point in the unit ball $\mathbf{B}$ of $\mathbf{C}^n$. Let $R : H^2(\mathbf{B}) \longrightarrow L^2(\delta_a+\delta_{-a})$ be the restriction operator. For every $p \geq 0$ there exists an analytically invariant subspace $V_p \subset H^2(\mathbf{B})$ of codimension p, such that*

$$\max_{f \in (V_p)_1} \|Rf\|^2 = \lambda_p(R^*R).$$

Proof. Let $S_w(z) = S(z, w) = [1 - \langle z, w\rangle]^{-n}$ be the Szegö kernel of the ball, with respect to the normalized area measure (so that the total mass of the sphere is 1). The operator R^*R has rank 2 and, by the preceding computations its eigenfunctions and eigenvalues are:

$$R^*R(S(\cdot, a) \pm S(\cdot, -a)) = [S(a, a) \pm S(a, -a)](S(\cdot, a) \pm S(\cdot, -a)).$$

For the largest eigenvalue, $\lambda_0 = S(a, a) + S(a, -a)$, the whole space $V_0 = H^2$ is obviously analytic and optimal in min-max. Similarly, V_2 can be chosen to be the analytically invariant subspace $M_{\pm a}$ of all functions vanishing at $\pm a$. And beyond that, since $\lambda_p = 0$ we can choose $V_p \subset V_2,\ p \geq 2$, to be any analytically invariant subspace of codimension p.

We will prove that V_1 can be chosen to be the space M_0 of analytic functions vanishing at 0. That is, knowing that

$$\langle R^*Rf, f\rangle = \int |f|^2(\delta_a + \delta_{-a}) = |f(a)|^2 + |f(-a)|^2,$$

we have to verify the inequality:

$$|f(a)|^2 + |f(-a)|^2 \leq [S(a, a) - S(a, -a)]\|f\|^2_{H^2},\ \ f \in H^2(\mathbf{B}),\ f(0) = 0. \tag{4.1}$$

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In order to prove this inequality we decompose:

$$M_0 = \mathbf{C}(S_a + S_{-a} - 2) \oplus \mathbf{C}(S_a - S_{-a}) \oplus (M_0 \cap M_{\pm a}).$$

Note that this decomposition is also orthogonal with respect to the metric of the space $L^2(\delta_a + \delta_{-a})$. Thus, it is sufficient to verify the inequality on each summand.

For a function $f \in M_0 \cap M_{\pm a}$ there is nothing to prove. In the case $f = S_a - S_{-a}$ the inequality is an equality, due to the eigenfunction equation. Finally, for $f = S_a + S_{-a} - 2$ we compute directly:

$$|f(a)|^2 + |f(-a)|^2 = 2(S(a,a) + S(a,-a) - 2)^2,$$

and

$$\|S_a + S_{-a} - 2\|^2 = 2[S(a,a) + S(a,-a) - 2].$$

Now the arithmetic/geometric mean inequality yields:

$$S(a,a) + S(a,-a) = \frac{1}{(1-\|a\|^2)^n} + \frac{1}{(1+\|a\|^2)^n} \geq \frac{2}{(1-\|a\|^4)^{n/2}} > 2.$$

Thus inequality (4.1) is equivalent to:

$$2(S(a,a) + S(a,-a) - 2)^2 \leq [S(a,a) - S(a,-a)]2(S(a,a) + S(a,-a) - 2),$$

or, after simplifications:

$$S(a,-a) = \frac{1}{(1+\|a\|^2)^n} \leq 1,$$

which is obviously true. $\square$

As a matter of fact the point 0 in the above proof can be replaced by any element $c \in \mathbf{B}$ which is orthogonal to a. Indeed, denoting by M_c the space of functions in H^2 vanishing at the point c, we have the analogous double orthogonal decomposition, with respect to the metrics of H^2 and $L^2(\delta_a + \delta_{-a})$:

$$M_c = \mathbf{C}(S_a + S_{-a} - 2) \oplus \mathbf{C}(S_a - S_{-a}) \oplus (M_c \cap M_{\pm a}), \;\; c \perp a.$$

The rest of the proof remains unchanged. Thus *a whole variety of analytically invariant subspaces of codimension* 1 *is optimal for the min-max computation of* λ_1.

5. Optimal subspaces in min-max. The simple computations of the last section invite us to a have a closer look at the optimal subspaces in the min-max principle applied to the modulus of the restriction operator R^*R.

We start by an elementary, and most likely known, remark.

PROPOSITION 5.1. *Let A be a non-negative compact operator acting on a Hilbert space H, with eigenvalues $\lambda_j(A)$ arranged in decreasing order. Assume that $\lambda_p(A) < \lambda_{p-1}(A)$ for some $p \geq 1$. Let f_p be a corresponding eigenvector: $Af_p\lambda_p(A)f_p$. Let V be a subspace of codimension p, optimal in min-max, that is*

$$\sup_{x \in (V)_1} \langle Ax, x \rangle \lambda_p(A). \tag{5.1}$$

Then $f_p \in V$.

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Proof. Assume by contradiction that the eigenvector $f = f_p$ does not belong to the optimal subspace V. If f is orthogonal to V, then the subspace $W = V \oplus \mathbf{C}f$ still satisfies (5.1) and has codimension $p-1$. But this contradicts, again via the min-max principle, the assumption $\lambda_p(A) < \lambda_{p-1}(A)$.

Assume that f is not orthogonal to V, and let g be the orthogonal projection of f onto V. Then $\langle g, f\rangle \neq 0$, and there exists a constant γ such that $\gamma g - f \perp f$. Consequently

$$\langle A(\gamma g - f + f),\ \gamma g - f + f\rangle \leq \lambda_p(A)(\|\gamma g - f\|^2 + \|f\|^2).$$

On the other hand, the orthogonality condition and the eigenvector equation satisfied by f imply

$$\langle A(\gamma g - f + f),\ \gamma g - f + f\rangle \langle A(\gamma g - f),\ \gamma g - f\rangle + \langle Af, f\rangle.$$

Hence

$$\langle A(\gamma g - f + f),\ \gamma g - f + f\rangle \leq \lambda_p(A)\|\gamma g - f\|^2.$$

By similar computations we infer that the vector $w = \gamma(g - f) = \gamma g - f + (1-\gamma)f$ satisfies

$$\langle Aw, w\rangle \leq \lambda_p(A)\|w\|^2.$$

And in addition $w \perp V$. Thus we can repeat the argument at the beginning of the proof, this time for the space $W = \mathbf{C}w \oplus V$, and obtain a contradiction. $\square$

A general result, proved for instance in [12], shows that every analytically invariant subspace V of finite codimension p in the Bergman (or Hardy) space H of the ball, or a domain with strictly pseudoconvex smooth boundary, consists of all functions vanishing at p points, taking into account multiplicities. To be more precise about multiplicities, such a space can always be written as

$$V = (P_1, P_2, \ldots, P_k)H,$$

where $P_1, P_2, \ldots, P_k$ are polynomials having all zeros in the ball and satisfying the codimension property

$$\dim \mathbf{C}[z]/(P_1, P_2, \ldots, P_k)\mathbf{C}[z] = p,$$

see [12] for details.

By putting together the above two facts we can state the following theorem.

THEOREM 5.2. *Let μ be a positive measure, compactly supported by a domain $\Omega \subset \mathbf{C}^n$ with smooth, strictly pseudoconvex boundary. Let $R : L^2_a(\Omega) \longrightarrow L^2(\mu)$ be the restriction operator. Assume that $\lambda_p(R^*R) < \lambda_{p-1}(R^*R)$ and that an analytically invariant subspace V of codimension p is optimal in the min-max computation of λ_p. Then there are polynomials $P_1, P_2, \ldots, P_k$ having p common zeros, all contained in Ω, such that:*

$$V = P_1 L^2_a(\Omega) + \cdots + P_k L^2_a(\Omega).$$

Moreover, every $\lambda_p(R^*R)$-eigenfunction $f \in L^2_a(\Omega)$ of R^*R can be written as:

$$f = P_1 g_1 + \cdots + P_k g_k, \quad g_j \in L^2_a(\Omega).$$

In other terms, the above observation implies that all eigenfunctions corresponding to $\lambda_p < \lambda_{p-1}$ have common zeros in Ω, as soon as there exists an analytically invariant subspace of codimension p, optimal in the min-max criterion.

It would be interesting to find an example of a positive measure μ, with the property that the restriction operator $R : L^2_a(\Omega) \longrightarrow L^2(\mu)$ exhibits the following pathology: there are eigenfunctions of R^*R associated to a single higher eigenvalue $\lambda_k, k > 0$ and without common zeros in Ω.

6. Perturbation methods. It is legitimate to ask whether the restrictions to finite atomic measures considered in the previous sections are not too extreme as natural examples. We show in this section how a rather general perturbation theory argument can replace finitely many points measures by volume measures supported by open regions, without altering the qualitative properties of the first eigenfunctions of the modulus of the restriction operator. A standard reference for the perturbation theory of linear operators we refer to below is Kato's book [10].

To fix ideas we will work with the Bergman space $L^2_a(\mathbf{B})$ of the unit ball $\mathbf{B}$ in $\mathbf{C}^n$, but a variety of other Hilbert spaces of analytic functions, with a reproducing kernel, can be considered. The $2n$-dimensional Lebesgue measure in $\mathbf{C}^n$ will be denoted by dv; we denote by $B_r(a)$ the open ball centered at a, of radius r. The $2n$-volume of a set A is denoted by $|A|$.

LEMMA 6.1. *Let $a \in \mathbf{B}$ be a fixed point, and let $r < \operatorname{dist}(a, \partial\Omega)$. The function*

$$r \mapsto \frac{1}{|B_r(a)|} \int_{B_r(a)} |f|^2 dv,$$

is increasing for every $f \in L^2_a(\mathbf{B})$.

Proof. The proof follows from the orthogonality of the monomials z^α with respect to the volume measure of the unit ball. Specifically, write

$$f(z) = \sum_{\alpha \in \mathbf{N}^n} c_\alpha (z-a)^\alpha,$$

and remark that

$$\frac{1}{|B_r(a)|} \int_{B_r(a)} |f|^2 dv \sum_{\alpha} |c_\alpha|^2 \frac{\alpha!\, n!}{(|\alpha|+n)!} r^{2|\alpha|}. \qquad \square$$

Moreover, for any $f \in L^2_a(\mathbf{B})$ we have:

$$\lim_{r \longrightarrow \infty} \frac{1}{|B_r(a)|} \int_{B_r(a)} |f|^2 dv = |f(a)|^2,$$

and even

$$\frac{1}{|B_r(a)|} \int_{B_r(a)} f dv = f(a).$$

We will use these simple facts for comparing the restriction operators associated to the point mass δ_a and the measure $\sigma_{a,r} = \frac{1}{|B_r(a)|} \chi_{B_r(a)} dv$. To be more precise, denote by

$$R_r : L^2_a(\mathbf{B}) \longrightarrow L^2(\sigma_{a,r}), \;\; r \geq 0$$

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the restriction operator, by adopting the convention $\sigma_{a,0} = \delta_a$. Then the above lemma implies

$$\langle R_r^* R_r f, f\rangle \geq \langle R_s^* R_s f, f\rangle,$$

whenever $r \geq s$ and for all $f \in L_a^2(\mathbf{B})$. Thus $R_r^* R_r \to R_0^* R_0$ when $r \to 0$, in the weak operator topology. As a matter of fact, this convergence is much stronger.

LEMMA 6.2. *Under the above notation,*

$$\lim_{r\to 0} \mathrm{trace}(R_r^* R_r - R_0^* R_0) = 0.$$

Proof. For a fixed $r \geq 0$, the operator $R_r^* R_r$ has finite trace due to the identity

$$\langle R_r^* R_r f, f\rangle = \frac{1}{|B_r(a)|}\int_{B_r(a)} |f|^2 dv,$$

and the nuclearity of $\int_{B_r(a)} |f|^2 dv$ with respect to the Bergman space norm. In addition, the operator $R_0^* R_0$ has rank-one, with the constant function 1 in its range.

Let $\epsilon > 0$ be small and choose an orthonormal basis $e_0 = 1, e_k, k \geq 1$, of $L_a^2(\mathbf{B})$. There is N depending on ϵ such that

$$\sum_{k>N} \langle R_r^* R_r e_k, e_k\rangle < \epsilon.$$

Then the same inequality holds for every $s \leq r$. On the other hand, for a fixed m,

$$\lim_{s\to 0} \langle R_s^* R_s e_m, e_m\rangle = \langle R_0^* R_0 e_m, e_m\rangle.$$

Therefore there exists $r_\epsilon > 0$ small enough with the property

$$\mathrm{trace}(R_r^* R_r - R_0^* R_0) \sum_{k\leq N} \langle (R_r^* R_r - R_0^* R_0) e_k, e_k\rangle + \sum_{k>N} \langle R_r^* R_r e_k, e_k\rangle < 2\epsilon,$$

for all $r \leq r_\epsilon$. □

The same argument applies to any finite atomic measure instead of a point mass. We state it as a separate proposition.

PROPOSITION 6.3. *Let*

$$\mu_0 = c_1\delta_{a_1} + \cdots + c_m\delta_{a_m}$$

be a finite atomic positive measure supported by the unit ball of $\mathbf{C}^n$. *For* $r < \mathrm{dist}(\{a_1, \ldots, a_m\}, \partial\mathbf{B})$ *one defines the measures:*

$$\mu_r = c_1 \frac{1}{|B_r(a_1)|}\chi_{B_r(a_1)} dv + \cdots + c_m \frac{1}{|B_r(a_m)|}\chi_{B_r(a_m)} dv,$$

and the associated restriction operators $R_r : L_a^2(\mathbf{B}) \longrightarrow L^2(\mu_r)$. *Then* $R_r^* R_r$ *is decreasing as a function of* r *and*

$$\lim_{r\to 0} \mathrm{trace}(R_r^* R_r - R_0^* R_0) = 0.$$

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From here we can derive the convergence of the spectral decompositions of the operators $R_r^* R_r$. For instance, assume, under the notation introduced in the proposition, that the spectrum of the limit operator is:

$$\sigma(R_0^* R_0) = \{0\} \cup \{\lambda_0, \ldots, \lambda_p\},$$

with

$$\lambda_0 > \lambda_1 > \cdots > \lambda_p > 0.$$

Since $\|(z - R_r^* R_r) - (z - R_0^* R_0)\| \to 0$, for every

$$\epsilon < \frac{1}{2} \min\{\lambda_0, \lambda_1 - \lambda_0, \ldots, \lambda_p - \lambda_{p-1}\}$$

there exists $r_\epsilon > 0$ with the property that the set

$$\sigma(R_r^* R_r) \cap (\lambda_k - \epsilon, \lambda_k + \epsilon),\ r \le r_\epsilon,$$

has the cardinality equal to the multiplicity of the eigenvalue λ_k of $R_0^* R_0$. Thus, by Dunford's functional calculus, the corresponding spectral projections converge in the operator norm:

$$\begin{aligned}\chi_{(\lambda_k - \epsilon, \lambda_k + \epsilon)}(R_r^* R_r) &= \frac{1}{2\pi i} \int_{|z - \lambda_k|\epsilon} (z - R_r^* R_r)^{-1} dz \\ &\longrightarrow \frac{1}{2\pi i} \int_{|z - \lambda_k|\epsilon} (z - R_0^* R_0)^{-1} dz = \chi_{(\lambda_k - \epsilon, \lambda_k + \epsilon)}(R_0^* R_0).\end{aligned}$$

See for instance [10] for such arguments.

We illustrate by a single application which contrasts to the Hardy space (of the disk) picture.

PROPOSITION 6.4. *There exists a relatively compact, open subset of the unit disk $U \subset \mathbf{D}$, such that the restriction operator between the associated Bergman spaces $R : L_a^2(\mathbf{D}) \longrightarrow L_a^2(U)$ has the following property. The largest eigenvalue of $R^* R$ is simple and the associated eigenfunction has zeros in the disk.*

Proof. We appeal to the comment following Example 4.4, and approximate the three point measure there as in the above proposition. Namely, fix a point $a \in \mathbf{D},\ |a| > 2^{-1/3}$ and consider the eigenfunction

$$f_0(z) = \frac{3 + 6\langle z, a\rangle^3}{(1 - \langle z, a\rangle^3)^2},$$

corresponding to the highest eigenvalue λ_0 of $R^* R$, where

$$R : L_a^2(\mathbf{D}) \longrightarrow L^2(\delta_a + \delta_{\epsilon a} + \delta_{\epsilon^2 a}).$$

Here we denote by ϵ the root of order three of unity.

Thus, λ_0 is a simple eigenvalue, and f_0 has zeros inside the disk. By enlarging the points to concentric disks, as in the proposition, we construct a sequence of operators $R_r^* R_r$, $r < \rho$, with the highest eigenvalue $\lambda_0(R_r^* R_r)$ being simple and converging to λ_0. The positive constant ρ is chosen sufficiently small. Then, for a fixed small $\epsilon > 0$,

$$f_0(r) = \frac{1}{2\pi i} \int_{|z - \lambda_0|\epsilon} (z - R_r^* R_r)^{-1} f_0 dz$$

is, up to a constant, the only non-trivial $\lambda_0(R_r^* R_r)$-eigenfunction of $R_r^* R_r$ and

$$\lim_{r\to 0} \|f_0(r) - f_0\|_{2,\mathbf{D}} = 0.$$

In conclusion the eigenfunctions $f_0(r)$ are all vanishing for r small enough. □

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