MARKOV OPERATORS ON THE SOLVABLE BAUMSLAG–SOLITAR GROUPS

FLORIAN MARTIN AND ALAIN VALETTE

Abstract. We consider the solvable Baumslag–Solitar group

$$\mathrm{BS}_n = \langle a, b \mid aba^{-1} = b^n \rangle,$$

for $n \geq 2$, and try to compute the spectrum of the associated Markov operators M_S, either for the oriented Cayley graph ($S = \{a, b\}$), or for the usual Cayley graph ($S = \{a^{\pm 1}, b^{\pm 1}\}$). We show in both cases that $\mathrm{Sp} M_S$ is connected.

For $S = \{a, b\}$ (nonsymmetric case), we show that the intersection of $\mathrm{Sp} M_S$ with the unit circle is the set C_{n-1} of $(n-1)$-st roots of 1, and that $\mathrm{Sp} M_S$ contains the $n-1$ circles

$$\{z \in \mathbb{C} : |z - \tfrac{1}{2}\omega| = \tfrac{1}{2}\}, \quad \text{for } \omega \in C_{n-1},$$

together with the $n+1$ curves given by

$$\left(\tfrac{1}{2}w^k - \lambda\right)\left(\tfrac{1}{2}w^{-k} - \lambda\right) - \tfrac{1}{4}\exp 4\pi i\theta = 0,$$

where $w \in C_{n+1}$, $\theta \in [0, 1]$.

Conditional on the Generalized Riemann Hypothesis (actually on Artin's conjecture), we show that $\mathrm{Sp} M_S$ also contains the circle $\{z \in \mathbb{C} : |z| = \frac{1}{2}\}$. This is confirmed by numerical computations for $n = 2, 3, 5$.

For $S = \{a^{\pm 1}, b^{\pm 1}\}$ (symmetric case), we show that $\mathrm{Sp} M_S = [-1, 1]$ for n odd, and $\mathrm{Sp} M_S = [-\frac{3}{4}, 1]$ for $n = 2$. For n even, at least 4, we only get $\mathrm{Sp} M_S = [r_n, 1]$, with

$$-1 < r_n \leq -\sin^2 \frac{\pi n}{2(n+1)}.$$

We also give a potential application of our computations to the theory of wavelets.