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Asymptotic analysis for stochastic volatility: Edgeworth expansion

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Abstract

The validity of an approximation formula for European option prices under a general stochastic volatility model is proved in the light of the Edgeworth expansion for ergodic diffusions. The asymptotic expansion is around the Black-Scholes price and is uniform in bounded payoff functions. The result provides a validation of an existing singular perturbation expansion formula for the fast mean reverting stochastic volatility model.

Key words: ergodic diffusion; fast mean reverting; implied volatility.

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1 Introduction

In the last decade, many results on asymptotic expansions of option prices for stochastic volatility models appeared in the literature. Such an expansion formula gives an approximation to theoretical price of option and sheds light to the shape of theoretical implied volatility surface. See e.g., Gatheral [13] for a practical guide. The primary objective of this article is not to introduce a new expansion formula but to prove the validity of an existing one which was introduced by Fouque et al. [8]. We suppose that the log price process Z satisfies the stochastic differential equation

$$\begin{aligned}\mathrm{d}Z_t &= \left\{r_t - \frac{1}{2}\varphi(X_t)^2\right\}\mathrm{d}t + \varphi(X_t)\left[\rho(X_t)\mathrm{d}W_t^1 + \sqrt{1-\rho(X_t)^2}\mathrm{d}W_t^2\right]\\ \mathrm{d}X_t &= b(X_t)\mathrm{d}t + c(X_t)\mathrm{d}W_t^1\end{aligned}\tag{1}$$

under a risk-neutral probability measure, where (W^1, W^2) is a 2-dimensional standard Brownian motion, $r = \{r_t\}$ stands for interest rate and is assumed to be deterministic, and b, c, φ, ρ are Borel functions with $|\rho| \leq 1$. Under mild conditions on the ergodicity of X, we validate an approximation

$$D\mathbb{E}[f(Z_T)] \approx D\mathbb{E}[(1+p(N))f(Z_0 - \log(D) - \Sigma/2 + \sqrt{\Sigma}N)]\tag{2}$$

for every bounded Borel function f, where $N \sim \mathcal{N}(0,1)$, $\Sigma = \Pi[\varphi^2]T$ and

$$\begin{aligned}\Pi(\mathrm{d}x) &= \frac{\mathrm{d}x}{\epsilon^2 s'(x)c(x)^2},\\ s'(x) &= \exp\left\{-2\int_0^x \frac{b(w)}{c(w)^2}\mathrm{d}w\right\},\\ \epsilon^2 &= \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{s'(x)c^2(x)},\\ D &= \exp\left\{-\int_0^T r_s \mathrm{d}s\right\},\\ p(z) &= \alpha\left\{1 - z^2 + \frac{1}{\sqrt{\Sigma}}(z^3 - 3z)\right\},\\ \alpha &= -\int_{-\infty}^{\infty}\int_{-\infty}^{x}\left\{\frac{\varphi(v)^2}{\Pi[\varphi^2]} - 1\right\}\Pi(\mathrm{d}v)\frac{\varphi(x)\rho(x)}{c(x)}\mathrm{d}x.\end{aligned}\tag{3}$$

Note that $s(x) = \int_0^x s'(y)\mathrm{d}y$ is the so-called scale function of X and that Π coincides with the ergodic distribution of X. In particular, we have a simple formula

$$D\mathbb{E}[(K - \exp(Z_T))_+] \approx P_{\mathrm{BS}}(K,\Sigma) - \alpha d_2(K,\Sigma)DK\phi(d_2(K,\Sigma))$$

for put option price with strike K, where $P_{\mathrm{BS}}(K,\Sigma)$ is the Black-Scholes price of the put option

$$\begin{aligned}P_{\mathrm{BS}}(K,\Sigma) &= DK\Phi(-d_2(K,\Sigma)) - \exp(Z_0)\Phi(-d_2(K,\Sigma) - \sqrt{\Sigma}),\\ d_2(K,\Sigma) &= -\frac{\log(K) - Z_0 + \log(D)}{\sqrt{\Sigma}} - \frac{\sqrt{\Sigma}}{2},\end{aligned}$$

and Φ and ϕ are the standard normal distribution function and density respectively. Notice that if $\alpha = 0$ then the right hand side of (2) coincides with the Black-Scholes price for the European payoff function $f \circ \log$ with volatility $\Pi[\varphi^2]^{1/2}$. The term with p is small if c is large, so that in such a case it should be regarded as a correction term to the Black-Scholes approximation. The right hand side of (2) is an alternative representation of the so-called fast mean reverting or singular perturbation expansion formula and its validity has been discussed by Fouque et al. [9][10], Conlon and Sullivan [5], Khasminskii and Yin [15] and Alòs [1] under restrictive conditions on the payoff function f or on the coefficients of the stochastic differential equation (1). Recently, Fukasawa [12] gave a general framework based on Yoshida's theory of martingale expansion to prove the validity of such an asymptotic expansion around the Black-Scholes price for a general stochastic volatility model with jumps, which in particular incorporates the fast mean reverting case with (1). This paper, on the other hand, concentrates on the particular standard model to improve the preceding results mainly in the following points:

i. conditions on the integrability of $\langle Z \rangle$ are weakened,

ii. precise order estimate of approximation error is given.

The framework of Fukasawa [12] is too general to give such a precise estimate of order of error. A PDE approach taken by Fouque et al. [9][10] gave order estimates which depend on the regularity of the payoff f. The order given in this article is more precise and does not depend on the regularity of f. We require no condition on the smoothness of f and a weaker condition on the coefficients φ, ρ, b and c. We exploit Edgeworth expansion for ergodic diffusions developed by Fukasawa [11].

The Edgeworth expansion is a refinement of the central limit theorem and has played an important role in statistics. There are three approaches to validate the Edgeworth expansion for ergodic continuous-time processes. Global(martingale) and local(mixing) approaches which were developed by Yoshida [20] and [21] respectively are widely applicable to general continuous-time processes. The third approach, which is called regenerative approach and was developed by Fukasawa [11] extending Malinovskii [16], is applicable only to strong Markov processes but requires weaker conditions of ergodicity and integrability. The martingale approach was applied to the validation problem of perturbation expansions by Fukasawa [12] as noted above. The present article is based on the regenerative approach that enables us to treat such an ergodic diffusion X that is not geometrically mixing. An extension to this direction is important because empirical studies such as Andersen et al. [2] showed that the volatility process appears "very slowly mean reverting", that is, the autocorrelation function decays slowly. Our model (1) under a condition of ergodicity given later is a natural extension of the fast mean reverting model of Fouque et al. [8][9] but does not necessarily imply a fast decay of the autocorrelation function. It admits a polynomial decay of α-mixing coefficient.

It should be noted that our approach in this article utilizes the fact that X is one-dimensional in (1). See Fukasawa [12] for multi-dimensional fast mean reverting stochastic volatility model with jumps. In Section 2, we review the fast mean reverting expansion technique. The main result is stated in Section 3 with examples. Basic results in the Edgeworth expansion theory are presented in Section 4 and then, the proof of the main result is given in Section 5. The proof of an important lemma used in Section 5 is deferred to Section 6.

2 Fast mean reverting stochastic volatility

2.1 PDE approach

Here we review an asymptotic method introduced by Fouque et al. [8], where a family of the stochastic volatility models

$$\begin{aligned} \mathrm{d}S_t^\eta &= rS_t^\eta \mathrm{d}t + \varphi(X_t^\eta)S_t^\eta \mathrm{d}W_t^\rho, \\ \mathrm{d}X_t^\eta &= \left\{ \frac{1}{\eta^2}(m - X_t^\eta) - \frac{\nu\sqrt{2}}{\eta}\Lambda(X_t^\eta) \right\} \mathrm{d}t + \frac{\nu\sqrt{2}}{\eta}\mathrm{d}W_t \end{aligned} \tag{4}$$

is considered, where $W = (W_t)$ and $W^\rho = (W_t^\rho)$ are standard Brownian motions with correlation $\langle W, W^\rho \rangle_t = \rho t$, $\rho \in [-1, 1]$. This is a special case of (1) with $\rho(x) \equiv \rho$, $b(x) = (m - x)/\eta^2 - \nu\sqrt{2}\Lambda(x)/\eta$, $c(x) \equiv \nu\sqrt{2}/\eta$, where m, ν are constants and Λ is a Borel function associated with the market price of volatility risk. For a given payoff function f and maturity T, the European option price at time $t < T$ defined as

$$P^\eta(t, s, \nu) = e^{-r(T-t)}\mathbb{E}[f(S_T^\eta)|S_t^\eta = s, X_t^\eta = \nu] \tag{5}$$

satisfies

$$\left(\frac{1}{\eta^2}\mathscr{L}_0 + \frac{1}{\eta}\mathscr{L}_1 + \mathscr{L}_2 \right) P^\eta = 0, \quad P^\eta(T, s, \nu) = f(s)$$

where

$$\begin{aligned} \mathscr{L}_0 &= \nu^2 \frac{\partial^2}{\partial \nu^2} + (m - \nu)\frac{\partial}{\partial \nu}, \\ \mathscr{L}_1 &= \sqrt{2}\rho\nu s\varphi(\nu)\frac{\partial^2}{\partial s \partial \nu} - \sqrt{2}\nu\Lambda(\nu)\frac{\partial}{\partial \nu}, \\ \mathscr{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2}\varphi(\nu)^2 s^2 \frac{\partial^2}{\partial s^2} + r(s\frac{\partial}{\partial s} - 1). \end{aligned}$$

Notice that $\mathscr{L}_0$ is the infinitesimal generator of the OU process

$$\mathrm{d}X_t^0 = (m - X_t^0)\mathrm{d}t + \nu\sqrt{2}\mathrm{d}W_t \tag{6}$$

and $\mathscr{L}_2$ is the Black-Scholes operator with volatility level $|\varphi(\nu)|$. By formally expanding P^η in terms of η and equating the same order terms of η in the PDE, one obtains

$$P^\eta = P_0 + \eta P_1 + \text{ higher order terms of } \eta \tag{7}$$

for the Black-Scholes price P_0 with constant volatility $\Pi_0[\varphi^2]^{1/2}$, where Π_0 is the ergodic distribution of the OU process X^0, and

$$P_0 + \eta P_1 = P_0 - (T - t)\left(V_2 s^2 \frac{\partial^2 P_0}{\partial s^2} + V_3 s^3 \frac{\partial^3 P_0}{\partial s^3} \right) \tag{8}$$

with constants V_2 and V_3 which are of $O(\eta)$.

As a practical application, Fouque et al. [8] proposed its use in calibration problem. They derived an expansion of the Black-Scholes implied volatility σ_{BS} of the form

$$\sigma_{\mathrm{BS}}(K, T-t) \approx a \frac{\log(K/S)}{T-t} + b \tag{9}$$

from (7), where K is the strike price, S is the spot price, $T-t$ is the time to the maturity, a and b are constants connecting to V_2 and V_3 as

$$V_2 = \bar{\sigma}((\bar{\sigma} - b) - a(r + \frac{3}{2}\bar{\sigma}^2)), \quad V_3 = -a\bar{\sigma}^3, \quad \bar{\sigma}^2 = \Pi_0[\varphi^2]. \tag{10}$$

The calibration methodology consists of (i) estimation of $\bar{\sigma}$ from historical stock returns, (ii) estimation of a and b by fitting (9) to the implied volatility surface, and (iii) pricing or hedging by using estimated $\bar{\sigma}$, a and b via (8) and (10). This approach captures the volatility skew as well as the term structure. It enables us to calibrate fast and stably due to parsimony of parameters; we have no more need to specify all the parameters in the underlying stochastic volatility model. The first step (i) can be eliminated because the number of essential parameters is 2 in light of (2); by using $\Pi_\eta[\varphi^2]^{1/2}$ instead of $\Pi_0[\varphi^2]^{1/2}$ for $\bar{\sigma}$, where Π_η is the ergodic distribution of X^η, we can see that the right hand side of (8) coincides with that of (2) with $V_3 = -\alpha\Pi_\eta[\varphi^2]$ and $V_2 = 2V_3$.

It should be explained what is the intuition of $\eta \to 0$. To fix ideas, let $\Lambda = 0$ for brevity. Then $\tilde{X}_t := X^\eta_{\eta^2 t}$ satisfies

$$\mathrm{d}\tilde{X}_t = (m - \tilde{X}_t)\mathrm{d}t + \nu\sqrt{2}\mathrm{d}\tilde{W}_t,$$

where $\tilde{W}_t = \eta^{-1}W_{\eta^2 t}$ is a standard Brownian motion, and it holds

$$\mathrm{d}S^\eta_t = rS^\eta_t\mathrm{d}t + \varphi(\tilde{X}_{t/\eta^2})S^\eta_t\mathrm{d}W^\rho_t.$$

Hence η stands for the volatility time scale. Note that

$$\langle \log(S^\eta)\rangle_t = \int_0^t \varphi(\tilde{X}_{s/\eta^2})^2\mathrm{d}s \sim \eta^2\int_0^{t/\eta^2} \varphi(X^0_s)^2\mathrm{d}s \to \Pi_0[\varphi^2]t$$

by the law of large numbers for ergodic diffusions, where X^0 is a solution of (6). This convergence implies that the log price $\log(S^\eta_t)$ is asymptotically normally distributed with mean $rt - \Pi_0[\varphi^2]t/2$ and variance $\Pi_0[\varphi^2]t$ by the martingale central limit theorem. The limit is nothing but the Black-Scholes model with volatility $\Pi_0[\varphi^2]^{1/2}$. The asymptotic expansion formula around the Black-Scholes price can be therefore regarded as a refinement of a normal approximation based on the central limit theorem for ergodic diffusions.

2.2 Martingale expansion

Note that a formal calculation as in (7) does not ensure in general that the asymptotic expansion formula is actually valid. A rigorous validation is not easy if the payoff f or a coefficient of the stochastic differential equation is not smooth. See e.g. Fouque et al. [9]. A general result on the validity is given by Fukasawa [12]. Here we state a simplified version of it. Consider a sequence of models of type (1):

$$\mathrm{d}Z^n_t = \left\{r_t - \frac{1}{2}\varphi(X^n_t)^2\right\}\mathrm{d}t + \varphi(X^n_t)\left[\rho(X^n_t)\mathrm{d}W^1_t + \sqrt{1-\rho(X^n_t)^2}\mathrm{d}W^2_t\right]$$
$$\mathrm{d}X^n_t = b_n(X^n_t)\mathrm{d}t + c_n(X^n_t)\mathrm{d}W^1_t,$$

where b_n and c_n, $n \in \mathbb{N}$ are sequences of Borel functions.

Theorem 2.1. *Suppose that for any $p > 0$, the L^p moments of*

$$\int_0^T \varphi(X_t^n)^2 \mathrm{d}t, \quad \left\{ \int_0^T \varphi(X_t^n)^2 (1 - \rho(X_t^n)^2) \mathrm{d}t \right\}^{-1} \tag{11}$$

are bounded in $n \in \mathbb{N}$ and that there exist positive sequences ϵ_n, Σ_n with $\epsilon_n \to 0$, $\Sigma_\infty := \lim_{n\to\infty} \Sigma_n > 0$ such that

$$\left(\frac{M_T^n}{\sqrt{\Sigma_n}}, \frac{\langle M^n \rangle_T - \Sigma_n}{\epsilon_n \Sigma_n} \right) \to \mathcal{N}(0, V) \tag{12}$$

in law with a 2×2 variance matrix $V = \{V_{ij}\}$ as $n \to \infty$, where M^n is the local martingale part of Z^n. Then, for every Borel function f of polynomial growth,

$$\mathbb{E}[f(Z_T^n)] = \mathbb{E}[(1 + p_n(N)) f(Z_0 - \log(D) - \Sigma_n/2 + \sqrt{\Sigma_n} N)] + o(\epsilon_n) \tag{13}$$

as $n \to \infty$, where $N \sim \mathcal{N}(0,1)$, D is defined as in (3) and

$$p_n(z) = \epsilon_n \frac{V_{12}}{2} \left\{ -\sqrt{\Sigma_n}(z^2 - 1) + (z^3 - 3z) \right\}.$$

An appealing point of this theorem is that it gives a validation of not only the singular perturbation but also regular perturbation expansions including the so-called small vol-of-vol expansion. It is also noteworthy that the asymptotic skewness V_{12} appeared in the expansion formula is represented as the asymptotic covariance between the log price and the integrated volatility. Our interest here is however to deal with the singular case only. Now, suppose that b_n and $(1 + c_n^2)/c_n$ are locally integrable and locally bounded on $\mathbb{R}$ respectively for each $n \in \mathbb{N}$; we take $\mathbb{R}$ as the state space of X^n by a suitable scale transformation. Further, we assume that $s_n(\mathbb{R}) = \mathbb{R}$ for each $n \in \mathbb{N}$, where

$$s_n(x) = \int_0^x \exp\left\{ -2 \int_0^v \frac{b_n(w)}{c_n(w)^2} \mathrm{d}w \right\} \mathrm{d}v$$

is the scale function of X^n. This assumption ensures that there exists a unique weak solution of (1). See e.g., Skorokhod [18], Section 3.1. It is also known that the ergodic distribution Π_n of X^n is, if exists, given by

$$\Pi_n(\mathrm{d}x) = \frac{\mathrm{d}x}{\epsilon_n^2 s_n'(x) c_n^2(x)}$$

with a normalizing constant ϵ_n^2:

$$\epsilon_n^2 = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{s_n'(x) c_n^2(x)}.$$

Theorem 2.2. *Suppose that*

i. for any $p > 0$, the L^p boundedness of the sequences (11) holds,

ii. $\epsilon_n \to 0$ as $n \to \infty$, *iii.* $\lim_{n\to\infty} \Pi_n[\varphi^2]$ *exists and is positive,*

iv. $\lim_{n\to\infty} \Pi_n[\varphi\rho\psi_n]$ *and* $\lim_{n\to\infty} \Pi_n[\psi_n^2]$ *exist, where*

$$\psi_n(x) = 2\epsilon_n c_n(x) s_n'(x) \int_{-\infty}^{x} (\varphi(\eta)^2 - \Pi_n[\varphi^2])\Pi_n(\mathrm{d}\eta),$$

v. the sequences

$$\int_{X_0^n}^{X_T^n} \frac{\psi_n(x)}{c_n(x)}\mathrm{d}x, \quad \frac{1}{T}\int_0^T \psi_n(X_t^n)^2\mathrm{d}t - \Pi_n[\psi_n^2]$$

and

$$\frac{1}{T}\int_0^T \psi_n(X_t^n)\rho(X_t^n)\varphi(X_t^n)\mathrm{d}t - \Pi_n[\psi_n\rho\varphi]$$

converge to 0 *in probability as* $n \to \infty$*.*

Then, the approximation (2) is valid in that (13) holds with $\Sigma_n = \Pi_n[\varphi^2]T$ *and* $p_n = p$ *defined as (3) with* $b = b_n$*,* $c = c_n$ *and* $\Sigma = \Sigma_n$*.*

Proof: Let us verify (12) with $\Sigma_n = \Pi_n[\varphi^2]T$ and

$$V_{12} = -2\lim_{n\to\infty} \Sigma_n^{-3/2}\Pi_n[\varphi\rho\psi_n].$$

Notice that by the Itô-Tanaka formula,

$$\begin{aligned}\langle M^n\rangle_T - \Pi_n[\varphi^2]T &= \int_0^T (\varphi(X_t^n)^2 - \Pi_n[\varphi^2])\mathrm{d}t \\ &= \epsilon_n \int_{X_0^n}^{X_T^n} \frac{\psi_n(x)}{c_n(x)}\mathrm{d}x - \epsilon_n \int_0^T \psi_n(X_t^n)\mathrm{d}W_t^1.\end{aligned}$$

It suffices then to prove the asymptotic normality of

$$\left(\int_0^T \varphi(X_t^n)\left[\rho(X_t^n)\mathrm{d}W_t^1 + \sqrt{1-\rho(X_t^n)^2}\mathrm{d}W_t^2\right], \int_0^T \psi_n(X_t^n)\mathrm{d}W_t^1\right).$$

This follows from the martingale central limit theorem under the fifth assumption. ////

The conditions are easily verified in such a case that both Π_n and s_n do not depend on $n \in \mathbb{N}$. The model (4) with $\Lambda \equiv 0$ and $\eta = \eta_n$, where η_n is a positive sequence with $\eta_n \to 0$, is an example of such an easy case.

3 Main results

3.1 Main theorem and remarks

Here we state the main results of this article. We treat (1) with Borel functions φ, ρ satisfying $|\rho| \leq 1$, b being a locally integrable function on $\mathbb{R}$ and c being a positive Borel function such that $(1+c^2)/c$ is locally bounded on $\mathbb{R}$. We suppose that φ also is locally bounded on $\mathbb{R}$ and that there exists a non-empty open set $U \subset \mathbb{R}$ such that it holds on U that

i. φ and ρ are continuously differentiable,

ii. $(1-\rho^2)\varphi^2 > 0$ and $|\varphi'| > 0$.

If φ is constant, then the approximation (2) is trivially valid. Since U can be any open set as long as it is not empty, this condition is not restrictive in the context of stochastic volatility models. This rules out, however, the case $|\rho| \equiv 1$. We can introduce alternative framework to include such a case although we do not go to the details in this article for the sake of brevity. We fix φ, ρ, U and assume $(Z_0, X_0) = (0, 0)$ without loss of generality.

Define the scale function $s : \mathbb{R} \to \mathbb{R}$ of X and the normalizing constant $\epsilon > 0$ as in Section 1. It is well-known that the stochastic differential equation for X in (1) has a unique weak solution which is ergodic if $\epsilon < \infty$ and $s(\mathbb{R}) = \mathbb{R}$. The ergodic distribution Π of X is given as in Section 1. See e.g., Skorokhod [18], Section 3.1. Denote by π the density of Π. Notice that X is completely characterized by (π, s, ϵ). In fact, we can recover b and c by $1/c^2 = \epsilon^2 s'\pi$ and $b = -c^2 s''/2s'$. Taking this into mind, denote by $\mathscr{C}$ the set of all triplets (π, s, ϵ) with π being a locally bounded probability density function on $\mathbb{R}$ such that $1/\pi$ is also locally bounded on $\mathbb{R}$, s being a bijection from $\mathbb{R}$ to $\mathbb{R}$ such that the derivative s' exists and is a positive absolutely continuous function, and ϵ being a positive finite constant.

For given $\gamma = (\gamma_+, \gamma_-) \in [0, \infty)^2$ and $\delta \in (0, 1)$, denote by $\mathscr{C}(\gamma, \delta)$ the set of $\theta = (\pi, s, \epsilon) \in \mathscr{C}$ satisfying Conditions 3.1, 3.2 below.

Condition 3.1. *It holds that*

$$(1+\varphi(x)^2)\pi(x)s'(y) \le \exp\{-\log(\delta) + \gamma_+ x - (4\gamma_+ + \delta)(x-y)\}$$

for all $x \ge y \ge 0$ and

$$(1+\varphi(x)^2)\pi(x)s'(y) \le \exp\{-\log(\delta) - \gamma_- x + (4\gamma_- + \delta)(x-y)\}$$

for all $x \le y \le 0$.

Condition 3.2. *There exist $x \in U$ and $a \in [\delta, 1/\delta]$ such that $|x| \le 1/\delta$, $[x-a, x+a] \subset U$, π is absolutely continuous on $[x-a, x+a]$ and it holds*

$$\sup_{y \in [x-a, x+a]} \left| \left(\sqrt{\frac{\pi}{s'}} \varphi \rho \right)'(y) \right| \vee s'(y) \vee \pi(y) \vee \frac{1}{s'(y)} \vee \frac{1}{\pi(y)} \le 1/\delta.$$

Given $\theta \in \mathscr{C}$, we write $\pi_\theta, s_\theta, \epsilon_\theta, b_\theta, c_\theta, Z^\theta$ for the elements of $\theta = (\pi, s, \epsilon)$, the corresponding coefficients b, c of the stochastic differential equations, and the log price process Z defined as (1) respectively.

Theorem 3.3. *Fix $\gamma = (\gamma_+, \gamma_-) \in [0, \infty)^2$ and $\delta \in (0, 1)$. Denote by $\mathscr{B}_\delta$ the set of the Borel functions bounded by $1/\delta$. Then,*

$$\sup_{f \in \mathscr{B}_\delta, \theta \in \mathscr{C}(\gamma, \delta)} \epsilon_\theta^{-2} \left| \mathbb{E}[f(Z_T^\theta)] - \mathbb{E}[(1 + p_\theta(N)) f(-\log(D) - \Sigma_\theta/2 + \sqrt{\Sigma_\theta} N)] \right|$$

is finite, where $N \sim \mathscr{N}(0, 1)$, $\Sigma_\theta = \Pi_\theta[\varphi^2]T$, $\Pi_\theta(\mathrm{d}x) = \pi_\theta(x)\mathrm{d}x$ and D, $p = p_\theta$ are defined by (3) with $\Sigma = \Sigma_\theta$, $\Pi = \Pi_\theta$, $c = c_\theta$.

Remark 3.4. The point of the definition of $\mathscr{C}(\gamma,\delta)$ is that it is written independently of ϵ. As a result, if $\theta \in \mathscr{C}(\gamma,\delta)$, then $(\pi_\eta, s_\eta, \epsilon_\eta)$ associated with the drift coefficient $b_\eta = b_\theta/\eta^2$ and the diffusion coefficient $c_\eta = c_\theta/\eta$ is also an element of $\mathscr{C}(\gamma,\delta)$ for any $\eta > 0$. In fact $\pi_\eta = \pi_\theta$ and $s_\eta = s_\theta$. On the other hand, $\epsilon_\eta = \eta\epsilon_\theta$, so that Theorem 3.3 implies, with a slight abuse of notation,

$$E[f(Z_T^\eta)] = E[(1+p_\eta(N))f(Z_0 - \log(D) - \Sigma_\eta/2 + \sqrt{\Sigma_\eta}N)] + O(\eta^2) \tag{14}$$

as $\eta \to 0$.

Remark 3.5. Given $\theta \in \mathscr{C}$, Condition 3.2 does not hold for any $\delta > 0$ only when considering vicious examples such as the case $(\sqrt{\pi_\theta/s'_\theta}\varphi\rho)'$ is not continuous at any point of U; a sufficient condition for Condition 3.2 to hold with some $\delta > 0$ is that $(\sqrt{\pi_\theta/s'_\theta}\varphi\rho)'$ is continuous at some point of U. If Condition 3.2 holds with some $\delta > 0$, then it holds with any $\hat{\delta} \in (0,\delta]$ as well.

3.2 Examples

Lemma 3.6. *Let $\theta \in \mathscr{C}$. If there exist $(\gamma_+,\gamma_-) \in [0,\infty)^2$ such that*

$$\kappa_\pm > 2\gamma_\pm, \quad \limsup_{v\to\pm\infty} \frac{1+\varphi(v)^2}{e^{\gamma_\pm |v|}c_\theta(v)^2} < \infty \tag{15}$$

with

$$\kappa_+ = -\limsup_{v\to\infty} \frac{b_\theta(v)}{c_\theta(v)^2}, \quad \kappa_- = \liminf_{v\to-\infty} \frac{b_\theta(v)}{c_\theta(v)^2},$$

then there exists $\delta_0 > 0$ such that for any $\delta \in (0,\delta_0 \wedge 1)$, Condition 3.1 holds for $\theta = (\pi,s,\epsilon)$ with $\gamma = (\gamma_+,\gamma_-)$ and δ.

Proof: This is shown in a straightforward manner by (3). ////

Example 3.7. Consider

$$\begin{aligned} \mathrm{d}Z_t &= \left\{ r_t - \frac{1}{2}V_t \right\}\mathrm{d}t + \sqrt{V_t}(\rho \mathrm{d}W_t^1 + \sqrt{1-\rho^2}\mathrm{d}W_t^2) \\ \mathrm{d}V_t &= \xi\eta^{-2}(\mu - V_t)\mathrm{d}t + \eta^{-1}|V_t|^\nu \mathrm{d}W_t^1 \end{aligned}$$

for positive constants $\xi,\mu,\eta > 0$, $\rho \in (-1,1)$ and $\nu \in [1/2,\infty)$. We assume $\xi\mu > 1/2$ if $\nu = 1/2$. Then, the scale function s^V of V satisfies $s^V((0,\infty)) = \mathbb{R}$, so that we can apply Itô's formula to $X = \log(V)$ to have

$$\mathrm{d}X_t = \eta^{-2}(\xi\mu e^{-X_t} - \xi - e^{-2(1-\nu)X_t}/2)\mathrm{d}t + \eta^{-1}e^{-(1-\nu)X_t}\mathrm{d}W_t^1.$$

In this scale, $\varphi(x) = \exp(x/2)$, so that we can take any open set as $U \subset \mathbb{R}$. We fix ξ,μ,ν,ρ arbitrarily. In the light of Remark 3.4, it suffices to verify Conditions 3.1 and 3.2 only when $\eta = 1$. It is trivial that Condition 3.2 holds with a sufficiently small $\delta > 0$. If $\nu = 1/2$, then (15) also holds with

$$\kappa_+ = \infty, \quad \kappa_- = \xi\mu - \frac{1}{2}, \quad \gamma_+ = 2, \quad \gamma_- = 0.$$

If $\nu \in (1/2, 1)$, then it holds with

$$\kappa_{\pm} = \infty, \ \ \gamma_+ = 3 - 2\nu, \ \ \gamma_- = 0.$$

If $\nu = 1$, then it holds with

$$\kappa_+ = \xi + \frac{1}{2}, \ \ \kappa_- = \infty, \ \ \gamma_+ = 1, \ \ \gamma_- = 0$$

provided that $\xi > 3/2$. Unfortunately, (15) does not hold if $\nu \in (1, 11/8]$. If $\nu > 11/8$, it then holds with

$$\kappa_+ = \frac{1}{2}, \ \ \kappa_- = \infty \ \ \gamma_+ = (3 - 2\nu)_+, \ \ \gamma_- = 2\nu - 2.$$

Note that the case $\nu = 1/2$ corresponds to the Heston model. In this case, we have a more explicit expression of the asymptotic expansion formula; we have (14) with

$$p_\eta(z) = \frac{\eta\rho}{2\xi}\left\{1 - z^2 + \frac{1}{\Sigma_\eta^{1/2}}(z^3 - 3z)\right\}, \ \ \Sigma_\eta = \mu T.$$

This is due to the fact that the ergodic distribution of the CIR process is a gamma distribution.

Example 3.8. Here we treat (4). In order to prove the validity of the singular expansion in the form (14) for $Z_T^\eta = \log(S_T^\eta)$, it suffices to show that there exist γ, δ and $\eta_0 > 0$ such that Conditions 3.1 and 3.2 hold for $\theta = (\pi, s, \epsilon) \in \mathscr{C}$ associated with

$$b_\theta(x) = m - x - \eta\nu\sqrt{2}\Lambda(x), \ \ c_\theta(x) = \nu\sqrt{2}$$

for any $\eta \in (0, \eta_0]$, in the light of Remark 3.4. Here we fix $m \in \mathbb{R}$ and $\nu \in (0, \infty)$. Suppose that there exists $(\gamma_+, \gamma_-) \in [0, \infty)^2$ such that

$$\limsup_{x \to \pm\infty} e^{-\gamma_\pm |x|}\varphi^2(x) < \infty$$

and that Λ is locally bounded on $\mathbb{R}$ with

$$\lambda_\infty := \liminf_{|x| \to \infty} \frac{\Lambda(x)}{x} > -\infty.$$

Then we have

$$-\mathrm{sgn}(v)\frac{b_\theta(v)}{c_\theta(v)^2} \to \infty,$$

as $|v| \to \infty$ uniformly in $\eta \in (0, \eta_0]$ with, say, $\eta_0 = 1 \wedge |1/(2\nu\lambda_\infty \wedge 0)|$. Hence, by Lemma 3.6, there exists $\delta \in (0, 1)$ such that Condition 3.1 holds for any $\eta \in (0, \eta_0]$ with $\gamma = (\gamma_+, \gamma_-)$ and δ. By, if necessary, replacing (δ, η_0) with a smaller one, Condition 3.2 also is verified for any $\eta \in (0, \eta_0]$ provided that there exists a non-empty open set U such that φ is continuously differentiable on U. Consequently, by Theorem 3.3, we have (14) for (4) if $|\rho| < 1$ and $|\varphi'| > 0$ on U in addition. The obtained estimate of error $O(\eta^2)$ is a stronger result than one obtained by Fouque et al. [9][10] and Alòs [1].

Example 3.9. Here we treat a diffusion which is not geometrically mixing. Consider the stochastic differential equation

$$\mathrm{d}X_t = -\frac{1}{\eta^2}\left(\frac{1}{2}+\xi\right)\frac{\tanh(Y_t)}{\cosh(Y_t)^2}\mathrm{d}t + \frac{1}{\eta}\frac{1}{\cosh(X_t)}\mathrm{d}W_t$$

with $\xi > 1/2$ and $\eta > 0$. Putting $Y_t = \sinh(X_t)$, we have

$$\mathrm{d}Y_t = -\frac{1}{\eta^2}\frac{\xi Y_t}{1+Y_t^2}\mathrm{d}t + \frac{1}{\eta}\mathrm{d}W_t$$

This stochastic differential equation has a unique weak solution which is ergodic. A polynomial lower bound for the α mixing coefficient is given in Veretennikov [19] which implies in particular that $X = \sinh^{-1}(Y)$ is not geometrically mixing for any ξ. Now, let us verify Conditions 3.1 and 3.2 for (1) with

$$b(x) = -\frac{1}{\eta^2}\left(\frac{1}{2}+\xi\right)\frac{\tanh(x)}{\cosh(x)^2},\quad c(x) = \frac{1}{\eta}\frac{1}{\cosh(x)}$$

for any $\eta > 0$. In the light of Remark 3.4, it suffices to deal with the case $\eta = 1$. Since

$$-\lim_{|x|\to\infty}\mathrm{sgn}(x)\frac{b(x)}{c(x)^2} = \frac{1}{2}+\xi,$$

we have (15) if there exists $\mu \geq 0$ such that

$$\sup_{|x|\to\infty} e^{-\mu|x|}\varphi(x)^2 < \infty,\quad \frac{1}{2}+\xi > 4+2\mu.$$

Condition 3.2 also is satisfied with a sufficiently small $\delta > 0$ under the condition on φ and ρ stated in the beginning of this section.

4 Edgeworth expansion

In this section, we present basic results of the Edgeworth expansion which play an essential role in the proof of Theorem 3.3 given in the next section. In Section 4.1, we give a validity theorem for the classical iid case with a brief introduction to the Edgeworth expansion theory. The theorem is applied to a non-iid case by the regenerative approach in Section 4.2 to establish a general validity theorem for regenerative functionals including additive functionals of ergodic diffusions.

4.1 The Edgeworth and Gram-Charlier expansions

The Edgeworth expansion is a rearrangement of the Gram-Charlier expansion. Let Y be a random variable with $\mathbb{E}[Y] = 0$ and $\mathbb{E}[Y^2] = 1$. If it has a density p_Y with an integrability condition

$$\int p_Y(z)^2\phi(z)^{-1}\mathrm{d}z < \infty, \tag{16}$$

where ϕ is the standard normal density, then we have

$$p_Y/\phi = \sum_{j=0}^{\infty} \frac{1}{j!} \mathbb{E}[H_j(Y)] H_j$$

in $L^2(\phi)$ with Hermite polynomials H_j defined as the coefficients of the Taylor series

$$e^{tx-t^2/2} = \sum_{j=0}^{\infty} H_j(x) \frac{t^j}{j!}, \quad (t,x) \in \mathbb{R}^2. \tag{17}$$

This is an orthonormal series expansion of $p_Y/\phi \in L^2(\phi)$ and implies that

$$\mathbb{E}[f(Y)] = \sum_{j=0}^{\infty} \frac{1}{j!} \mathbb{E}[H_j(Y)] \int f(z) H_j(z) \phi(z) \mathrm{d}z \tag{18}$$

for $f \in L^2(\phi)$. The Edgeworth formula is obtained by rearranging this Gram-Charlier series. For example, if $Y = m^{-1/2} \sum_{j=1}^m X_j$ with an iid sequence X_j, then the j-th cumulant κ_j^Y of Y is of $O(m^{1-j/2})$. This is simply because

$$\partial^j \log(\psi_Y(u)) = m \partial^j \log(\psi_X(m^{-1/2}u)),$$

where ψ_Y and ψ_X are the characteristic functions of Y and X_j respectively. Even if Y is not an iid sum, $\kappa_j^Y = O(m^{1-j/2})$ often remains true in cases where Y converges in law to a normal distribution as $m \to \infty$. Since $\mathbb{E}[H_0(Y)] = 1$, $\mathbb{E}[H_1(Y)] = \mathbb{E}[H_2(Y)] = 0$ and for $j \geq 3$,

$$\mathbb{E}[H_j(Y)] = \sum_{k=1}^{[j/3]} \sum_{r_1+\cdots+r_k=j,\ r_j \geq 3} \frac{\kappa_{r_1}^Y \ldots \kappa_{r_k}^Y}{r_1! \ldots r_k!} \frac{j!}{k!}$$

by (17), it follows from (18) that

$$\mathbb{E}[f(Y)] = \sum_{j=0}^{J} m^{-j/2} \int f(z) q_j(z) \phi(z) \mathrm{d}z + o(m^{-J/2})$$

with suitable polynomials q_j. Taking $J = 0$, we have the central limit theorem; in this sense, the Edgeworth expansion is a refinement of the central limit theorem. This asymptotic expansion can be validated under weaker conditions than (16); see Bhattacharya and Rao [3] and Hall [14] for iid cases. Here we give one of the validity theorems which is used in the next subsection.

Theorem 4.1. *Let X_j^n be a triangular array of d-dimensional independent random variables with mean 0. Assume that $X_j^n \sim X_1^n$ for all j and that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|X_1^n|^{\xi}] < \infty$$

for an integer $\xi \geq 4$,

$$\sup_{|u| \geq b, n \in \mathbb{N}} |\Psi^n(u)| < 1, \quad \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d} |\Psi^n(u)|^{\eta} \mathrm{d}u < \infty,$$

for all $b > 0$ and for some $\eta \geq 1$ respectively, where

$$\Psi^n(u) = \mathbb{E}[\exp\{iu \cdot X_1^n\}].$$

Then, there exists m_0 such that $S_m^n = m^{-1/2}\sum_{j=1}^m X_j^n$ has a bounded density p_m^n for all $m \geq m_0$, $n \in \mathbb{N}$. Further, it holds that

$$\sup_{x\in\mathbb{R}^k, m\geq m_0, n\in\mathbb{N}} m(1+|x|^\xi)|p_m^n(x) - q_m^n(x)| < \infty,$$

where

$$q_m^n(x) = \phi(x;0,v^n) - \frac{1}{6\sqrt{m}}\sum_{i,j,k=1}^d \kappa_{ijk}^n \partial_i\partial_j\partial_k \phi(x;0,v^n)$$

with the variance matrix v^n of X_1^n and the third moment κ_{ijk}^n of X_1^n.

Proof: This result is a variant of Theorem 19.2 of Bhattacharya and Rao [3]. Although the distribution of X_1^n depends on n, the assertion is proved in a similar manner with the aid of Theorem 9.10 of Bhattacharya and Rao [3], due to our assumptions. For example, we have (and use)

$$0 < \inf_{|u|=1,n\in\mathbb{N}} \mathbb{E}[|u\cdot X_1^n|^2] \leq \sup_{|u|=1,n\in\mathbb{N}} \mathbb{E}[|u\cdot X_1^n|^2] < \infty.$$

////

Remark that an Edgeworth-type result typically requires the existence of moments up to a sufficiently large order and the smoothness of a distribution to hold. In an iid case, such conditions are verifiable because they are usually given in terms of the identical distribution of the summands, as in the above theorem. It is not the case when considering non-iid summands. The technique of the regenerative approach presented in the next subsection is to decompose a sum or integral of a dependent process into iid blocks.

4.2 Edgeworth expansion for regenerative functionals

We have seen that the fast mean reverting expansion gives a correction term to the Black-Scholes price that corresponds to the central limit of an additive functional of ergodic diffusion in Section 2.1. In order to prove the validity of the expansion, it is therefore natural to apply the Edgeworth expansion theory for ergodic diffusions. Here we present a general result for triangular arrays of regenerative functionals, which extends a result for additive functionals of ergodic diffusions given by Fukasawa [11]. Let $\mathbb{P}^n = (\Omega^n, \mathscr{F}^n, \{\mathbb{F}_t^n\}, P^n)$ be a family of filtered probability spaces satisfying the usual assumptions and $K^n = (K_t^n)$ be an $\{\mathbb{F}_t^n\}$-adapted cadlag process defined on $\mathbb{P}^n$. What we treat in the proof of the main result is essentially of the form

$$K_t^n = \left(\int_0^t h(X_s)\mathrm{d}s, \int_0^t \varphi(X_s)[\rho(X_s)\mathrm{d}W_s^1 + \sqrt{1-\rho(X_s)^2}\mathrm{d}W_s^2]\right)$$

with $h = T\varphi^2 - \Sigma$ for each $n \in \mathbb{N}$, where X, φ, ρ and Σ are the same as in Section 1. We however work for a while in more general framework of regenerative functionals in order to clarify the essence of the argument. For a given sequence of increasing $\{\mathbb{F}^n_t\}$-stopping times $\{\tau^n_j\}$ with $\tau^n_0=0$ and $\lim_{j\to\infty}\tau^n_j=\infty$, put

$$\mathscr{K}^n_j=\left(\mathscr{K}^n_{j,t}\right)_{t\geq 0},\ \ \mathscr{K}^n_{j,t}=K^n_{t+\tau^n_j}-K^n_{\tau^n_j},\ \ l^n_j=\tau^n_{j+1}-\tau^n_j,\ \ j=0,1,2,\dots$$

We say that K^n is a **regenerative functional** if there exists $\{\tau^n_j\}$ such that

(i) $(\mathscr{K}^n_j,l^n_j)$ is independent of $\mathbb{F}^n_{\tau^n_j}$ for each $j=1,2,\dots,$
(ii) $(\mathscr{K}^n_j,l^n_j)$, $j=1,2,\dots$ are identically distributed.

Let K^n be a d-dimensional regenerative functional and put $\bar{\mathscr{K}}^n_j=(\mathscr{K}^n_{j,l^n_j},l^n_j)$ for $j=0,1,\dots.$ The idea of the regenerative approach is to use the fact that $\bar{\mathscr{K}}^n_j$, $j\geq 1$ is an iid sequence and independent of $\bar{\mathscr{K}}^n_0$. Denote by $E^n[\cdot]$ and $\mathrm{Var}^n[\cdot]$ the expectation and variance with respect to P^n respectively. Assume that $\mathrm{Var}^n[\bar{\mathscr{K}}^n_j]$ exists and is of rank $d'+1$ with $1\leq d'\leq d$ for all $j\geq 1$. Without loss of generality, assume that there exists a d'-dimensional iid sequence G^n_j, $j\geq 1$ such that the variance matrix of (G^n_j,l^n_j) is of full rank and that

$$\bar{\mathscr{K}}^n_j=(G^n_j,R^n_j,l^n_j) \tag{19}$$

with a $d-d'$ dimensional sequence R^n_j. Put

$$\begin{aligned}&m^n_L=E^n[l^n_1],\ \ m^n_G=E^n[G^n_1],\ \ m^n_R=E^n[R^n_1],\\&\mu^n=(\mu^n_k)=(m^n_G,m^n_R)/m^n_L,\end{aligned}$$

and

$$\mathbb{K}^n_j=(\mathbb{G}^n_j,l^n_j),\ \ \mathbb{G}^n_j=G^n_j-l^n_jm^n_G/m^n_L,\ \ j\in\mathbb{N}.$$

Due to the definition, it is not difficult to see a law of large numbers holds:

$$K^n_T/T\to\mu^n$$

in probability as $T\to\infty$. Further, a central limit theorem

$$\sqrt{T}(K^n_T/T-\mu^n)\Rightarrow\mathscr{N}(0,V^n)$$

holds with a suitable matrix V^n. Our aim here is to give a refinement of this central limit theorem. More precisely, for a given function $A^n:\mathbb{R}^d\to\mathbb{R}$ and a positive sequence $T_n\to\infty$, we present a valid approximation of the distribution of

$$\sqrt{T_n}(A^n(K^n_{T_n}/T_n)-A^n(\mu^n))$$

up to $O(T_n^{-1})$ as $n\to\infty$. As far as considering this form, we can assume without loss of generality that $E^n[|R^n_j|]=0$ for all $j\geq 1$ in (19). Put

$$(\mu^n_{k,l})=\mathrm{Var}^n[\mathbb{G}^n_1]/m^n_L,\ \ \rho^n=(\rho^n_k)=\mathrm{Cov}^n[\mathbb{G}^n_1,l^n_1]$$

and

$$\mu^n_{k,l,m}=(\kappa^n_{k,l,m}-\rho^n_k\mu^n_{l,m}-\rho^n_l\mu^n_{m,k}-\rho^n_m\mu^n_{k,l})/m^n_L,$$

where $(\kappa^n_{k,l,m})$ is the third moment of $\mathbb{G}^n_1$.

We have decomposed $(K^n_{T_n},T_n)$ into the blocks $\bar{\mathscr{K}}^n_j$. Notice that the number of blocks obtained up to T_n is random. To control its distribution, we put the following condition on m^n_L.

Condition 4.2. *It holds that*

$$\inf_{n\in\mathbb{N}} m_L^n > 0.$$

The next condition corresponds to the assumption on the existence of moments of iid summands that is required in the classical Edgeworth theory.

Condition 4.3. *For $\xi = (d'+2)\vee 4$, it holds that*

$$\sup_{n\in\mathbb{N}}\left\{E^n[|\bar{\mathscr{K}}_0^n|^2] + E^n[|\mathbb{K}_1^n|^\xi] + E^n\left[\left|\int_{\tau_1^n}^{\tau_2^n}|\mathscr{K}_{1,t}^n|^2\mathrm{d}t\right|\right]\right\} < \infty.$$

Under Conditions 4.2 and 4.3, the sequences μ^n, $(\mu_{k,l}^n)$, $(\mu_{k,l,m}^n)$ are bounded in $n\in\mathbb{N}$. The next condition corresponds to the assumption on the smoothness of the identical distribution of summands in the classical Edgeworth theory.

Condition 4.4. *Let Ψ^n be the characteristic function of $\mathbb{K}_1^n$:*

$$\Psi^n(u) = E^n[\exp\{iu\cdot\mathbb{K}_1^n\}].$$

It holds

$$\sup_{|u|\geq b, n\in\mathbb{N}} |\Psi^n(u)| < 1$$

for all $b>0$ and there exists $\eta\geq 1$ such that

$$\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^{d'+1}}|\Psi^n(u)|^\eta \mathrm{d}u < \infty.$$

Note that under Conditions 4.3 and 4.4, it holds

$$0 < \inf_{|a|=1,n\in\mathbb{N}} E^n[|a\cdot\mathbb{K}_1^n|^2] \leq \sup_{|a|=1,n\in\mathbb{N}} E^n[|a\cdot\mathbb{K}_1^n|^2] < \infty,$$

that is, the largest and smallest eigenvalues of the variance matrix of $\mathbb{K}_1^n$ is bounded and bounded away from 0 in $n\in\mathbb{N}$.
Let $B_n(\zeta) = \{x\in\mathbb{R}^d; |x-\mu^n| < \zeta\}$ for $\zeta > 0$,

$$a_i^n = \partial_i A^n(\mu^n), \quad a_{i,j}^n = \partial_i\partial_j A^n(\mu^n), \quad 1\leq i,j\leq d$$

for a given function $A^n:\mathbb{R}^d\to\mathbb{R}$ which is twice differentiable at the point μ^n and

$$a^n = (a_k^n)\in\mathbb{R}^d, \quad v^n = \sum_{k,l=1}^{d'}\mu_{k,l}^n a_k^n a_l^n.$$

We put the following condition on A^n.

Condition 4.5. *There exists $\zeta > 0$ such that*

i. $A^n:\mathbb{R}^d\to\mathbb{R}$ is four times continuously differentiable on $B_n(\zeta)$ for all n, *ii. all the derivatives up to fourth order are bounded on $B_n(\zeta)$ uniformly in n,*

iii. it holds that

$$0 < \inf_{n\in\mathbb{N}} v^n \le \sup_{n\in\mathbb{N}} v^n < \infty.$$

Denote by ι the natural inclusion: $\mathbb{R}^{d'} \ni v \mapsto (v,0,\dots,0) \in \mathbb{R}^d$.

Theorem 4.6. *Let M be a positive constant and $\mathscr{B}_M$ be the set of Borel functions on $\mathbb{R}$ which are bounded by M. Under Conditions 4.2, 4.3, 4.4 and 4.5, it holds that*

$$\sup_{H\in\mathscr{B}_M, n\in\mathbb{N}} T_n \left| E^n[H(\sqrt{T_n}(A^n(K^n_{T_n}/T_n) - A^n(\mu^n)))] - \int H(z)q^n(z)dz \right| < \infty,$$

where q^n is defined as

$$q^n(z) = \phi(z;v^n) + T_n^{-1/2}\left\{A_1^n q_1(z;v^n) + \frac{A_3^n}{6}q_3(z;v^n)\right\} \tag{20}$$

with $\phi(z;v^n)$ being the normal density with mean 0 and variance v^n,

$$q_1(z;v^n) = -\partial\phi(z;v^n), \;\; q_3(z;v^n) = -\partial^3\phi(z;v^n),$$

and

$$\begin{aligned} A_1^n &= \frac{1}{2}\sum_{k,l=1}^{n'} a^n_{k,l}\mu^n_{k,l} + a^n\cdot\left\{E^n[K^n_{\tau^n_1}] + \frac{1}{m^n_L}E^n\left[\int_{\tau^n_1}^{\tau^n_2} K^n_t \mathrm{d}t\right] - \frac{\iota(\rho^n)}{m^n_L}\right\}, \\ A_3^n &= \sum_{k,l,m=1}^{n'} a^n_k a^n_l a^n_m \mu^n_{k,l,m} + 3\sum_{j,k,l,m=1}^{n'} a^n_j a^n_k a^n_{l,m}\mu^n_{j,l}\mu^n_{k,m}. \end{aligned} \tag{21}$$

Proof: The proof is a repetition of the proof of Theorem 4.1 in Fukasawa [11] with the aid of Theorem 4.1 in the previous subsection and so is omitted. ////

5 Proof of Theorem 3.3

Here we give the proof of Theorem 3.3. We formulate the problem in terms of the distribution of a regenerative functional and verify all the conditions for Theorem 4.6 to hold. We are considering (1) with φ, ρ, U satisfying the condition stated in the beginning of Section 3. The initial value $(Z_0, X_0) = (0,0)$ and the time to maturity T are fixed. Now, to obtain a contradiction, let us suppose that the supremum in Theorem 3.3 is infinite. Then there exists a sequence $\theta_n \in \mathscr{C}(\gamma,\delta)$ such that

$$\epsilon_n^{-2}\left|\mathbb{E}[f(Z^n_T)] - \mathbb{E}[(1+p_n(N))f(-\log(D) - \Sigma_n/2 + \sqrt{\Sigma_n}N)]\right| \to \infty \tag{22}$$

as $n\to\infty$, where $\epsilon_n = \epsilon_{\theta_n}$, $p_n = p_{\theta_n}$, $\Sigma_n = \Sigma_{\theta_n}$. Put $\hat{b}_n = \epsilon_n^2 b_{\theta_n}$, $\hat{c}_n = \epsilon_n c_{\theta_n}$ and denote by $\mathbb{E}^n_x$ the expectation operator with respect to the law of $\hat{X}^n$ determined by the stochastic differential equation

$$\mathrm{d}\hat{X}^n_t = \hat{b}_n(\hat{X}^n_t)\mathrm{d}t + \hat{c}_n(\hat{X}^n_t)\mathrm{d}\hat{W}^1_t, \;\; \hat{X}^n_0 = x \in \mathbb{R},$$

where $\hat{W}^1$ is a standard Brownian motion. It is easy to see that the law of $X = \{X_t\}$ in (1) is the same as that of $\left\{\hat{X}^n_{t/\epsilon_n^2}\right\}$ with $\hat{X}^n_0 = 0$. Hence, the law of $Z^n = Z^{\theta_n}$ is the same as that of

$$-\log(D) - \frac{1}{2}\Sigma_n + \sqrt{\Sigma_n}\sqrt{T_n}A^n(K^n_{T_n}/T_n)$$

under $\mathbb{E}^n_0$, where

$$T_n = \frac{T}{\epsilon_n^2}, \quad A^n(x,y) = \frac{\sqrt{T}y - T_n^{-1/2}x/2}{\sqrt{\Sigma_n}}, \quad h_n = T\varphi^2 - \Sigma_n, \tag{23}$$

$$K^n_t = \left(\int_0^t h_n(\hat{X}^n_s)\mathrm{d}s, \int_0^t \varphi(\hat{X}^n_s)\left[\rho(\hat{X}^n_s)\mathrm{d}\hat{W}^1_s + \sqrt{1-\rho(\hat{X}^n_s)^2}\mathrm{d}\hat{W}^2_s\right]\right)$$

and $(\hat{W}^1, \hat{W}^2)$ is a 2-dimensional standard Brownian motion. By the strong Markov property, K^n is a regenerative functional in the sense given in the previous section with the stopping times $\{\tau^n_j\}$ defined as

$$\tau^n_0 = 0, \quad \tau^n_{j+1} = \inf\left\{ t > \tau^n_j; \hat{X}^n_t = x^n_0, \sup_{s\in[\tau^n_j,t]} \hat{X}^n_s \geq x^n_1 \right\}, \tag{24}$$

with an arbitrarily fixed point $(x^n_0, x^n_1) \in \mathbb{R}^2$ with $x^n_0 < x^n_1$. Let us take $x^n_0 = x$, $x^n_1 = x + a$ with (x,a) which satisfies Condition 3.2; recall that $\theta_n \in \mathscr{C}(\gamma,\delta)$, so that we can find such a pair (x,a) for each n.

Put $s_n = s_{\theta_n}$, $\pi_n = \pi_{\theta_n}$ and $\Pi_n = \Pi_{\theta_n}$. To verify all the conditions for Theorem 4.6 to hold, we use the following more-or-less known identities. The first one is that

$$\Pi_n[g] = \frac{1}{\mathbb{E}_{x^n_0}[\tau^n_1]}\mathbb{E}\left[\int_0^{\tau^n_1} g(\hat{X}^n_t)\mathrm{d}t\right] \tag{25}$$

for all integrable function g; see e.g., Skorokhod [18]. Section 3.1. The second one is Kac's moment formula [7]: for given a positive Borel function g, define

$$G^k_g(y;z) = \mathbb{E}^n_y\left[\int_0^{\tau(z)} g(\hat{X}^n_t)G^{k-1}_g(\hat{X}^n_t;z)\mathrm{d}t\right]$$

recursively for $k \in \mathbb{N}$, where $y,z \in \mathbb{R}$, $G^0_g(y;z) \equiv 1$ and

$$\tau(z) = \inf\{t > 0; \hat{X}^n_t = z\}.$$

Then, it holds that

$$\mathbb{E}^n_y\left[\left|\int_0^{\tau(z)} g(\hat{X}^n_t)\mathrm{d}t\right|^k\right] = k!G^k_g(y;z) \tag{26}$$

for any $y,z \in \mathbb{R}$. The third one is that

$$G^1_g(y;z) = 2\int_y^z (s_n(z) - s_n(x))g(x)\Pi_n(\mathrm{d}x) + 2(s_n(z) - s_n(y))\int_{-\infty}^y g(x)\Pi_n(\mathrm{d}x)$$

if $y \le z$, and

$$G_g^1(y;z) = 2(s_n(y) - s_n(z)) \int_y^\infty g(x)\Pi_n(\mathrm{d}x) + 2\int_z^y (s_n(x) - s_n(z))g(x)\Pi_n(\mathrm{d}x)$$

if $y > z$. See Skorokhod [18], Section 3.1 for the details.

Lemma 5.1. *Condition 4.2 holds.*

Proof: By the strong Markov property and the above identities,

$$m_L^n = \mathbb{E}[\tau_2^n - \tau_1^n] = \mathbb{E}^n_{x_0^n}[\tau(x_1^n)] + \mathbb{E}^n_{x_1^n}[\tau(x_0^n)] = 2(s_n(x_1^n) - s_1(x_0^n)).$$

The result then follows from Condition 3.2. ////

Lemma 5.2. *Condition 4.3 holds.*

Proof: By the Burkholder-Davis-Gundy inequality and the strong Markov property, it suffices to show

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n_y \left[|\tau(z)|^4 + \left| \int_0^{\tau(z)} |h_n(\hat{X}^n_t)|\mathrm{d}t \right|^4 + \left| \int_0^{\tau(z)} \varphi(\hat{X}^n_t)^2 \mathrm{d}t \right|^2 \right] < \infty$$

for $(y,z) = (0, x_0^n)$, $(y,z) = (x_0^n, x_1^n)$ and $(y,z) = (x_1^n, x_0^n)$. We only need to show

$$\sup_{n\in\mathbb{N}} \mathbb{E}^n_y \left[\left| \int_0^{\tau(z)} (1 + \varphi(\hat{X}^n_t)^2)\mathrm{d}t \right|^4 \right] < \infty. \tag{27}$$

because $h_n = T\varphi^2 - \Sigma_n$,

$$\Sigma_n = T\Pi_n[\varphi^2] = \frac{T}{m_L^n}\mathbb{E}^n_{x_0^n}\left[\int_0^{\tau_1^n} \varphi(\hat{X}^n_t)^2 \mathrm{d}t \right]$$

and $\inf_n m_L^n > 0$ by Lemma 5.1. By Condition 3.1, we have

$$s_n'(w)g_k(v)\pi_n(v) \le \frac{1}{\delta} e^{(k+1)\gamma_\pm|v| - (4\gamma_\pm + \delta)|v-w|)}$$

for $g_k(v) = (1 + \varphi(v)^2)\exp(k\gamma_\pm|v|)$, $k \in \mathbb{Z}$ if $|v| \ge |w|$ and $vw \ge 0$, where $\gamma_\pm = \gamma_+$ if $v \ge 0$ and $\gamma_\pm = \gamma_-$ otherwise. Hence, by Condition 3.2, there exists a constant C (independent of n) such that

$$G^1_{g_k}(u;z) \le Ce^{(k+1)\gamma_\pm|u|}.$$

for any $u \in \mathbb{R}$ as long as $k \le 3$, where $z = x_0^n$ or $z = x_1^n$. This inequality implies (27) with the aid of (26). ////

Lemma 5.3. *Condition 4.4 holds.*

Proof: The proof is lengthy so is deferred to Section 6. ////

Lemma 5.4. *Condition 4.5 holds.*

Proof: Note that

$$0 < \inf_{n\in\mathbb{N}} \Sigma_n \le \sup_{n\in\mathbb{N}} \Sigma_n < \infty$$

by Lemmas 5.1, 5.2 and 5.3. The first two properties are then obvious from (23). To see the third, notice that

$$\begin{aligned} v^n =& \frac{T}{m_L^n \Sigma_n} \mathbb{E}\left[\int_{\tau_1^n}^{\tau_2^n} \varphi^2(\hat{X}_t^n)\mathrm{d}t\right] \\ &+ \frac{\epsilon_n}{m_L^n\Sigma_n}\mathbb{E}\left[\int_{\tau_1^n}^{\tau_2^n} \psi_n(\hat{X}_t^n)\varphi(\hat{X}_t^n)\rho(X_t^n)\mathrm{d}t\right] + O(\epsilon_n^2) \\ =& 1 + \frac{\epsilon_n}{\Sigma_n}\Pi_n[\psi_n\varphi\rho] + O(\epsilon_n^2) \end{aligned} \tag{28}$$

and $\Pi_n[\psi_n\varphi\rho] = O(1)$, in the light of Lemma 5.2, where

$$\psi_n(y) = 2s_n'(y)\hat{c}_n(y)\int_{-\infty}^{y} h_n(w)\Pi_n(\mathrm{d}w). \tag{29}$$

Here we used the fact that

$$0 = \int_{\hat{X}^n_{\tau_1^n}}^{\hat{X}^n_{\tau_2^n}} \frac{\psi_n(x)}{\hat{c}_n(x)}\mathrm{d}x = \int_{\tau_1^n}^{\tau_2^n}\psi_n(\hat{X}_t^n)\mathrm{d}\hat{W}_t^1 + \int_{\tau_1^n}^{\tau_2^n} h_n(\hat{X}_t^n)\mathrm{d}t,$$

which follows from the Itô-Tanaka formula. ////

Now we are ready to apply Theorem 4.6. In the light of Lemma 3 of Fukasawa [11] and Lemma 5.2, we have $A_1^n = O(\epsilon_n)$. Further, by the Itô-Tanaka formula and Lemma 5.2, we obtain

$$\begin{aligned} &\mathbb{E}^n_{x_0^n}\left[\left\{\int_0^{\tau_1^n}\varphi(\hat{X}_t^n)\left\{\rho(\hat{X}_t^n)\mathrm{d}\hat{W}_t^1 + \sqrt{1-\rho(\hat{X}_t^n)}\mathrm{d}\hat{W}_t^2\right\}\right\}^3\right] \\ &= 3\mathbb{E}^n_{x_0^n}\left[\int_0^{\tau_1^n}\varphi(\hat{X}_t^n)\left\{\rho(\hat{X}_t^n)\mathrm{d}\hat{W}_t^1 + \sqrt{1-\rho(\hat{X}_t^n)}\mathrm{d}\hat{W}_t^2\right\}\int_0^{\tau_1^n}\varphi(\hat{X}_t^n)^2\mathrm{d}t\right] \\ &= 3\mathbb{E}^n_{x_0^n}\left[\int_0^{\tau_1^n}\varphi(\hat{X}_t^n)\rho(\hat{X}_t^n)\mathrm{d}\hat{W}_t^1\int_0^{\tau_1^n}\varphi(\hat{X}_t^n)^2\mathrm{d}t\right] \\ &= 3\frac{\mathbb{E}^n_{x_0^n}[\tau_1^n]}{T}\left\{-\Pi_n[\varphi\rho\psi_n] + \frac{\Sigma_n}{\mathbb{E}^n_{x_0^n}[\tau_1^n]}\mathbb{E}^n_{x_0^n}\left[\tau_1^n\int_0^{\tau_1^n}\varphi(\hat{X}_t^n)\rho(\hat{X}_t^n)\mathrm{d}\hat{W}_t^1\right]\right\}, \end{aligned}$$

where ψ_n is defined by (29). This implies

$$A_2^n = -\frac{3\Pi_n[\varphi\rho\psi_n]}{\Pi_n[\varphi^2]^{3/2}T} + O(\epsilon_n) = \frac{6\alpha_n\sqrt{T_n}}{\sqrt{\Sigma_n}} + O(\epsilon_n),$$

where $\alpha = \alpha_n$ is defined as (3) with $c = c_{\theta_n} = \hat{c}_n/\epsilon_n$, $\Sigma = \Sigma_n$ and $\Pi = \Pi_n$. Since

$$\begin{aligned}\int g(z)\phi(z;v^n)\mathrm{d}z &= \int g(z)\phi(z;1)\mathrm{d}z + \frac{\epsilon_n}{\Sigma_n}\Pi_n[\psi_n\varphi\rho]\int g(z)\frac{\partial}{\partial v}\phi(z;v)\Big|_{v=1}\mathrm{d}z + O(\epsilon_n^2)\\ &= \int g(z)\phi(z;1)\mathrm{d}z - \alpha_n\int g(z)\phi(z;1)(z^2-1)\mathrm{d}z + O(\epsilon_n^2)\end{aligned}$$

for any Borel function g of polynomial growth by (28), we conclude

$$\int g(z)q^n(z)\mathrm{d}z = \int g(z)\phi(z;1)(1+p_n(z))\mathrm{d}z + O(\epsilon_n^2),$$

where q^n is defined by (20). By Theorem 4.6, we obtain a contradiction to (22).

6 Proof of Lemma 5.2

Here we prove that the characteristic function $\Psi^n(u)$ of

$$\left(\tau_1^n, \int_0^{\tau_1^n} h_n(\hat{X}_t^n)\mathrm{d}t, \int_0^{\tau_1^n}\varphi(\hat{X}_t^n)\left[\rho(\hat{X}_t^n)\mathrm{d}\hat{W}_t^1 + \sqrt{1-\rho(\hat{X}_t^n)^2}\mathrm{d}\hat{W}_t^2\right]\right)$$

under $\mathbb{E}^n_{x_0^n}$ satisfies the inequalities of Condition 4.4. By the strong Markov property, it suffices to prove the same inequalities for the characteristic function $\hat{\Psi}^n(u)$ of

$$\left(\tau(x_1^n), \int_0^{\tau(x_1^n)} h_n(\hat{X}_t^n)\mathrm{d}t, \int_0^{\tau(x_1^n)}\varphi(\hat{X}_t^n)\left[\rho(\hat{X}_t^n)\mathrm{d}\hat{W}_t^1 + \sqrt{1-\rho(\hat{X}_t^n)^2}\mathrm{d}\hat{W}_t^2\right]\right)$$

under $\mathbb{E}^n_{x_0^n}$ instead of $\Psi^n(u)$.

Note that $Y^n := s_n(\hat{X}^n)$ is a local martingale by the Itô-Tanaka formula, so that there exists a standard Brownian motion B^n such that $Y^n = B^n_{\langle Y^n\rangle}$ by the martingale representation theorem. Under $\mathbb{E}^n_{x_0^n}$, $B_0^n = s_n(x_0^n)$. Note also that

$$\mathrm{d}Y_t^n = s_n'(\hat{X}_t^n)\hat{c}_n(\hat{X}_t^n)\mathrm{d}\hat{W}_t^1 = \sqrt{\frac{s_n'}{\pi_n}}(\hat{X}_t^n)\mathrm{d}\hat{W}_t^1 = \frac{1}{\sigma_n(Y_t^n)}\mathrm{d}\hat{W}_t^1,$$

where $\sigma_n(y) = \sqrt{\pi_n(s_n^{-1}(y))}/\sqrt{s_n'(s_n^{-1}(y))}$. It follows that

$$\begin{aligned}\int_0^\tau g(\hat{X}_t^n)\mathrm{d}t &= \int_0^{\langle Y^n\rangle_\tau} g(s_n^{-1}(B_u^n))\sigma_n(B_u^n)^2\mathrm{d}u,\\ \int_0^\tau g(\hat{X}_t^n)\mathrm{d}\hat{W}^1 &= \int_0^{\langle Y^n\rangle_\tau} g(s_n^{-1}(B_u^n))\sigma_n(B_u^n)\mathrm{d}B_u^n\end{aligned}$$

for every finite stopping time τ and locally bounded Borel function g. When considering the hitting time $\tau = \tau(x_1^n)$ of $\hat{X}^n$, we have

$$\langle Y^n\rangle_\tau = \hat{\tau}_n := \inf\{s>0; B_s^n = s_n(x_1^n)\}. \tag{30}$$

Put $y_i^n = s_n(x_i^n)$ for $i = 0, 1$ and $y_{-1}^n = s_n(2x_0^n - x_1^n)$. Notice that by definition,

$$\inf_{n\in\mathbb{N}} |y_1^n - y_0^n| > 0, \quad \inf_{n\in\mathbb{N}} |y_0^n - y_{-1}^n| > 0, \quad \sup_{n\in\mathbb{N}} |y_1^n - y_{-1}^n| < \infty.$$

Lemma 6.1. *Let B^n be a standard Brownian motion with $B_0^n = y_0^n$ and define $\hat{\tau}_n$ as (30). Let Λ be a set and $g_n(\cdot, \lambda) : \mathbb{R} \to \mathbb{R}$ be a sequence of Borel functions for each $\lambda \in \Lambda$ with*

$$\sup_{\lambda\in\Lambda, n\in\mathbb{N}, v\in[y_{-1}^n, y_1^n]} |g_n(v, \lambda)| < \infty. \tag{31}$$

Then there exist positive constants a_1 and a_2 such that for all $\lambda \in \Lambda$ and $n \in \mathbb{N}$, the distribution of

$$\int_0^{\hat{\tau}_n} g_n(B_t^n, \lambda)\mathrm{d}t$$

is infinite divisible with Lévy measure L satisfying for all $z > 0$,

$$L((-\infty, -z]) \vee L((z, \infty)) \leq a_1 + \frac{a_2}{\sqrt{z}}.$$

Moreover,

- *if there exists a sequence of intervals $I_n \subset [y_0^n, y_1^n]$ such that*

$$\inf_{n\in\mathbb{N}} |I_n| > 0, \quad \inf_{\lambda\in\Lambda, n\in\mathbb{N}, v\in I_n} g_n(v, \lambda) > 0,$$

  then there exist positive constants a_3 and a_4 such that

$$-a_3 + \frac{a_4}{\sqrt{z}} \leq L((z, \infty))$$

  holds for all $z > 0$, $\lambda \in \Lambda$ and $n \in \mathbb{N}$,

- *if there exists a sequence of intervals $I_n \subset [y_0^n, y_1^n]$ such that*

$$\inf_{n\in\mathbb{N}} |I_n| > 0, \quad \sup_{n\in\mathbb{N}, \lambda\in\Lambda, v\in I_n} g_n(v, \lambda) < 0,$$

  then there exist another positive constants a_3 and a_4 such that

$$-a_3 + \frac{a_4}{\sqrt{|z|}} \leq L((\infty, z])$$

  holds for all $z < 0$, $\lambda \in \Lambda$ and $n \in \mathbb{N}$.

Proof: This can be proved by the same argument as in the proof of Lemma 3 of Borisov [4]. ////

Lemma 6.2. *Let $(B^n, \check{B}^n)$ be a 2-dimensional standard Brownian motion with $B_0^n = y_0^n$ and define $\hat{\tau}_n$ as (30). Let g_n be a sequence of locally bounded Borel functions with*

$$\sup_{n\in\mathbb{N}, v\in[y_{-1}^n, y_1^n]} |g_n(v)| < \infty, \quad \inf_{n\in\mathbb{N}, v\in[y_{-1}^n, y_1^n]} |g_n(v)| > 0.$$

Then there exist positive constants a_1, a_2, a_3, a_4 such that the distribution of

$$\int_0^{\hat{\tau}_n} g_n(B_t^n)\mathrm{d}\check{B}_t^n$$

is infinite divisible with Lévy measure L satisfying

$$-a_1 + \frac{a_2}{z} \leq L((-\infty, -z]) = L((z, \infty)) \leq a_3 + \frac{a_4}{z}$$

for all $z > 0$, $n \in \mathbb{N}$.

Proof: Put $\Delta_n = y_1^n - y_0^n$. Let $\tau_{i/m}$, $i = 1, \ldots, m$ be the times at which B^n first attains the levels $y_0^n + \Delta_n i/m$ respectively. Put

$$J_n = \sum_{i=1}^m J_n^{mi}, \quad J_n^{mi} = \int_{\tau_{(i-1)/m}}^{\tau_{i/m}} g_n(B_t^n)\mathrm{d}\check{B}_t^n.$$

Note that J_n^{mi}, $i = 1, \ldots, m$ are independent by the strong Markov property. Besides, $\{J_n^{mi}\}_{1\leq i\leq m}$ is a null array for each $n \in \mathbb{N}$ since for all $\epsilon > 0$,

$$\sup_{1\leq i\leq m} P[|J_n^{mi}| > \epsilon] \leq P[M\tau_{1/m}N^2 > \epsilon^2] + A_m$$

which converges to 0 as $m \to \infty$, where M is a constant, N is a standard normal variable independent of B^n, and

$$A_m = \sup_{1\leq i\leq m, n\in\mathbb{N}} P\left[\left\{\inf_{\tau_{(i-1)/m}\leq t\leq\tau_{i/m}} B_t^n \leq y_{-1}^n\right\}\right] = \frac{1}{m}\frac{1}{\inf_{n\in\mathbb{N}}|y_0^n - y_{-1}^n|}. \tag{32}$$

Hence, J_n is infinite divisible for each $n \in \mathbb{N}$. Denoting by L its Lévy measure, it holds that for every continuity point $z > 0$,

$$\lim_{m\to\infty}\sum_{i=1}^m P[J_n^{mi} > z] = L((z, \infty)) \tag{33}$$

and for every continuity point $z < 0$ of L,

$$\lim_{m\to\infty}\sum_{i=1}^m P[J_n^{mi} \leq z] = L((-\infty, z]), \tag{34}$$

for which see e.g., Feller [6], XVII.7. Observe that for $z > 0$,

$$\begin{aligned}
&P\left[J_n^{mi} < -z\right] = P\left[J_n^{mi} > z\right] \\
&\leq A_m + \int_0^\infty \int_{z/\sqrt{Mt}}^\infty \phi(y;1)\mathrm{d}y \frac{\Delta_n}{m\sqrt{2\pi t^3}} \exp\left\{-\frac{\Delta_n^2}{2tm^2}\right\}\mathrm{d}t \\
&= A_m + \int_0^\infty \int_{z/\sqrt{M}}^\infty \frac{\Delta_n}{2\pi m t^2} \exp\left\{-\frac{\Delta_n^2 + m^2u^2}{2tm^2}\right\}\mathrm{d}u\mathrm{d}t \\
&= A_m + \int_{z/\sqrt{M}}^\infty \int_0^\infty \frac{\Delta_n}{2\pi m} \exp\left\{-\frac{s}{2}\left\{u^2 + \frac{\Delta_n^2}{m^2}\right\}\right\}\mathrm{d}s\mathrm{d}u \\
&= A_m + \frac{1}{\pi}\int_{mz/(\Delta_n\sqrt{M})}^\infty \frac{\mathrm{d}v}{1+v^2}
\end{aligned}$$

where M is a constant. Hence, by L'Hopital's rule and (33),

$$L((-\infty,-z]) = L((z,\infty)) \leq a + \frac{\sup_{n\in\mathbb{N}} \Delta_n \sqrt{M}}{\pi z}$$

with a constant $a > 0$. By the same calculation, we have also

$$L((-\infty,-z]) = L((z,\infty)) \geq -a + \frac{\inf_{n\in\mathbb{N}} \Delta_n \sqrt{M'}}{\pi z}$$

with another constant $M' > 0$. ////

Lemma 6.3. *Let $(B^n, \check{B}^n)$ be a 2-dimensional standard Brownian motion with $B_0^n = y_0^n$ and define $\hat{\tau}_n$ as (30). Let Λ be a set, $g_{n,1}(\cdot,\lambda)$ be Borel functions for each $\lambda \in \Lambda$, $g_{n,2}$ be Borel functions which are absolutely continuous on $[y_{-1}^n, y_1^n]$ respectively, and $g_{n,3}$ be Borel functions with*

$$\sup_{\lambda\in\Lambda, n\in\mathbb{N}, v\in[y_{-1}^n, y_1^n]} |g_{n,1}(v,\lambda)| \vee |g_{n,2}(v)| \vee |g'_{n,2}(v)| \vee |g_{n,3}(v)| \vee \frac{1}{|g_{n,3}(v)|} < \infty.$$

Assume that there exists a sequence of intervals $I_n \subset [y_0^n, y_1^n]$ with

$$\inf_{n\in\mathbb{N}} |I_n| > 0$$

such that

$$\inf_{\lambda\in\Lambda, n\in\mathbb{N}, v\in I_n} g_{n,1}(v,\lambda) > 0 \quad \text{or} \quad \sup_{\lambda\in\Lambda, n\in\mathbb{N}, v\in I_n} g_{n,1}(v,\lambda) < 0 \tag{35}$$

holds. Denote by $\hat{g}_n(\cdot; u, \lambda)$ the characteristic function of J defined as

$$J = u_1 \int_0^{\hat{\tau}_n} g_{n,1}(B_t^n, \lambda)\mathrm{d}t + u_2 \int_0^{\hat{\tau}_n} g_{n,2}(B_t^n)\mathrm{d}B_t^n + u_2 \int_0^{\hat{\tau}_n} g_{n,3}(B_t^n)\mathrm{d}\check{B}_t^n,$$

where $u = (u_1, u_2) \in \mathbb{R}^2$ with $|u| = 1$. Then, there exists a constant $C \in (0,\infty)$ such that for every $t \in \mathbb{R}$, it holds

$$\sup_{\lambda\in\Lambda, n\in\mathbb{N}, u;|u|=1} |\hat{g}_n(t; u, \lambda)| \leq C e^{-\sqrt{|t|}/C}.$$

Proof: Put $\Delta_n = y_1^n - y_0^n$ and let $\tau_{i/m}$, $i = 1, \ldots, m$ be the times at which B^n first attains the levels $y_0^n + \Delta_n i/m$ respectively as in the previous proof. Put

$$\begin{aligned}
J_n^{mi,1} &= u_1 \int_{\tau_{(i-1)/m}}^{\tau_{i/m}} g_{n,1}(B_t^n, \lambda)\mathrm{d}t + u_2 \int_{\tau_{(i-1)/m}}^{\tau_{i/m}} g_{n,2}(B_t^n)\mathrm{d}B_t^n, \\
J_n^{mi,2} &= u_2 \int_{\tau_{(i-1)/m}}^{\tau_{i/m}} g_{n,3}(B_t^n)\mathrm{d}\check{B}_t^n
\end{aligned}$$

and $J_n^{mi} = J_n^{mi,1} + J_n^{mi,2}$. By the same argument as before, we conclude that J is infinitely divisible for each $n \in \mathbb{N}$, $\lambda \in \Lambda_n$ and $u \in \mathbb{R}^2$. We have (33) and (34) with its Lévy measure L. Notice that

$$\int_{\tau_{(i-1)/m}}^{\tau_{i/m}} g_{n,2}(B_t^n)\mathrm{d}B_t^n = \int_{y_0^n+(i-1)\Delta_n/m}^{y_0^n+i\Delta_n/m} g_{n,2}(y)\mathrm{d}y - \frac{1}{2}\int_{\tau_{(i-1)/m}}^{\tau_{i/m}} g'_{n,2}(B_t^n)\mathrm{d}t \tag{36}$$

on the set

$$\left\{\inf_{\tau_{(i-1)/m}\le t\le \tau_{i/m}} B_t^n > y_{-1}^n\right\}$$

by the Itô-Tanaka formula. Since, for example,

$$P[J_n^{mi} > z] \ge P\left[J_n^{mi,2} > 2z\right] - P[J_n^{mi,1} \le -z],$$
$$P[J_n^{mi} > z] \le P\left[J_n^{mi,2} > z/2\right] + P[J_n^{mi,1} > z/2],$$

there exist positive constants a_i, $i = 1, 2, \dots, 6$ such that

$$-a_1 - \frac{a_2}{\sqrt{z}} + \frac{a_3|u_2|}{z} \le L((z,\infty)) \le a_4 + \frac{a_5}{\sqrt{z}} + \frac{a_6|u_2|}{z} \tag{37}$$

for all $z > 0$ and

$$-a_1 - \frac{a_2}{\sqrt{|z|}} + \frac{a_3|u_2|}{|z|} \le L((-\infty, z]) \le a_4 + \frac{a_5}{\sqrt{|z|}} + \frac{a_6|u_2|}{|z|}$$

for all $z < 0$ by Lemmas 6.1, 6.2 and (32), (33), (34).

In case we have the first inequality in (35), if

$$|u_2| \le \beta_0 := \frac{1}{\sqrt{2}} \wedge \frac{\beta_1}{2\beta_2}$$

with $\beta_1 = \inf_{\lambda\in\Lambda, n\in\mathbb{N}, v\in I_n} g_{n,1}(v,\lambda)$ and $\beta_2 = \sup_{n\in\mathbb{N}, v\in I_n} |g'_{n,2}(v)|$, then

$$\begin{aligned}&\inf_{n\in\mathbb{N},\lambda\in\Lambda_n, v\in I_n} u_1 g_{n,1}(v;\lambda) - u_2 g'_{n,2}(v)/2\\ &\ge \sqrt{1-|u_2|^2}\beta_1 - |u_2|\beta_2/2 \ge \beta_1/4 > 0,\end{aligned}$$

so that by Lemma 6.1 and (36),

$$\lim_{m\to\infty}\sum_{i=1}^m P[J_n^{mi,1} > z] \ge -\tilde{a}_1 + \frac{\tilde{a}_2}{\sqrt{z}}$$

for all $z > 0$, where $\tilde{a}_i, i = 1, 2$ are positive constants. In addition, we have

$$P[J_n^{mi} > z] \ge P[J_n^{mi} > z; J_n^{mi,1} > z] \ge \frac{1}{2} P[J_n^{mi,1} > z]$$

for all $z > 0$. Hence, when $|u_2| \le \beta_0$, there exist another constants a'_i, $i = 1, 2$ such that

$$-a'_1 + \frac{a'_2}{\sqrt{z}} \le L((z,\infty)) \le a_4 + \frac{a_5}{\sqrt{z}} + \frac{a_6|u_2|}{z} \tag{38}$$

for all $z > 0$. Now, note that by the Lévy-Khinchin expression, there exists a constant $\sigma^2 \ge 0$ such that

$$\operatorname{Re}\log(\hat{g}_n(t;u,\lambda)) = -\sigma^2 t^2/2 - 2\int_{\mathbb{R}} \sin^2(zt/2) L(\mathrm{d}z).$$

Take $z_1 > 0$ such that $z \mapsto \sin^2(z)$ is increasing on $[0, z_1/2]$. Then, observe that for sufficiently small $z_0 \in (0, z_1)$, it holds that

$$\frac{a_2'/\sqrt{z_0} - a_5/\sqrt{z_1}}{a_6/z_1} > \frac{a_2/\sqrt{z_0} + a_5/\sqrt{z_1}}{a_3/z_0 - a_6/z_1} > 0.$$

Fix such a point z_0 and take β_3 such that

$$\frac{a_2'/\sqrt{z_0} - a_5/\sqrt{z_1}}{a_6/z_1} > \beta_3 > \frac{a_2/\sqrt{z_0} + a_5/\sqrt{z_1}}{a_3/z_0 - a_6/z_1}.$$

Then we have for the case that $|u_2|\sqrt{t} \geq \beta_3$,

$$\begin{aligned}
&\int_{\mathbb{R}} \sin^2(zt/2) L(\mathrm{d}z) \\
&\geq \int_{(z_0/|t|, z_1/|t|]} \sin^2(zt/2) L(\mathrm{d}z) \\
&\geq \sin^2(z_0/2) L((z_0/|t|, z_1/|t|]) \\
&\geq \sin^2(z_0/2) \left\{ \left(\frac{a_3}{z_0} - \frac{a_6}{z_1} \right) |u_2||t| - \left(\frac{a_2}{\sqrt{z_0}} + \frac{a_5}{\sqrt{z_1}} \right) \sqrt{|t|} - a_1 - a_4 \right\} \\
&\geq \sqrt{|t|}/C - \log(C)
\end{aligned} \tag{39}$$

for sufficiently large constant C by (37). Further, by (38), we have for the case that $|u_2|\sqrt{t} \leq \beta_3$,

$$\begin{aligned}
&\sin^2(z_0/2) L((z_0/|t|, z_1/|t|]) \\
&\geq \sin^2(z_0/2) \left\{ \left(\frac{a_2'}{\sqrt{z_0}} - \frac{a_5}{\sqrt{z_1}} \right) \sqrt{|t|} - \frac{a_6}{z_1}|u_2||t| - a_1' - a_4 \right\} \\
&\geq \sqrt{|t|}/C - \log(C).
\end{aligned}$$

The same conclusion is obtained also in the case that we have the second inequality instead of the first in (35). For example, we define β_1 alternatively as $\beta_1 = -\sup_{\lambda \in \Lambda, n \in \mathbb{N}, \nu \in I_n} g_{n,1}(\nu, \lambda)$ and observe

$$\lim_{m \to \infty} \sum_{i=1}^{m} P[J_n^{mi,1} \leq z] \geq -\tilde{a}_1 + \frac{\tilde{a}_2}{\sqrt{|z|}}$$

for all $z < 0$ with positive constants $\tilde{a}_1$, $\tilde{a}_2$ when $|u_2| \leq \beta_0$. Then use

$$P[J_n^{mi} \leq z] \geq P[J_n^{mi} \leq z; J_n^{mi,1} \leq z] \geq \frac{1}{2} P[J_n^{mi,1} \leq z]$$

to obtain

$$-a_1' + \frac{a_2'}{\sqrt{|z|}} \leq L((\infty, z]) \leq a_4 + \frac{a_5}{\sqrt{|z|}} + \frac{a_6|u_2|}{|z|}$$

for all $z < 0$. The rest is a straightforward translation. ////

Now we are ready to prove Lemma 5.2. By Petrov's lemma (see Petrov [17], p.10), it suffices to prove that there exists a constant $C \in (0,\infty)$ such that

$$|\hat{\Psi}^n(u)| \leq Ce^{-|u|/C},$$

where $\hat{\Psi}^n$ is what was defined at the beginning of this section. Put

$$\begin{aligned} g_{n,1}(v,\lambda) &= (\lambda_1 + \lambda_2 h_n(s_n^{-1}(v)))\sigma_n(v), \\ g_{n,2}(v) &= \varphi(s_n^{-1}(v))\rho(s_n^{-1}(v))\sigma_n(v), \\ g_{n,3}(v) &= \varphi(s_n^{-1}(v))\sqrt{1-\rho(s_n^{-1}(v))^2}\sigma_n(v) \end{aligned}$$

and

$$M(\lambda) = \sup_{n\in\mathbb{N}, v\in[y_{-1}^n, y_1^n]} |g_{n,1}(v,\lambda))| \vee |g_{n,2}(v)| \vee |g_{n,2}'(v)| \vee |g_{n,3}(v)| \vee \frac{1}{|g_{n,3}(v)|}$$

for $\lambda = (\lambda_1, \lambda_2) \in \mathbb{S}$, the 1-dimensional unit sphere. It is not difficult to see that for all $\lambda \in \mathbb{S}$, we have $M(\lambda) < \infty$ and that there exists a sequence of intervals $I_n(\lambda) \subset [y_0^n, y_1^n]$ with

$$\inf_{n\in\mathbb{N}} |I_n(\lambda)| > 0$$

such that

$$\inf_{n\in\mathbb{N}, v\in I_n(\lambda)} g_{n,1}(v,\lambda) > 0 \quad \text{or} \quad \sup_{n\in\mathbb{N}, v\in I_n(\lambda)} g_{n,1}(v,\lambda) < 0$$

holds. If the first inequality holds for $\lambda = \lambda_0$, put

$$m(\lambda_0) = \inf_{n\in\mathbb{N}, v\in I_n(\lambda_0)} g_{n,1}(v,\lambda_0)$$

and

$$\Lambda(\lambda_0) = \left\{\lambda \in \mathbb{S}; \inf_{n\in\mathbb{N}, v\in I_n(\lambda_0)} g_{n,1}(v,\lambda) > m(\lambda_0)/2, M(\lambda) < 2M(\lambda_0)\right\}.$$

If the second inequality holds for $\lambda = \lambda_0$, put

$$m(\lambda_0) = \sup_{n\in\mathbb{N}, v\in I_n(\lambda_0)} g_{n,1}(v,\lambda_0)$$

and

$$\Lambda(\lambda_0) = \left\{\lambda \in \mathbb{S}; \sup_{n\in\mathbb{N}, v\in I_n(\lambda_0)} g_{n,1}(v,\lambda) < m(\lambda_0)/2, M(\lambda) < 2M(\lambda_0)\right\}.$$

Now, notice that $\Lambda(\lambda_0), \lambda_0 \in \mathbb{S}$ is an open covering of $\mathbb{S}$, so that it has a finite subcovering $\Lambda(\lambda_1), \cdots, \Lambda(\lambda_J)$. For each λ_j, we can apply Lemma 6.3 with $\Lambda = \Lambda(\lambda_j)$ to obtain that there exists $C_j > 0$ such that

$$\sup_{\lambda\in\Lambda(\lambda_j), u\in\mathbb{S}, n\in\mathbb{N}} |\hat{\Psi}^n(tu_1\lambda_1, tu_1\lambda_2, tu_2)| \leq C_j e^{-|t|/C_j}$$

for all $t \in \mathbb{R}$. Since $J < \infty$, we conclude that there exists $C > 0$ such that

$$\sup_{\lambda\in\mathbb{S}, u\in\mathbb{S}, n\in\mathbb{N}} |\hat{\Psi}^n(tu_1\lambda_1, tu_1\lambda_2, tu_2)| \leq Ce^{-|t|/C}$$

for all $t \in \mathbb{R}$, which completes the proof of Lemma 5.2.

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