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$\mathbb{C}^*$-ACTIONS ON $\mathbb{C}^3$ ARE LINEARIZABLE

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ABSTRACT. We give the outline of the proof of the linearization conjecture: every algebraic $\mathbb{C}^*$-action on $\mathbb{C}^3$ is linear in a suitable coordinate system.

1. INTRODUCTION

The purpose of this note is to outline the main ingredients in a proof of the following

Linearization Theorem. *Every algebraic action of the torus $T = \mathbb{C}^*$ on affine space $X = \mathbb{C}^3$ is linearizable, that is linear in suitably chosen coordinates for X.*

It is known that the action has a fixpoint $0 \in X$ ([B-B]). The *weights* of the action are the weights

$$a_1,\ a_2,\ a_3$$

of the (diagonalized) action on the tangent space T_0X. (They are independent of the choice of fixpoint [KbR].) We will assume tacitly that the action is effective, or, equivalently, that $\mathrm{GCD}(a_1, a_2, a_3) = 1$. Put

$$\delta = \dim X//T, \quad \tau = \dim X^T.$$

Then $2 \geq \delta \geq \tau \geq 0$.

It is known that *fixpointed* actions, that is those for which all weights have the same sign, are linearizable [KbR]. This settles the following cases:

$\delta = 0 = \tau$, or three nonzero weights of the same sign;

$\delta = 1 = \tau$, or one zero weight, two nonzero weights of the same sign;

$\delta = 2 = \tau$, or two zero weights, one nonzero weight.

The case

$\delta = 2, \tau = 1$, or one zero weight, two nonzero weights of opposite sign,

was settled in [KR1].

It remains to consider the

Hyperbolic Case. $\delta = 2, \tau = 0$, or three nonzero weights, not all of the same sign.

A program to settle this case was proposed in [KR2]. It has two quite distinct components.

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Step I, the quotient. Show that $X//T$ is as expected for a linear action, i.e.

$$X//T \simeq T_0X//T.$$

Let $\omega_\alpha \subset \mathbb{C}^*$ be the group of α-roots of 1. Linearizability follows from Step I (see 1.4 below) in the case $\dim X^{w_\alpha} \leq 1$ for all $\alpha > 1$, or equivalently, if the weights are pairwise relatively prime. This leads to

Step II, reduction of weights. Reduction of the proof to the case of pairwise relatively prime weights.

If $\alpha > 1$ and α divides two weights, then $X' = X/\omega_\alpha$ is a smooth, affine threefold, but only after linearizability has been established is it at all clear that $X' \simeq \mathbb{C}^3$. We are therefore led to study more general $\mathbb{C}^*$-threefolds.

1.1. **Standard conditions.** Let X be a $\mathbb{C}^*$-threefold. We consider the following conditions.

(i) X is smooth and the action of $T = \mathbb{C}^*$ is *hyperbolic*,

$$\text{i.e. there is a unique fixpoint } 0 \text{ and } \dim X//T = 2.$$

(ii) X is contractible.

(iii) $\overline{\kappa}(X) = -\infty$ ($\overline{\kappa}$ = logarithmic Kodaira dimension).

If we have 1.1 (i), the weights of the action are defined as above for $X = \mathbb{C}^3$, and we assume

$$a_1 < 0,\ a_2 > 0,\ a_3 > 0, \qquad \mathrm{GCD}(a_1, a_2, a_3) = 1.$$

We put

$$\alpha_i = \mathrm{GCD}(\{a_1, a_2, a_3\} - \{a_i\}).$$

Then

$$-a_1 = a\alpha_2\alpha_3, \quad a_2 = b\alpha_1\alpha_3, \quad a_3 = c\alpha_1\alpha_2$$

with $a, b, c > 0$ and *reduced* (pairwise relatively prime).

1.2. Proposition ([KR3], 2.5)**.** *Let X satisfy 1.1 (i).*

(i) Suppose $\alpha_i > 1$. Then $\dim X^{\omega_{\alpha_i}} = 2$ and

$$X' = X/\omega_{\alpha_i}$$

satisfies 1.1 (i) for $T' = T/\omega_{\alpha_i} \simeq \mathbb{C}^*$ with weights a_i and a_j/α_i for $j \neq i$.

(ii) $X^\# = X/\omega_{\alpha_1\alpha_2\alpha_3}$ satisfies 1.1 (i) for $T^\# = T/\omega_{\alpha_1\alpha_2\alpha_3} \simeq \mathbb{C}^*$ and reduced *weights $-a, b, c$.*

(iii) $X//T = X'//T' = X^\#//T^\#$.

(iv) If X satisfies 1.1 (ii) or (iii), then so do X' and $X^\#$.

Let X satisfy 1.1 (i) and (ii). We put ([KR3], 1.4)

$$X^+ = \{x \in X \,|\, \lim_{t \to 0} t \cdot x = 0\}.$$

Then $X^+ \simeq \mathbb{C}^2$ and $X^+ = F^{-1}(0)$, where F is semiinvariant of weight a_1. ω_{a_1} acts on

$$X_1 = F^{-1}(1)$$

and we have ([KR1], Lemma 2)

1.3. $X//T \simeq X_1/w_{a_1}$.

The reduction of the proof to Steps I and II is now contained in

1.4. Proposition ([KR3], 2.3, 2.8, and 1.10). *Let X satisfy 1.1 (i) and (ii) and suppose the weights are reduced. If*

$$X//T \simeq T_0X//T,$$

or equivalently

$$X_1/\omega_a \simeq \mathbb{C}^2/\omega_a,$$

where ω_a acts diagonally on $\mathbb{C}^2$ with weights $\equiv b, c \mod a$, then

$$X_1 \simeq \mathbb{C}^2,$$

and

$$X \simeq_e \mathbb{C}^3$$

(X is equivariantly isomorphic to $\mathbb{C}^3 = T_0X$).

2. The quotient

2.1. Theorem ([KR4], 1.2). *Suppose X satisfies all conditions of 1.1. Then*

$$S' = X//\mathbb{C}^* \simeq T_0X//\mathbb{C}^*.$$

By 1.2, we may assume the weights are reduced when studying the quotient. Also, 2.1 is known ([KR2]) when S' is smooth, or equivalently $a = 1$. So we assume $a > 1$. Then by [KR4], 2.4

2.2. S' is contractible, $\overline{\kappa}(S') = -\infty$, S' has a unique singular point q, q is analytically of the type of the origin in $\mathbb{C}^2/\omega_a$, and $\mathrm{Pic}(S' - q) \simeq \mathbb{Z}/a\mathbb{Z}$.

2.3. Theorem ([K]). *If S' is as in 2.2, then*
(i) if $\overline{\kappa}(S' - q) = -\infty$, then $S' \simeq \mathbb{C}^2/\omega_a$,
(ii) $\overline{\kappa}(S' - q) \neq 0, 1$.

It remains to rule out $\overline{\kappa}(S' - q) = 2$ to complete Step I.

2.4. Theorem ([KR4], 1.1). *Let*

$$S' = X//T$$

with X satisfying all conditions of 1.1. Then

$$\overline{\kappa}(S' - q) < 2.$$

The proof is rather involved. It relies in a crucial way on the theory of open algebraic surfaces, in particular the inequalities of Miyaoka [M] and Kobayashi [Ko] and the results on the existence of affine rulings of Miyanishi and Tsunoda [MT].

2.5. Proposition ([KR4], 2.8). *Let S' be as in 2.4. There exists a desingularization S of S' admitting an $\mathbb{A}^1$-ruling with all but one component E of the exceptional locus $\hat{E}$ in fibres. Moreover, $S - \Delta$ is simply connected, where $\Delta = \hat{E} - E$.*

The proof of 2.4 proceeds by a detailed analysis of such "good" rulings under the conditions of 2.2.

3. Reduction of weights and "exotic affine spaces"

Step II, the reduction of weights, is achieved in a roundabout way. In [KR3], an explicit construction is given of a class of smooth, contractible $\mathbb{C}^*$-threefolds that encompasses, in the equivariant sense, all possible counterexamples to linearization. It is then shown in [KM-L] that only the "obviously" equivariantly trivial threefolds in the class are isomorphic to $\mathbb{C}^3$ (without reference to the $\mathbb{C}^*$-action). The others are in themselves interesting examples of "exotic affine spaces" (algebraic varieties homeomorphic to $\mathbb{C}^3$). They include the threefolds described in [D], 4.36.

3.1. Theorem ([KR3], 4.1). *The threefolds*

$$X = \operatorname{Spec} A$$

satisfying 1.1 (i) and (ii) and

$$X//\mathbb{C}^* \simeq T_0X//\mathbb{C}^*$$

are precisely the ones obtained as follows.

(1) Let

$$-a = a_1', \ b = a_2', \ c = a_3'$$

be a triple of reduced weights with $a, b, c > 0$. (These define a hyperbolic $\mathbb{C}^*$-action on

$$W = \operatorname{Spec} B \simeq \mathbb{C}^3$$

with $B = \mathbb{C}[\eta, \xi, \zeta]$ and η, ξ, ζ homogeneous of weight $-a, b, c$).

(2) Let

$$\alpha_1, \alpha_2, \alpha_3$$

be a reduced triple of positive integers with $\operatorname{GCD}(\alpha_i, a_i') = 1, \ i = 1, 2, 3$.

(3) Let C_2 and C_3 be ω_a-homogeneous "lines" (curves isomorphic to $\mathbb{C}$) in $W_1 = \operatorname{Spec} k[\xi, \zeta] \simeq \mathbb{C}^2$, identified with $\eta^{-1}(1) \subset W$, such that

(i) C_2 and C_3 meet normally in $r \geq 1$ points, including the origin,

(ii) $U_i = \overline{\mathbb{C}^* \cdot C_i} \subset W$ is smooth, $i = 2, 3$.

(4) Let $U_1 = W^+ = \eta^{-1}(0)$.

Then X is the "tri-cyclic" cover of W ramified to order α_i over $U_i, \ i = 1, 2, 3$, that is,

$$A = B[z_1, z_2, z_3],$$

where $z_i^{\alpha_i} = u_i$ with $u_1 = \eta$ and for $i = 2, 3, \ u_i$ is an equation for U_i and uniquely determined by

$$s^{-a_i'} f_i(\xi s^{a_2'}, \zeta s^{a_3'}) = u_i(s^{-a_1'}, \xi, \zeta),$$

where f_i is an equation for $C_i \subset W_1$.

Moreover,

$$B = \mathbb{C}[u_1, u_2, u_3^*] = \mathbb{C}[u_1, u_2^*, u_3],$$

with u_i and u_i^* homogeneous of weight a_i' and if

$$u_2 = G_2(u_1, u_2^*, u_3) \quad \text{and} \quad u_3 = G_3(u_1, u_2, u_3^*),$$

then the equations

$$z_2^{\alpha_2} = G_2(z_1^{\alpha_1}, z_2^*, z_2^{\alpha_3}) \quad \text{and} \quad z_3^{\alpha_3} = G_3(z_1^{\alpha_1}, z_2^{\alpha_2}, z_3^*)$$

describe X (in two ways) as a hypersurface in $\mathbb{C}^4$.

3.1.1. *Remark.* 1) (3)(ii) imposes a rather mild restriction that can be made quite explicit ([KR3], 1.11.1).

2) Possibilities for f_2, f_3, and hence for G_2, G_3, can be worked out explicitly with the help of the *epimorphism theorem* of Abhyankar, Moh and Suzuki [AM], [S].

The key to 3.1 is the following observation.

3.2. Proposition ([KR3], 2.6, 2.7)**.** *Suppose X is as in 3.1 and $\alpha_2 = 1,\ \alpha_3 > 1$. Then*

$$X/\omega_{\alpha_3} \simeq_e \mathbb{C}^3 \textit{ implies } X \simeq_e \mathbb{C}^3.$$

A similar result holds if $\alpha_2 > 1,\ \alpha_3 = 1$. Also,

$$X/\omega_{\alpha_1} \simeq_e \mathbb{C}^3 \textit{ implies } X \simeq_e \mathbb{C}^3.$$

In view of 1.4 we obtain a commutative diagram

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
\mathbb{C}^3 \simeq_e X/\omega_{\alpha_2} \arrow[dr] & & X/\omega_{\alpha_3} \simeq_e \mathbb{C}^3 \arrow[dl] \\
 & X/\omega_{\alpha_2\alpha_3} \simeq_e \mathbb{C}^3 \arrow[d] & \\
 & X/\omega_{\alpha_1\alpha_2\alpha_3} \simeq_e \mathbb{C}^3 &
\end{tikzcd}$$

3.1 is an elaboration of the possibilities for such a diagram. It is not difficult to decide when X is equivariantly isomorphic to $\mathbb{C}^3$ (see 3.4). The question of just isomorphism with $\mathbb{C}^3$, on the other hand, proved to be much more elusive.

3.3. Let us for instance choose $a = b = c = 1,\ \alpha_2 = 2$ and $\alpha_3 = 3$ and a parabola and straight line for C_2 and C_3. Then in suitable coordinates X is defined in $\mathbb{C}^4$ by

$$x + x^2y + z^2 + t^3 = 0.$$

X is dominated birationally by $\mathbb{C}^3$ and there exists a surjective quasi-finite map $\mathbb{C}^3 \to X$ ([KR3], 7.7 and 7.8). It is shown in [M-L1] that, nevertheless, X is not isomorphic to $\mathbb{C}^3$. The proof is based implicitly on the computation of the following invariant:

$$AK(X) = \bigcap_{\partial \in LND(X)} \operatorname{Ker} \partial$$

where $LND(X)$ is the set of locally nilpotent derivations on the ring $\mathbb{C}[X]$ of regular functions on X. For this hypersurface $AK(X) \neq \mathbb{C}$, but clearly $AK(\mathbb{C}^3) = \mathbb{C}$.

Let X be as in 3.1. We define

$$\varepsilon = (r-1)(\alpha_2 - 1)(\alpha_3 - 1)$$

($\varepsilon = \operatorname{rank} \pi_2(X - X^+)$ is an invariant of the higher-dimensional knot (X, X^+) ([KR3], 4.8)).

3.4. Theorem ([KR3], remark following 5.1)**.** *Let X be as in 3.1. Then $X \simeq_e \mathbb{C}^3$ if and only if $\varepsilon = 0$.*

3.5. Theorem ([KM-L]). *Let X be as in 3.1. If $\varepsilon > 0$, then $X \not\simeq \mathbb{C}^3$.*

If now X is $\mathbb{C}^3$ with a hyperbolic $\mathbb{C}^*$-action, then by 2.1 it is one of the X in 3.1 and hence $X \simeq_e \mathbb{C}^3$ by 3.4 and 3.5.

4. The computation of $AK(X)$

4.1. Theorem 3.5 is again the consequence of the fact that $AK(X) \neq \mathbb{C}$ [KM-L]. More precisely, $AK(X) = \mathbb{C}[X]$ unless X is isomorphic to a hypersurface in $\mathbb{C}^4$ given by one of the following equations:

$$\text{(i)} \qquad x + x^k y + z^{\alpha_2} + t^{\alpha_3} = 0 \quad \text{or}$$

$$\text{(ii)} \qquad x + y(x^k + z^{\alpha_2})^l + t^{\alpha_3} = 0$$

where $k \geq 2, l \geq 1$, and in the second equation $(kl, \alpha_3) = 1$. In case (i) $AK(X)$ is the restriction of $\mathbb{C}[x]$ to X and in case (ii) $AK(X)$ is the restriction of $\mathbb{C}[x, z]$ to X.

4.2. The scheme of the computation of $AK(X)$ is discussed below. Every X from 3.1 is the hypersurface $P(x, y, z, t) = 0$ where

$$(x, y, z, t) = (z_3^*, z_1, z_2, z_3) \quad \text{and} \quad P(x, y, z, t) = t^{\alpha_3} - G_3(y^{\alpha_1}, z^{\alpha_2}, x).$$

The polynomials from 4.1 (i) and (ii) are examples of such P. A derivation ∂ on $\mathbb{C}[X]$ is said to be of *Jacobian type* if $\partial(f)$ coincides with the restriction of $J_{x,y,z,t}(P, \varphi_1, \varphi_2, \varphi)$ to X where $\varphi_1, \varphi_2 \in \mathbb{C}[x, y, z, t]$ are fixed and the restriction of $\varphi \in \mathbb{C}[x, y, z, t]$ to X coincides with $f \in \mathbb{C}[X]$.

4.3. Proposition ([KM-L]). *Let $\delta \in LND(X)$ be nontrivial and let φ_1, φ_2 be such that $\varphi_1|_X, \varphi_2|_X \in \operatorname{Ker} \delta$ and P, φ_1, φ_2 are algebraically independent. Then ∂ has the same kernel as δ.*

Furthermore, since the transcendence degree of the field of fractions of $\operatorname{Ker} \delta$ is 2 [M-L1], one can always find φ_1, φ_2, and therefore ∂ as above.

4.4. We consider degree functions L on $\mathbb{C}[x, y, z, t]$ obtained by assigning real weights to the variables. The L-quasi-leading part φ^L of a nonzero polynomial φ is the sum of the terms from φ whose L-degree coincides with $L(\varphi)$. Suppose, given φ_1, there exists a degree function L_1 with positive values such that for any other degree function L_2 with positive values each nonzero monomial from φ^{L_2} is also present in φ^{L_1}. We then call

$$\hat{\varphi} := \varphi^{L_1}$$

the quasi-leading part of φ. In cases 4.1 (i) and (ii) $\hat{P}$ coincides with $x^k y + z^{\alpha_2} + t^{\alpha_3}$ and $y(x^k + z^{\alpha_2})^l + t^{\alpha_3}$ respectively. In all other cases $\hat{P}$ also exists and can be computed explicitly by virtue of the Abhyankar-Moh-Suzuki theorem (see 3.1.1 (ii)). Consider further only those degree functions (may be with negative values) which satisfy the condition

$$P^L = \hat{P}.$$

4.5. Proposition ([KM-L]). *Let ∂ be a nontrivial locally nilpotent derivation of Jacobian type on $\mathbb{C}[X]$. Then polynomials φ_1, φ_2 can be chosen so that $\varphi_1|_X, \varphi_2|_X \in \operatorname{Ker} \partial$ and $\hat{P}, \varphi_1^L, \varphi_2^L$ are algebraically independent.*

4.6. With ∂ as in 4.5, suppose that $\hat{X}$ is the hypersurface $\hat{P}(x,y,z,t)=0$ in $\mathbb{C}^4$ and that ∂^L is the derivation on $\mathbb{C}[\hat{X}]$ such that $\partial^L(f)$ coincides with restriction of $J_{x,y,z,t}(\hat{P},\varphi_1^L,\varphi_2^L,\varphi)$ to $\hat{X}$, where the restriction of $\varphi\in\mathbb{C}[x,y,z,t]$ to $\hat{X}$ coincides with $f\in\mathbb{C}[\hat{X}]$. Then ∂^L is nontrivial and also locally nilpotent [M-L1].

4.7. Since $\hat{P}$ is known explicitly, we can find all nontrivial locally nilpotent derivations of Jacobian type on $\mathbb{C}[\hat{X}]$. If P is not as in 4.1 (i) or (ii) there are no such derivations. By 4.3 and 4.6 there is no nontrivial locally nilpotent derivation on $\mathbb{C}[X]$, that is, $AK(X)=\mathbb{C}[X]$.

4.8. In case 4.1 (ii) the kernel of any nontrivial locally nilpotent derivation ∂^L on $\mathbb{C}[\hat{X}]$ is contained in $\mathbb{C}[x,z]|_{\hat{X}}$. Since this is true for every L satisfying condition 4.4, it follows ([KM-L], Theorem 8.4) that the kernel of the corresponding nontrivial locally nilpotent derivation ∂ on $\mathbb{C}[X]$ is contained in $\mathbb{C}[x,z]|_X$. The transcendence degree of the field of fractions of $\operatorname{Ker}\partial$ is 2 and $\operatorname{Ker}\partial$ is algebraically closed in $\mathbb{C}[X]$ [M-L1]. Hence $\operatorname{Ker}\partial=\mathbb{C}[x,z]$. In case 4.1 (ii) nontrivial locally nilpotent derivations on $\mathbb{C}[X]$ exist, for instance $J_{x,y,z,t}(P,x,z,\varphi)|_X$. This yields $AK(X)=\mathbb{C}[x,z]|_X$.

4.9. In case 4.1 (i) nontrivial locally nilpotent derivations on $\mathbb{C}[X]$ exist as well. Examples are $J_{x,y,z,t}(P,x,z,\varphi)|_X$ and $J_{x,y,z,t}(P,x,t,\varphi)|_X$. The intersection of the kernels of these locally nilpotent derivations is $\mathbb{C}[x]|_X$, whence it suffices to show that $x\in\operatorname{Ker}\partial$ for every nontrivial $\partial\in LND(X)$. It can be shown that $\operatorname{Ker}\partial^L\subset\mathbb{C}[x,z,t]|_{\hat{X}}$ [KM-L]. Varying L under the condition 4.4 we prove that $\operatorname{Ker}\partial\subset\mathbb{C}[x,z,t]|_X$. From this we deduce that $\partial(\mathbb{C}[x,z,t]|_X)\subset x^k\mathbb{C}[x,z,t]|_X$ with k as in 4.1. Since $\mathbb{C}[x,z,t]|_X\not\subset\operatorname{Ker}\partial$, there exists $f\in\operatorname{Ker}\partial\setminus 0$ which is divisible by x, and then $x\in\operatorname{Ker}\partial$ by [FLN], that is, $AK(X)=\mathbb{C}[x]$.

5. Further results

Once 2.5 and 2.2 are established, the fact that $S'=X//T$ can be forgotten in the proof of 2.4. In special cases, a geometric characterization of $\mathbb{C}^2/\omega_a$ is obtained.

5.1. Theorem ([KR4], 10.1). *Suppose S' is as in 2.2. If either q is an ordinary a-fold point, that is $b\equiv c \mod a$, or the minimal resolution of q is a single $(-a)$-curve, or q is a rational double point, that is, $b\equiv -c \mod a$, or the minimal resolution of q is a chain of $a-1$ (-2)-curves, then*

$$S'\simeq\mathbb{C}^2/\omega_a.$$

We do not know whether the restriction on the analytic type of q is needed in 5.1.

Extending the arguments of [KP], Popov [P] recently proved that any effective action of a noncommutative, connected reductive group on $\mathbb{C}^3$ is linearizable. Since effective actions of $(\mathbb{C}^*)^r$, $r>1$, are linearizable by [B-B], we obtain

5.2. Theorem. *Any action of a connected, reductive group G on $\mathbb{C}^3$ is linearizable.*

It is an open question whether the connectedness assumption in 5.2 can be removed, and in particular, whether finite group actions on $\mathbb{C}^3$ are linearizable.

It is reasonable to expect that our methods and results will shed some light in general on codimension 2 torus actions on $\mathbb{C}^n$. As an illustration, consider the possibility that $X\times\mathbb{C}\simeq\mathbb{C}^4$, where X is a $\mathbb{C}^*$-threefold. For linearizability of the obvious $(\mathbb{C}^*)^2$-action the following *weak cancellation conjecture* is required: *Let X be an affine threefold such that $X \times \mathbb{C} \simeq \mathbb{C}^4$. Then $X \simeq \mathbb{C}^3$ or X does not admit an effective $\mathbb{C}^*$-action.*

This is known for nonhyperbolic actions. For hyperbolic actions, one would have to show that $X \times \mathbb{C} \not\simeq \mathbb{C}^4$ for the threefolds in 3.1. This is true in the case when X is not isomorphic to a hypersurface of the form 4.1 (i) or (ii) since $AK(Y \times \mathbb{C}) = \mathbb{C}[Y]$ for every algebraic manifold Y with $AK(Y) = \mathbb{C}[Y]$ [M-L2].

We remark that linearizability of $\mathbb{G}_m$-actions on $\mathbb{A}^3$ in positive characteristic is an open question, even in certain nonhyperbolic cases.

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