Construction of the Fourfold Cover of the Mathieu Group M_{22}

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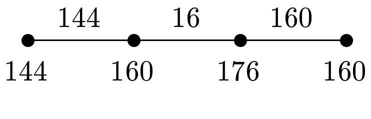
CONTENTS

We give an explicit construction for the two faithful, irreducible, 16-dimensional representations of $4{}^{\cdot}M_{22}$ over the field GF(49). Then we extend them to the 32-dimensional representation of $4{}^{\cdot}M_{22}{:}2$ over GF(7). Explicit matrices are given on page 14.

INTRODUCTION

At one time [Burgoyne and Fong 1968] it was believed that the Schur multiplier of the Mathieu group M_{22} was 3. Later this was amended to 6, which was believed for several years to be the correct answer. Now there are several independent proofs that the answer is 12. For example, it was noted in [Gagola and Garrison 1982] that the standard construction of a spin representation gives an easy proof of the existence of a proper fourfold cover $4{}^{\cdot}M_{22}$. Namely, $2{}^{\cdot}M_{22}$ has a 210-dimensional faithful irreducible real orthogonal representation, in which the central involution obviously has 210 eigenvalues -1. Since the number of eigenvalues -1 is congruent to 2 (mod 4), this element lifts to elements of order 4 in the spin group Spin$(210, \mathbf{R})$, giving rise to a proper 4-fold cover $4{}^{\cdot}M_{22}$.

In this paper we describe an explicit construction of this group, in its 16-dimensional representation over GF(49). The existence of such a representation is easy to prove from the ordinary character table, as there is only one possibility for the Brauer tree of the faithful 7-block of defect 1 for $4{}^{\cdot}M_{22}$, shown here (see also [Parker et al.]):



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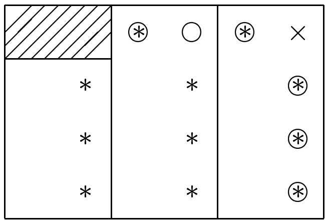
For the computations we used R. A. Parker's Meat-axe programs [Parker 1984], together with arithmetic subroutines written by M. van Meegen of RWTH, Aachen. The programs ran on a SUN SPARCstation, whose purchase was assisted by a grant from the SERC Computational Science Initiative. The general method we adopt is that described in [Parker and Wilson 1990].

CONSTRUCTION

The generating subgroups

We note first of all that $4{\cdot}M_{22}$ may be generated by subgroups $2{\cdot}L_2(11)$ and $2{\cdot}A_6$ intersecting in $2{\cdot}A_5$. This follows from the fact that M_{22} is generated by subgroups $L_2(11)$ and A_6 intersecting in A_5. Specifically, $L_2(11)$ is the stabilizer of an endecad (marked $*$ in the diagram below: see [Curtis 1976] for the notation) and A_6 is the stabilizer of a hexad (marked $\bigcirc$) and a point outside it (marked $\times$).



As these subgroups have order prime to 7, the representation restricts to each as a direct sum of ordinary irreducibles reduced modulo 7. In *Atlas* notation [Conway et al. 1985], the representation of $2{\cdot}L_2(11)$ is $6b \oplus 10c$ (that is, $\chi_{10} \oplus \chi_{11}$), while that of $2{\cdot}A_6$ is $8c \oplus 8d$ (again, coincidentally, $\chi_{10} \oplus \chi_{11}$). The restriction to $2{\cdot}A_5$ is $6a \oplus 6a \oplus 2a \oplus 2b$, or $\chi_6 \oplus \chi_7 \oplus 2\chi_9$.

Constructing $2{\cdot}L_2(11)$

From $\mathrm{SL}_2(11) \cong 2{\cdot}L_2(11)$ written as 2×2 matrices over GF(11) it is easy to obtain a faithful permutation action on 24 points, for example generated by the two permutations

$$(2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12)(14\,15\,16\,17\,18\,19\,20\,21\,22\,23\,24)$$

and

$$(1\,2\,13\,14)(3\,12\,15\,24)(7\,4\,19\,16)$$
$$(5\,9\,17\,21)(6\,10\,18\,22)(11\,8\,23\,20).$$

Writing this as a 24-dimensional matrix representation over GF(49) we can use the Meat-axe to chop out a copy of the representation $6b$. The exterior square of $6b$ is $5a + 10b$, and $5a \otimes 6b = 10c + 10d + 10e$, so we can obtain the desired representation $6b \oplus 10c$ over GF(49).

Constructing $2{\cdot}A_6$

From $\mathrm{SL}_2(9) \cong 2{\cdot}A_6$ written as 2×2 matrices over GF(9), we obtain a faithful permutation action on the 80 nonzero vectors. Writing this over GF(49), we can chop out copies of $5a$ and $10b$, and then chop $8c \oplus 8d$ from $5a \otimes 10b$.

Restricting to $2{\cdot}A_5$

Finding subgroups $2{\cdot}A_5$ in $2{\cdot}A_6$ and $2{\cdot}L_2(11)$ is straightforward. For example, if all else fails, we can search at random for elements x and y satisfying $x^2 = -1$ and $y^3 = (xy)^5 = 1$. We arrange that in both cases the group $2{\cdot}A_5$ is represented by block diagonal matrices, with blocks of sizes 6, 6, 2, 2. Moreover we use the Standard Base program of the Meat-axe to find bases with respect to which the two copies of $2{\cdot}A_5$ are represented by the same matrices. We write each of the groups $2{\cdot}L_2(11)$ and $2{\cdot}A_6$ with respect to the corresponding such basis.

Checking the cases

We now have matrices generating the two groups $H \cong 2{\cdot}L_2(11)$ and $K \cong 2{\cdot}A_6$, intersecting in a group $L \cong 2{\cdot}A_5$. We can conjugate either H or K by any matrix commuting with L, and the same situation will obtain, although the group generated by these two groups may change. Now $\langle H^g, K\rangle \cong \langle H, K^{g^{-1}}\rangle$, so it does not matter whether we conjugate H or K. Moreover, matrices commuting with H will have no effect on H, and similarly for K. Thus the cases we need to consider correspond to the double cosets of $C(H)$ and $C(K)$

in $C(L)$, where the centralizers are computed in $\mathrm{GL}_{16}(49)$. We have $C(H) \cong C(K) \cong 48^2$ and $C(L) \cong 48^2 \times \mathrm{GL}_2(49)$. More precisely, $C(L)$ consists of all invertible block matrices of the shape

$$\begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & F \end{pmatrix},$$

while $C(H)$ consists of all diagonal matrices of the form $\mathrm{diag}(P, Q, Q, Q)$ and $C(K)$ consists of those of shape $\mathrm{diag}(R, S, R, S)$. Since conjugation by a scalar matrix has no effect, we need only consider elements with $F = 1$, $Q = 1$, and $S = 1$. Then we can choose double coset representatives with $E = F = 1$, by putting $R = E^{-1}$. Thus we need to compute the $48 \times 49 \times 50 = 117600$ cosets in $\mathrm{GL}_2(49)$ of the subgroup of all matrices of the form $\left(\begin{smallmatrix} P & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Finally, we eliminate 117599 of the cases by showing that the group so generated contains elements of order greater than 44. Thus the remaining case must generate the group $4{\cdot}M_{22}$.

Extending to $4{\cdot}M_{22}{:}2$

The representation we have constructed is not invariant under the outer automorphism of $4{\cdot}M_{22}$, but is taken to its dual. Therefore, in order to construct the holomorph $4{\cdot}M_{22}{:}2$, we must begin by taking the direct sum of these two representations. Then we find "standard generators" for the group: for our purposes that means finding elements $x \in 2A$ and $y \in 4A$ with xy of order 11. We put the representation into a "standard basis" defined by (x, y). Then we find words in x and y that give us a new pair of generators (x', y'), which we guess to be automorphic to (x, y). We prove this isomorphism using the standard basis algorithm, as described in [Parker 1984]. The algorithm produces a matrix P which conjugates (x, y) to (x', y'). Furthermore, by applying the algorithm to the irreducible representations we can tell whether the isomorphism between (x, y) and (x', y') is realized by an inner or an outer automorphism, so we can ensure that it is outer. Now adjoining P to $4{\cdot}M_{22}$ gives a group which is isoclinic to $4{\cdot}M_{22}{:}2$. There are 48 such matrix groups in this isoclinism class, two of which are isomorphic to $4{\cdot}M_{22}{:}2$. They can all be obtained by multiplying P by a matrix which acts trivially on one of the 16-dimensional constituents, and as a scalar on the other. Moreover, it is easy to identify the two cases which are $4{\cdot}M_{22}{:}2$ simply by looking at the orders of elements.

The matrices

The sidebar on the next page exhibits two matrices generating $4{\cdot}M_{22}{:}2$.

ACKNOWLEDGEMENTS

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65105060325335443166106033333052	50305103030010211165513551204366
23635515203140246655504166052210	50026050314523455204352243315660

Generators for $4{\cdot}M_{22}{:}2$, in the 32-dimensional representation over GF(7).

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