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Cramér Type Moderate deviations for the Maximum of Self-normalized Sums

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Abstract

Let $\{X, X_i, i \geq 1\}$ be i.i.d. random variables, S_k be the partial sum and $V_n^2 = \sum_{i=1}^n X_i^2$. Assume that $E(X) = 0$ and $E(X^4) < \infty$. In this paper we discuss the moderate deviations of the maximum of the self-normalized sums. In particular, we prove that $P(\max_{1\leq k\leq n} S_k \geq x\, V_n)/(1-\Phi(x)) \to 2$ uniformly in $x \in [0, o(n^{1/6}))$.

Key words: Large deviation; moderate deviation ; self-normalized sum.

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1 Introduction and main results

Let $X, X_1, X_2, \cdots$ be a sequence of i.i.d. random variables with mean zero. Set

$$S_n = \sum_{j=1}^{n} X_j \quad \text{and} \quad V_n^2 = \sum_{j=1}^{n} X_j^2.$$

The past decade has witnessed a significant development on the limit theorems for the so-called self-normalized sum S_n/V_n. Griffin and Kuelbs (1989) obtained a self-normalized law of the iterated logarithm for all distributions in the domain of attraction of a normal or stable law. Shao (1997) showed that no moment conditions are needed for a self-normalized large deviation result $P(S_n/V_n \geq x\sqrt{n})$ and that the tail probability of S_n/V_n is Gaussian like when X_1 is in the domain of attraction of the normal law and sub-Gaussian like when X is in the domain of attraction of a stable law, while Giné, Götze and Mason (1997) proved that the tails of S_n/V_n are uniformly sub-Gaussian when the sequence is stochastically bounded. Shao (1999) established a Cramér type result for self-normalized sums only under a finite third moment condition. Jing, Shao and Wang (2003) proved a Cramér type large deviation result (for independent random variables) under a Lindeberg type condition. Jing, Shao and Zhou (2004) obtained the saddlepoint approximation without any moment condition. Other results include Wang and Jing (1999) as well as Robinson and Wang (2005) for an exponential non-uniform Berry-Esseen bound, Csörgő, Szyszkowicz and Wang (2003a, b) for Darling-Erdős theorems and Donsker's theorems, Wang (2005) for a refined moderate deviation, Hall and Wang (2004) for exact convergence rates, and Chistyakov and Götze (2004) for all possible limiting distributions when X is in the domain of attraction of a stable law. These results show that the self-normalized limit theorems usually require fewer moment conditions than the classical limit theorems do. On the other hand, self-normalization is commonly used in statistics. Many statistical inferences require the use of classical limit theorems. However, these classical results often involve some unknown parameters, one needs to first estimate the unknown parameters and then substitute the estimators into the classical limit theorems. This commonly used practice is exactly the self-normalization. Hence, the development on self-normalized limit theorems not only provides a theoretical foundation for statistical practice but also gives a much wider applicability of the results because they usually require much less moment assumptions.

In contrast with the achievements for the self-normalized partial sum S_n/V_n, there is little work on the maximum of self-normalized sums. This paper is part of our efforts to develop limit theorems for the maximum of self-normalized sums. Using a different approach from some known techniques for self-normalized sum, we establish a Cramér type large deviation result for the maximum of self-normalized sums under a finite fourth moment. Note that the Cramér type large deviation result holds under a finite third moment for partial sum, we conjecture that a finite third moment is sufficient for (1.1). Our main result is as follows.

Theorem 1.1. *If $EX^4 < \infty$, then*

$$\lim_{n\to\infty} \frac{P\left(\max_{1\leq k\leq n} S_k \geq x\, V_n\right)}{1-\Phi(x)} = 2 \tag{1.1}$$

uniformly for $x \in [0,\ o(n^{1/6}))$.

This theorem is comparable to the large deviation result for the maximum of partial sum given in Aleshkyavichene (1979). However the latter requires a finite exponential moment condition. We also note that Berry-Esseen type results for the maximum of non-self-normalized sums are available in literature and one can refer to Arak (1974) and Arak and Nevzorov (1973). For a Chernoff type large deviation, the fourth moment condition can be relaxed to a finite second moment condition. Indeed, we have the following theorem.

Theorem 1.2. *If X is in the domain of attraction of the normal law, then*

$$\lim_{n\to\infty} x_n^{-2}\log P\big(\max_{1\le k\le n} S_k \ge x_n V_n\big) = -\frac{1}{2} \tag{1.2}$$

for any $x_n \to \infty$ with $x_n = o(\sqrt{n})$.

This paper is organized as follows. In the next section, we give the proof of Theorem 1.1 as well as two propositions. The proofs of the two propositions are postponed to Section 3. Finally, the proof of Theorem 1.2 is given in Section 4.

2 Proof of Theorem 1.1

Throughout this section, without loss of generality, we assume $E(X^2) = 1$. The proof is based on the following two propositions. Their proofs will be given in the next section.

Proposition 2.1. *If $E|X|^3 < \infty$, then for $0 \le x \le n^{1/6}$,*

$$P(\max_{1\le k\le n} S_k \ge xV_n) \quad\le\quad C\,e^{-x^2/2}, \tag{2.1}$$

where C is a constant not depending on x and n.

Let $x \ge 2$ and $0 < \alpha < 1$. Write

$$\bar{X}_i = X_i I\left[|X_i| \le (\sqrt{n}/x)^\alpha\right], \qquad \bar{S}_n = \sum_{i=1}^n \bar{X}_i, \qquad \bar{V}_n^2 = \sum_{i=1}^n \bar{X}_i^2. \tag{2.2}$$

Proposition 2.2. *If $E|X|^{\max\{(\alpha+2)/\alpha,\,4\}} < \infty$, then*

$$\lim_{n\to\infty} \frac{P\big(\max_{1\le k\le n} \bar{S}_k \ge x\,\bar{V}_n\big)}{1-\Phi(x)} \quad=\quad 2, \tag{2.3}$$

uniformly in $x \in [2, \epsilon_n n^{1/6})$, where $\epsilon_n \to 0$ is any sequence of constants.

We are now ready to prove Theorem 1.1. Let $M_n = \max_{1\le k\le n} S_k$. It is well-known that if the second moment of X is finite, then by the law of large numbers and the weak convergence

$$\sup_{x\ge 0} |P(M_n \ge xV_n) - 2(1-\Phi(x))| \to 0.$$

Hence (1.1) holds for $x \in [0,\ 2]$ and we can assume $2 \le x = o(n^{1/6})$. We first prove

$$\lim_{n\to\infty} \frac{P\big(M_n \ge x\,V_n\big)}{1-\Phi(x)} \quad\le\quad 2, \tag{2.4}$$

uniformly in the range $2 \le x = o(n^{1/6})$. Put

$$S_k^{(j)} = \begin{cases} S_k, & \text{if } 1 \le k \le j-1 \\ S_k - X_j, & \text{if } j \le k \le n \end{cases}$$

and $V_n^{(j)} = \sqrt{V_n^2 - X_j^2}$. Noting that for any real numbers s and t and non-negative number c and $x \ge 1$

$$\{s + t \ge x\sqrt{c + t^2}\} \subset \{s \ge \sqrt{x^2 - 1}\, \sqrt{c}\}$$

(see p. 2181 in Jing, Shao and Wang (2003)), we have

$$\{M_n \ge xV_n\} \subset \{\max_{1\le k\le n} S_k^{(j)} \ge \sqrt{x^2-1}\, V_n^{(j)}\}$$

for each $1 \le j \le n$. Let $\alpha = 7/8$ in (2.2). It is readily seen by the independence of X_j and $\max_{1\le k\le n} S_k^{(j)}/V_n^{(j)}$, the iid properties of X_i and Proposition 2.1 that if $EX^4 < \infty$, then for each j

$$\begin{aligned}
&P\big(M_n \ge xV_n, X_j \ne \bar{X}_j\big) \\
&\le P\big(\max_{1\le k\le n} S_k^{(j)} \ge \sqrt{x^2-1}\, V_n^{(j)}, X_j \ne \bar{X}_j\big) \\
&= P(|X_j| > (\sqrt{n}/x)^{7/8}) P\big(\max_{1\le k\le n} S_k^{(j)} \ge \sqrt{x^2-1}\, V_n^{(j)}\big) \\
&= P(|X| > (\sqrt{n}/x)^{7/8}) P\big(M_{n-1} \ge \sqrt{x^2-1}\, V_{n-1}\big) \\
&= O(1)(x/\sqrt{n})^{7/2} e^{-x^2/2} \\
&= O(1)x^{9/2} n^{-7/4}(1 - \Phi(x)) \\
&= o(n^{-1})(1 - \Phi(x))
\end{aligned}$$

uniformly in $x \in [2, o(n^{1/6}))$. This, together with Proposition 2.2 yields

$$\begin{aligned}
P\big(M_n \ge x\,V_n\big) &\le P\big(\max_{1\le k\le n} \bar{S}_k \ge x\,\bar{V}_n\big) + \sum_{j=1}^{n} P\big(M_n \ge x\,V_n, X_j \ne \bar{X}_j\big) \\
&= (2 + o(1))(1 - \Phi(x)) \qquad (2.5)
\end{aligned}$$

uniformly in $x \in [2, o(n^{1/6}))$. This proves (2.4).

We next prove

$$\lim_{n\to\infty} \frac{P\big(M_n \ge x\,V_n\big)}{1 - \Phi(x)} \ge 2, \qquad (2.6)$$

uniformly in the range $2 \le x \le o(n^{1/6})$. Let $\alpha = 3/4$ in (2.2). It follows from $EX^4 < \infty$ and the iid properties of X_i that

$$
\begin{aligned}
& P\big(\max_{1\le k\le n} \bar S_k \ge x\,\bar V_n\big) \\
\le\; & P\big(M_n \ge x\,V_n\big) + \sum_{j=1}^n P\big(\max_{1\le k\le n} \bar S_k \ge x\,\bar V_n, X_j \ne \bar X_j\big) \\
\le\; & P\big(\max_{1\le k\le n} S_k \ge x\,V_n\big) \\
& + \sum_{j=1}^n P(|X_j| > (\sqrt{n}/x)^{3/4}) P\big(\max_{1\le k\le n} \bar S_k^{(j)} \ge \sqrt{x^2-1}\,\bar V_n^{(j)}\big) \\
\le\; & P\big(\max_{1\le k\le n} S_k \ge x\,V_n\big) \\
& + o(1) x^3 n^{-1/2} P\big(\max_{1\le k\le n-1} \bar S_k \ge \sqrt{x^2-1}\,\bar V_{n-1}\big), \qquad (2.7)
\end{aligned}
$$

where $\bar S_k^{(j)}$ and $\bar V_n^{(j)}$ are defined similarly as $S_k^{(j)}$ and $V_n^{(j)}$ with $\bar X_i$ to replace X_i. By (2.7), result (2.6) follows immediately from Proposition 2.2. The proof of Theorem 1.1 is now complete. □

3 Proofs of Propositions

3.1 Preliminary lemmas

This subsection provides several preliminary lemmas. Some of which are interesting by themselves. For each $n \ge 1$, let $X_{n,i}, 1 \le i \le n$, be a sequence of independent random variables with zero mean and finite variance. Write $S^*_{n,k} = \sum_{i=1}^k X_{n,i}, k = 1, 2, ...n$, $B_n^2 = \sum_{i=1}^n EX_{n,i}^2$ and

$$L(t) = \sum_{i=1}^n E\big[|X_{n,i}|^3 \max\{e^{tX_{n,i}}, 1\}\big].$$

Lemma 3.1. *If $\mathscr{L}_{3n} := \sum_{i=1}^n E|X_{n,i}|^3/B_n^3 \to 0$, and there exists an R_0 (that may depend on x and n) such that $R_0 \ge 2\max\{x, 1\}/B_n$ and $L(R_0) \le C_0 \sum_{i=1}^n E|X_{n,i}|^3$, where $C_0 \ge 1$ is a constant, then*

$$\lim_{n\to\infty} \frac{P(S^*_{n,n} \ge xB_n)}{1-\Phi(x)} = 1, \qquad (3.1)$$

uniformly in $0 \le x \le \epsilon_n \mathscr{L}_{3n}^{-1/3}$, where $0 < \epsilon_n \to 0$ is any sequence of constants. Furthermore we also have

$$\lim_{n\to\infty} \frac{P(\max_{1\le k\le n} S^*_{n,k} \ge xB_n)}{1-\Phi(x)} = 2, \qquad (3.2)$$

uniformly in $0 \le x \le \epsilon_n \mathscr{L}_{3n}^{-1/3}$.

Proof. The result (3.1) follows immediately from (4) and (5) of Sakhanenko (1991). In order to prove (3.2), without loss of generality, assume $x \ge 1$. The result for $0 \le x \le 1$ follows from the well-known Berry-Esseen bound. See Nevzorov (1973), for instance. For each $\epsilon > 0$, write, for $k = 1, 2, ..., n$,

$$S_{n,k}^{(1)} = \sum_{i=1}^{k} X_{n,i} I_{(X_{n,i} \le \epsilon)}, \quad S_{n,k}^{(2)} = \sum_{i=1}^{k} X_{n,i} I_{(X_{n,i} \ge -\epsilon)}$$

and $A_n = \sum_{i=1}^n \left(|EX_{n,i} I_{(X_{n,i} \ge -\epsilon)}| + |EX_{n,i} I_{(X_{n,i} \le \epsilon)}| \right)$. We have

$$\begin{aligned} & 2P\left(S_{n,n}^{(1)} \ge xB_n + (c_0+1)\epsilon + A_n\right) \\ \le \ & P(\max_{1\le k\le n} S_{n,k}^* \ge xB_n) \le 2P\left(S_{n,n}^{(2)} \ge xB_n - c_0\epsilon - A_n\right), \end{aligned} \tag{3.3}$$

where $c_0 > 0$ is an absolute constant.

The statement (3.3) has been established in (8) of Nevzorov (1973). We present a proof here for the convenience of the reader. Let $S_{n,k}^{(3)} = \sum_{i=1}^k X_{n,i} I_{(|X_{n,i}| \le \epsilon)}$ for $k = 1, 2, ..., n$. We first claim that there exists an absolute constant $c_0 > 0$ such that, for any $n \ge 1$, $1 \le l \le n$ and $\epsilon > 0$,

$$I_{1n} \equiv P\{S_{n,n}^{(3)} - S_{n,l}^{(3)} - E(S_{n,n}^{(3)} - S_{n,l}^{(3)}) \ge c_0\,\epsilon\} \le 1/2, \tag{3.4}$$

$$I_{2n} \equiv P\{S_{n,n}^{(3)} - S_{n,l}^{(3)} - E(S_{n,n}^{(3)} - S_{n,l}^{(3)}) \ge -c_0\epsilon\} \ge 1/2. \tag{3.5}$$

In fact, by letting $s_n^2 = var(S_{n,n}^{(3)} - S_{n,l}^{(3)})$ and $Y_i = X_{n,i} I_{(|X_{n,i}| \le \epsilon)} - EX_{n,i} I_{(|X_{n,i}| \le \epsilon)}$, it follows from the non-uniform Berry-Esseen bound that, for any $n \ge 1$, $1 \le l \le n$ and $\epsilon > 0$,

$$\begin{aligned} I_{1n} &\le \left[1 - \Phi(c_0\epsilon/s_n)\right] + A_0 (1 + c_0\epsilon/s_n)^{-3} \sum_{j=l+1}^{n} E|Y_j|^3/s_n^3 \\ &\le \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{t_0} e^{-s^2/2} ds \ + \ \frac{2A_0}{c_0} (1 + t_0)^{-3} t_0, \end{aligned}$$

where A_0 is an absolute constant and $t_0 = c_0\epsilon/s_n$. Note that $\int_0^t e^{-s^2/2} ds \ge t(1+t)^{-3}/2$ for any $t \ge 0$. Simple calculations yield (3.4), by choosing $c_0 \ge 4A_0\sqrt{2\pi}$. The proof of (3.5) is similar, we omit the details. Now, by noting $X_{n,i} I_{(|X_{n,i}| \le \epsilon)} \le X_{n,i} I_{(X_{n,i} \ge -\epsilon)}$ and $X_{n,i} \le X_{n,i} I_{(X_{n,i} \ge -\epsilon)}$, we obtain, for all $1 \le l \le n$,

$$P\{S_{n,n}^{(2)} - S_{n,l}^{(2)} \ge -c_0\epsilon - A_n\} \ \ge \ P\{S_{n,n}^{(3)} - S_{n,l}^{(3)} - E(S_{n,n}^{(3)} - S_{n,l}^{(3)}) \ge -c_0\epsilon\} \ge \frac{1}{2},$$

and hence for $y = x\,B_n$,

$$\begin{aligned} P\{\max_{1\le k\le n} S_{n,k}^* \ge y\} &\le P\{\max_{1\le k\le n} S_{n,k}^{(2)} \ge y\} \\ &= \sum_{k=1}^{n} P\{S_{n,1}^{(2)} < y, \cdots, S_{n,k}^{(2)} \ge y\} \\ &\le 2\sum_{k=1}^{n} P\{S_{n,1}^{(2)} < y, \cdots, S_{n,k}^{(2)} \ge y\} \times P\{S_{n,n}^{(2)} - S_{n,k}^{(2)} \ge -c_0\epsilon - A_n\} \\ &\le 2\sum_{k=1}^{n} P\{S_{n,1}^{(2)} < y, \cdots, S_{n,k}^{(2)} \ge y,\, S_{n,n}^{(2)} \ge y - c_0\epsilon - A_n\} \\ &\le 2\,P\{S_{n,n}^{(2)} \ge y - c_0\epsilon - A_n\}. \end{aligned}$$

Similarly, it follows from $X_{n,i}I_{(X_{n,i}\le\epsilon)}\le X_{n,i}I_{(|X_{n,i}|\le\epsilon)}$ and $X_{n,i}I_{(X_{n,i}\le\epsilon)}\le X_{n,i}$ that, for all $1\le l\le n$,

$$P\{S^{(1)}_{n,n}-S^{(1)}_{n,l}\ge c_0\epsilon+A_n\} \quad\le\quad P\{S^{(3)}_{n,n}-S^{(3)}_{n,l}-E(S^{(3)}_{n,n}-S^{(3)}_{n,l})\ge c_0\epsilon\}\le\frac{1}{2}.$$

and hence for $y=xB_n$,

$$\begin{aligned}
P\{\max_{1\le k\le n}S^{*}_{n,k}\ge y\} &\ge P\{\max_{1\le k\le n}S^{(1)}_{n,k}\ge y\}\\
&= \sum_{k=1}^{n}P\{S^{(1)}_{n,1}<y,\cdots,S^{(1)}_{n,k-1}<y,S^{(1)}_{n,k}\ge y\}\\
&= \sum_{k=1}^{n}P\{S^{(1)}_{n,1}<y,\cdots,S^{(1)}_{n,k-1}<y,\,y\le S^{(1)}_{n,k}\le y+\epsilon\}\\
&\ge 2\sum_{k=1}^{n}P\{S^{(1)}_{n,1}<x,\cdots,S^{(1)}_{n,k-1}<y,\,y\le S^{(1)}_{n,k}\le y+\epsilon\}\\
&\qquad\qquad\times P\{S^{(1)}_{n,n}-S^{(1)}_{n,k}\ge c_0\epsilon+A_n\}\\
&\ge 2\sum_{k=1}^{n}P\{S^{(1)}_{n,1}<y,\cdots,S^{(1)}_{n,k-1}<y,\,y\le S^{(1)}_{n,k}\le y+\epsilon,\\
&\qquad\qquad S^{(1)}_{n,n}\ge y+(1+c_0)\epsilon+A_n\}\\
&= 2P\{S^{(1)}_{n,n}\ge y+(1+c_0)\epsilon+A_n\}.
\end{aligned}$$

This completes the proof of (3.3).

In the following proof, take $\epsilon=\epsilon_nB_n/x$ in (3.3). By recalling $EX_{n,i}=0$, we have $|ES^{(t)}_{n,n}|\le A_n\le\epsilon^{-2}\sum_{i=1}^{n}E|X_{n,i}|^3\le\epsilon_nB_n/x$ and

$$var(S^{(t)}_{n,n})=B_n^2+r(n),\quad t=1,2, \tag{3.6}$$

where $|r(n)|\le 2\epsilon^{-1}\sum_{i=1}^{n}E|X_{n,i}|^3\le 2\epsilon_n^2B_n^2/x^2$, whenever $1\le x\le\epsilon_n\mathscr{L}_{3n}^{-1/3}$. Therefore, for n large enough such that $\epsilon_n\le 1/4$,

$$\begin{aligned}
P\big(S^{(2)}_{n,n}\ge xB_n-c_0\epsilon-A_n\big) &\le P\big[S^{(2)}_{n,n}-ES^{(2)}_{n,n}\ge xB_n[1-(c_0+1)\epsilon_n/x^2]\big]\\
&= P\big[S^{(2)}_{n,n}-ES^{(2)}_{n,n}\ge x\sqrt{var(S^{(2)}_{n,n})}(1+\eta_{2n})\big],
\end{aligned}\tag{3.7}$$

where $|\eta_{2n}|=|\sqrt{\frac{B_n^2}{var(S^{(2)}_{n,n})}}[1-(c_0+1)\epsilon_n/x^2]-1|\le 2(c_0+2)\epsilon_n/x^2$ by (3.6), uniformly in $1\le x\le\epsilon_n\mathscr{L}_{3n}^{-1/3}$. Similarly, we have

$$\begin{aligned}
&P\big(S^{(1)}_{n,n}\ge xB_n+(c_0+1)\epsilon+A_n\big)\\
&\quad\ge P\big[S^{(1)}_{n,n}-ES^{(1)}_{n,n}\ge x\sqrt{var(S^{(1)}_{n,n})}(1+\eta_{1n})\big],
\end{aligned}\tag{3.8}$$

where $|\eta_{1n}|\le 2(c_0+3)\epsilon_n/x^2$ by (3.6), uniformly in $1\le x\le\epsilon_n\mathscr{L}_{3n}^{-1/3}$. By virtue of (3.3), (3.7)-(3.8) and the well-known fact that if $|\delta_n|\le C\epsilon_n/x^2$, then

$$\lim_{n\to\infty}\frac{1-\Phi\big[x(1+\delta_n)\big]}{1-\Phi(x)}=1, \tag{3.9}$$

uniformly in $x \in [1, \infty)$, the result (3.2) will follow if we prove

$$\lim_{n\to\infty} \frac{P\big(S_{n,n}^{(t)} - ES_{n,n}^{(t)} \geq x\sqrt{var(S_{n,n}^{(t)})}\big)}{1-\Phi(x)} = 1, \quad \text{for } t = 1, 2, \tag{3.10}$$

uniformly in $0 \leq x \leq \epsilon_n \mathscr{L}_{3n}^{-1/3}$. In fact, by noting $var(S_{n,n}^{(t)}) \geq B_n^2/2$, for $t = 1, 2$, whenever $1 \leq x \leq \epsilon_n \mathscr{L}_{3n}^{-1/3}$ and n sufficient large, (3.10) follows immediately from (3.1). The proof of Lemma 3.1 is now complete. □

Lemma 3.1 will be used to establish Craḿer type large deviation result for truncated random variables under finite moment conditions. Indeed, as a consequence of Lemma 3.1, we have the following lemma.

Lemma 3.2. *If $EX^2 = 1$ and $E|X|^{(\alpha+2)/\alpha} < \infty$, $0 < \alpha \leq 1$, then we have*

$$\lim_{n\to\infty} \frac{P\big(\bar{S}_n \geq x\sqrt{n}\big)}{1-\Phi(x)} = 1 \tag{3.11}$$

and

$$\lim_{n\to\infty} \frac{P\big(\max_{1\leq k\leq n} \bar{S}_k \geq x\sqrt{n}\big)}{1-\Phi(x)} = 2. \tag{3.12}$$

uniformly in the range $2 \leq x \leq \epsilon_n n^{1/6}$, where $0 < \epsilon_n \to 0$ is any sequence of constants, and $\bar{X}_j$ and $\bar{S}_k$ are defined as in (2.2).

Proof. Write $X_{n,i} = X_i I\big[|X_i| \leq (\sqrt{n}/x)^\alpha\big] - EX_i I\big[|X_i| \leq (\sqrt{n}/x)^\alpha\big]$ and $B_n^2 = n\,Var(XI\big[|X| \leq (\sqrt{n}/x)^\alpha\big])$. It is easy to check that, for $R_0 = 2\max\{x, 1\}/B_n$,

$$\sum_{i=1}^n E\big[|X_{n,i}|^3 \max\{e^{R_0 X_{n,i}}, 1\}\big] \leq C_0 \sum_{i=1}^n E|X_{n,i}|^3 \leq C_0\, nE|X|^3,$$

and $\mathscr{L}_{3n} := nE|X_{1n}|^3/B_n^3 \sim n^{-1/2}E|X|^3$, uniformly in the range $2 \leq x \leq \epsilon_n n^{1/6}$, where C_0 is a constant not depending on x and n. So, by (3.1) in Lemma 3.1,

$$\lim_{n\to\infty} \frac{P\big(\bar{S}_n - E\bar{S}_n \geq x\sqrt{var(\bar{S}_n)}\big)}{1-\Phi(x)} = 1, \tag{3.13}$$

uniformly in the range $2 \leq x \leq \epsilon_n n^{1/6}$. Now, by noting

$$\begin{aligned} |\eta_n| &:= \Big|\sqrt{\frac{n}{var(\bar{S}_n)}}\Big(1 - \frac{\sqrt{n}}{x}E(XI\big[|X| \leq (\sqrt{n}/x)^\alpha\big])\Big) - 1\Big| \\ &\leq O(1)\Big[E(X^2 I\big[|X| \geq (\sqrt{n}/x)^\alpha\big]) + \frac{\sqrt{n}}{x}E(|X|I\big[|X| \geq (\sqrt{n}/x)^\alpha\big])\Big] \\ &= o(x/\sqrt{n}), \end{aligned}$$

it follows from (3.13) and then (3.9) that

$$\lim_{n\to\infty} \frac{P\big(\bar{S}_n \geq x\sqrt{n}\big)}{1-\Phi(x)} = \lim_{n\to\infty} \frac{P\big[\bar{S}_n - E\bar{S}_n \geq x\sqrt{var(\bar{S}_n)}(1+\eta_n)\big]}{1-\Phi(x)} = 1,$$

uniformly in the range $1 \le x \le \epsilon_n n^{1/6}$. This proves (3.11). The proof of (3.12) is similar except we make use of (3.2) in replacement of (3.1) and hence the details are omitted. The proof of Lemma 3.2 is now complete. □

Lemma 3.3. *Suppose that $EX^2 = 1$ and $E|X|^3 < \infty$.*

(i). For any $0 < h < 1$, $\lambda > 0$ and $\theta > 0$,

$$f(h) := Ee^{\lambda hX - \theta h^2 X^2} = 1 + (\lambda^2/2 - \theta)h^2 + O_{\lambda,\theta} h^3 E|X|^3, \tag{3.14}$$

where $|O_{\lambda,\theta}| \le e^{\lambda^2/\theta} + 2\lambda + \theta + (\lambda+\theta)^3 e^{\lambda}$.

(ii). For any $0 < h < 1$, $\lambda > 0$ and $\theta > 0$,

$$Ee^{\lambda h \max_{1\le k\le n} S_k - \theta h^2 V_n^2} \quad = \quad f^n(h) + \sum_{k=1}^{n-1} f^k(h)\,\varphi_{n-k}(h), \tag{3.15}$$

where

$$\varphi_k(h) = Ee^{-\theta h^2 V_k^2}\big(1 - e^{\lambda h \max_{1\le j\le k} S_j}\big) I_{(\max_{1\le j\le k} S_j < 0)}.$$

(iii). Write $b = x/\sqrt{n}$. For any $\lambda > 0$ and $\theta > 0$, there exists a sufficient small constant b_0 (that may depend on λ and θ) such that, for all $0 < b < b_0$,

$$\begin{aligned} &Ee^{\lambda b \max_{1\le k\le n} S_k - \theta b^2 V_n^2} \\ &\quad \le \quad C \exp\{(\lambda^2/2 - \theta)x^2 + O_{\lambda,\theta} x^3 E|X|^3/\sqrt{n}\}, \end{aligned} \tag{3.16}$$

where C is a constant not depending on x and n.

Proof. We only prove (3.15) and (3.16). The result (3.14) follows from (2.7) of Shao (1999). Set

$$\begin{aligned} \gamma_{-1}(h) &= 0, \quad \gamma_0(h) = 1, \quad \gamma_n(h) = Ee^{\lambda h \max\{0, \max_{1\le k\le n} S_k\} - \theta h^2 V_n^2}, \quad \text{for } n \ge 1, \\ \Phi(h,z) &= \sum_{n=0}^{\infty} \gamma_n(h) z^n, \qquad \Psi(h,z) = (1 - f(h)z)\Phi(h,z). \end{aligned}$$

Also write $\bar{\gamma}_n(h) = \gamma_n(h) - f(h)\gamma_{n-1}(h)$, $\widehat{S}_k = \max_{1\le j\le k} S_j$ and $f^*(h) = Ee^{\lambda h|X| - \theta h^2 X^2}$. Note that $f^*(h) \le e^{\lambda^2/4\theta}$ and $\gamma_n(h) \le \big[f^*(h)\big]^n$ by independence of X_j. It is readily seen that, for all $|z| < \min\{1, 1/f^*(h)\}$,

$$\begin{aligned} \Psi(h,z) &= 1 + \sum_{n=0}^{\infty} \gamma_{n+1}(h) z^{n+1} - \sum_{n=0}^{\infty} f(h)\gamma_n(h) z^{n+1} \\ &= 1 + \sum_{n=0}^{\infty} \big[\gamma_{n+1}(h) - f(h)\gamma_n(h)\big] z^{n+1} \\ &= \sum_{n=0}^{\infty} \bar{\gamma}_n(h) z^n. \end{aligned}$$

This, together with $f(h) \le f^*(h)$, implies that, for all z satisfying $|z| < \min\{1, 1/|f(h)|\}$,

$$\begin{aligned}[1-f(h)z]^{-1}\Psi(h,z) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f(h)^m \bar{\gamma}_n(h) z^{m+n} \\ &= \sum_{n=0}^{\infty}\Big[\sum_{k=0}^{n} f^k(h)\bar{\gamma}_{n-k}(h)\Big] z^n. \qquad (3.17)\end{aligned}$$

By virtue of (3.17) and the identity $\Phi(h,z) = [1-f(h)z]^{-1}\Psi(h,z)$, for all $n \ge 0$,

$$\gamma_n(h) = \sum_{k=0}^{n} f^k(h)\bar{\gamma}_{n-k}(h). \qquad (3.18)$$

Now, by noting that

$$\begin{aligned}&\lambda h \max_{1\le k\le n} S_k - \theta h^2 V_n^2 \\ &\quad = (\lambda h X_1 - \theta h^2 X_1^2) + \max\{0, \lambda h(S_2 - X_1), ..., \lambda h(S_n - X_1)\} - \theta h^2(V_n^2 - X_1^2),\end{aligned}$$

it follows easily from (3.18) and the i.i.d. properties of $X_j, j \ge 1$ that

$$E e^{\lambda h \max_{1\le k\le n} S_k - \theta h^2 V_n^2} = f(h)\gamma_{n-1}(h) = \sum_{k=1}^{n} f^k(h)\bar{\gamma}_{n-k}(h).$$

This, together with the facts that $\bar{\gamma}_0(h) = 1$ and

$$\begin{aligned}\bar{\gamma}_k(h) &= \gamma_k(h) - f(h)\gamma_{k-1}(h) \\ &= E e^{\lambda h \widehat{S}_k - \theta h^2 V_k^2} I_{(\widehat{S}_k \ge 0)} + E e^{-\theta h^2 V_k^2} I_{(\widehat{S}_k < 0)} - E e^{\lambda h \widehat{S}_k - \theta h^2 V_k^2} \\ &= \varphi_k(h), \quad \text{for } 1 \le k \le n-1,\end{aligned}$$

gives (3.15).

We next prove (3.16). In view of (3.14) and (3.15), it is enough to show that

$$\sum_{k=1}^{n} f^{-k}(b)\varphi_k(b) \le C, \qquad (3.19)$$

where C is a constant not depending on x and n. In order to prove (3.19), let ϵ_0 be a constant such that $0 < \epsilon_0 < \min\{1, \lambda^2/(6\theta)\}$, and $t_0 > 0$ sufficiently small such that

$$\varepsilon_1 := \varepsilon_0 t_0 - 2t_0^{3/2}(1 + E|X|^3)e^{t_0} > 0.$$

First note that

$$\begin{aligned}\varphi_k(b) &= E e^{-\theta b^2 V_k^2}\big(1 - e^{\lambda b \widehat{S}_k}\big) I_{(\widehat{S}_k < 0)} \\ &\le P\big[V_k^2 \le (1-\varepsilon_0)k\big] + e^{-(1-\varepsilon_0)\theta k b^2} E\big(1 - e^{\lambda b \widehat{S}_k}\big) I_{(\widehat{S}_k < 0)}. \qquad (3.20)\end{aligned}$$

It follows from the inequality $e^x - 1 - x \le |x|^{3/2} e^{x\vee 0}$ and the iid properties of X_j that

$$\begin{aligned} P\big[V_k^2 \le (1-\varepsilon_0)k\big] &= P(k - V_k^2 \ge \varepsilon_0 k) \le e^{-\varepsilon_0 t_0 k}(Ee^{t_0(1-X^2)})^k \\ &\le e^{-\varepsilon_0 t_o k}(1 + t_0^{3/2} E(|1-X^2|)^{3/2} e^{t_0})^k \\ &\le \exp\big\{ -\varepsilon_0 t_0 k + 2t_0^{3/2}(1+E|X|^3) k e^{t_0}\big\} = e^{-\varepsilon_1 k}. \end{aligned}$$

As in the proof of Lemma 1 of Aleshkyavichene(1979) [see (26) and (28) there], we have

$$E\big(1 - e^{\lambda b \widehat{S}_k}\big) I_{(\widehat{S}_k < 0)} = \lambda b\Big(\frac{1}{\sqrt{2\pi k}} + r_k\Big) + O\Big(\frac{\lambda^2 b^2}{\sqrt{k}}\Big) \le C\, b\, k^{-1/2} + \lambda b |r_k|,$$

for all $k \ge 1$ and sufficient small b, where $\sum_{k=1}^{\infty} |r_k| = O(1)$ and C is a constant not depending on x and n. Taking these estimates into (3.20), we obtain

$$\varphi_k(b) \le e^{-\varepsilon_1 k} + b\,\big(C\, k^{-1/2} + \lambda |r_k|\big)\, e^{-(1-\varepsilon_0)\theta k b^2},$$

for sufficient small b. On the other hand, it follows easily from (3.14) that there exists a sufficient small b_0 such that for all $0 < b \le b_0$

$$f(b) = Ee^{\lambda b X - \theta (bX)^2} \ge 1 + (\lambda^2/2 - \theta - \varepsilon_0 \theta/2) b^2 \ge e^{(\lambda^2/2 - \theta - \varepsilon_0\theta) b^2},$$

and also $f(b) \ge e^{-\varepsilon_1/2}$. Now, by recalling $\epsilon_0 < \lambda^2/(6\theta)$ and using the relation [see (39) in Nagaev (1969)]

$$\frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{z^k}{\sqrt{k}} = \frac{1}{\sqrt{1-z}} + \sum_{k=0}^{\infty} \rho_k z^k = \frac{1}{\sqrt{1-z}} + O(1), \quad |z| < 1,$$

where $\rho_k = O(k^{-3/2})$, simple calculations show that there exists a sufficient small b_0 such that for all $0 < b \le b_0$

$$\begin{aligned} \sum_{k=1}^{n} f^{-k}(b)\varphi_k(b) &\le \sum_{k=1}^{n} e^{-\varepsilon_1 k/2} + b \sum_{k=1}^{n} \big(C\, k^{-1/2} + \lambda |r_k|\big)\, e^{(2\varepsilon_0\theta - \lambda^2/2) k b^2} \\ &\le \frac{1}{1 - e^{-\epsilon_1/2}} + C\, b \sum_{k=1}^{n} k^{-1/2} e^{-\varepsilon_0 \theta k b^2} + \lambda b \sum_{k=1}^{\infty} |r_k| \\ &\le C_1 + C\sqrt{\pi} b / \sqrt{1 - e^{-\epsilon_0 \theta b^2}} \le C_2, \end{aligned}$$

where C_1 and C_2 are constants not depending on x and n. This proves (3.19) and hence completes the proof of (3.16). □

3.2 Proofs of propositions.

Without loss of generality, assume $EX^2 = 1$. Otherwise we only need to replace X_j by $X_j/\sqrt{EX^2}$.

Proof of Proposition 2.1. Since (2.1) is trivial for $0 \le x < 1$, we assume $x \ge 1$. Write $b = x/\sqrt{n}$. Observe that

$$\begin{aligned} P(\max_{1\le k\le n} S_k \ge xV_n) &\le P\big(2b \max_{1\le k\le n} S_k \ge b^2V_n^2 + x^2 - 1\big) \\ &\quad + P\big(\max_{1\le k\le n} S_k \ge xV_n,\ 2bxV_n \le b^2V_n^2 + x^2 - 1\big) \\ &= P\big(2b \max_{1\le k\le n} S_k \ge b^2V_n^2 + x^2 - 1\big) \\ &\quad + P\big(\max_{1\le k\le n} S_k \ge xV_n,\ (bV_n - x)^2 \ge 1\big) \\ &\le P\big(2b \max_{1\le k\le n} S_k - b^2V_n^2 \ge x^2 - 1\big) \\ &\quad + P\big(\max_{1\le k\le n} S_k \ge xV_n,\ \ b^2V_n^2 \ge x^2 + x\big) \\ &\quad + P\big(\max_{1\le k\le n} S_k \ge xV_n,\ \ b^2V_n^2 \le x^2 - x\big) \\ &:= \Lambda_{1,n}(x) + \Lambda_{2,n}(x) + \Lambda_{3,n}(x), \quad \text{say.} \end{aligned} \tag{3.21}$$

By (3.16) with $\lambda = 1$ and $\theta = 1/2$ in Lemma 3.3 and the exponential inequality, we have

$$\begin{aligned} \Lambda_{1,n}(x) &\le \sqrt{e}\, e^{-x^2/2} E \exp\big\{b \max_{1\le k\le n} S_k - \frac{1}{2} b^2 V_n^2\big\} \\ &\le C\, e^{-x^2/2}, \end{aligned} \tag{3.22}$$

whenever $0 \le x \le n^{1/6}$, where C is a constant not depending on x and n. By (3.16) again, it follows from the similar arguments as in the proofs of (2.12) and (2.28) in Shao (1999) that

$$\begin{aligned} \Lambda_{2,n}(x) &\le P\Big(\max_{1\le k\le n} S_k \ge xV_n,\ b^2V_n^2 \ge 9x^2\Big) \\ &\quad + P\Big(\max_{1\le k\le n} S_k \ge xV_n,\ x^2 + x \le b^2V_n^2 \le 9x^2\Big) \\ &\le C\exp\{-x^2 + O(x^3/\sqrt{n})\} + C\exp\big\{-x^2/2 - x/4 + O(x^3/\sqrt{n})\big\} \\ &\le C_1 e^{-x^2/2}, \end{aligned}$$

whenever $0 \le x \le n^{1/6}$, where C and C_1 are constants not depending on x and n. Similarly to (2.29) of Shao (1999), we obtain $\Lambda_{3,n}(x) \le Ce^{-x^2/2}$ whenever $0 \le x \le n^{1/6}$. Taking these estimates into (3.21), we prove (2.1). The proof of Proposition 2.1 is now complete. □

Proof of Proposition 2.2. First note that, by (3.9) and Lemma 3.2, we have

$$\lim_{n\to\infty} \frac{P\big[\max_{1\le k\le n} \bar{S}_k \ge x\sqrt{n}(1+\delta_n)\big]}{1 - \Phi(x)} = 2, \tag{3.23}$$

uniformly in the range $2 \le x \le \epsilon_n n^{1/6}$, where $|\delta_n| \le \epsilon_n/x^2$. Also note that it follows from

$$Ee^{t(1-\bar{X}^2)} \le 1 + tEX^2 I_{\{|X|\ge(\sqrt{n}/x)^\alpha\}} + t^2 EX^4 e^t$$

and $E|X|^{\max\{(\alpha+2)/\alpha,4\}} < \infty$ that (by letting $t = x/\sqrt{n}$)

$$\begin{aligned} P\Big(n - \bar{V}_n^2 \ge \epsilon_n n/x^2\Big) &\le e^{-t\epsilon_n n/x^2}\, \Pi_{j=1}^n E e^{t(1-\bar{X}_j^2)} \\ &\le \exp\big\{-t\epsilon_n n/x^2 + ntEX^2 I_{|X|\ge(\sqrt{n}/x)^\alpha} + nt^2 EX^4 e^t\big\} \\ &\le \exp\big\{-t\epsilon_n n/x^2 + o(1)n(x/\sqrt{n})^{2-\alpha} t + nt^2 EX^4 e^t\big\} \\ &= \exp\Big\{-\epsilon_n\sqrt{n}/x + O(x^2)\Big\} \\ &\le o(1)\big[1-\Phi(x)\big], \end{aligned}$$

uniformly in the range $2 \le x \le \epsilon_n n^{1/6}$. It is now readily seen that

$$\begin{aligned} P\big(\max_{1\le k\le n} \bar{S}_k \ge x\,\bar{V}_n\big) &\le P\big(\max_{1\le k\le n} \bar{S}_k \ge x\sqrt{n}(1-\epsilon_n/x^2)^{1/2}\big) + P\Big(n - \bar{V}_n^2 \ge \epsilon_n n/x^2\Big) \\ &= \Big[2 + o(1)\Big]\big[1-\Phi(x)\big], \end{aligned}$$

uniformly in the range $2 \le x \le \epsilon_n\, n^{1/6}$. Therefore, in order to prove Proposition 2.2, it suffices to show that

$$\lim_{n\to\infty} \frac{P\big(\max_{1\le k\le n} \bar{S}_k \ge x\,\bar{V}_n\big)}{1-\Phi(x)} \quad \ge \quad 2, \tag{3.24}$$

In fact, by noting $x\bar{V}_n \le (x^2 + b^2\bar{V}_n^2)/2b$ where $b = x/\sqrt{n}$, we have

$$\begin{aligned} P\big(\max_{1\le k\le n} \bar{S}_k \ge x\,\bar{V}_n\big) &\ge P\big(\max_{1\le k\le n} \bar{S}_k - x\bar{V}_n^2/(2\sqrt{n}) \ge x\sqrt{n}/2, \bar{V}_n^2 \le n(1+\epsilon_n/x^2)\big) \\ &\ge P\big(\max_{1\le k\le n} \bar{S}_k \ge x\sqrt{n}(1+\epsilon_n/x^2),\ \bar{V}_n^2 \le n(1+\epsilon_n/x^2)\big) \\ &= P\big(\max_{1\le k\le n} \bar{S}_k \ge x\sqrt{n}(1+\epsilon_n/x^2)\big) - R_n, \end{aligned}$$

where

$$R_n = P\big(\max_{1\le k\le n} \bar{S}_k \ge x\sqrt{n}(1+\epsilon_n/x^2),\ \bar{V}_n^2 \ge n(1+\epsilon_n/x^2)\big).$$

This, together with (3.23), implies that (3.24) will follow if we prove

$$R_n \quad = \quad o(1)\Big[1-\Phi(x)\Big], \tag{3.25}$$

uniformly in the range $2 \le x \le \epsilon_n\, n^{1/6}$.

In order to prove (3.25), write $\xi = \bar{X}^2 - E\bar{X}^2$, $\eta = \bar{X} - E\bar{X}$, $W_n = \bar{V}_n^2 - E\bar{V}_n^2$ and $T_n = \max_{1\le k\le n}(\bar{S}_k - E\bar{S}_k)$. Recall that $|\xi| \le (\sqrt{n}/x)^{2\alpha}$ and $|\eta| \le 2(\sqrt{n}/x)^\alpha$. For any $0 < \theta_0 \le (\sqrt{n}/x)^{1-\alpha}$ and $b = x/\sqrt{n}$, it follows easily from $EX^2 = 1$, $E|X|^{\max\{(2+\alpha)/\alpha,4\}} < \infty$ and $|e^s - 1 - s - s^2/2| \le |s|^3 e^{s\vee 0}$ that

$$\begin{aligned} \bar{f}(b) &:= E e^{b\eta + \theta_0 b^2\xi} \\ &= 1 + \frac{1}{2}E(b\eta + \theta_0 b^2\xi)^2 + O(1)E|b\eta + \theta_0 b^2\xi|^3 e^3 \\ &= 1 + \frac{1}{2}b^2 E\eta^2 + \theta_0 b^3 E(\eta\xi) + \frac{1}{2}\theta_0^2 b^4 E\xi^2 + O(1)e^3(b^3 E|\eta|^3 + \theta_0^3 b^6 E|\xi|^3) \\ &= 1 + \frac{1}{2}b^2(E\bar{X}^2 - (E\bar{X})^2) + O(1)(1+\theta_0)\, b^3 \\ &= 1 + \frac{1}{2}b^2 + O(1)(1+\theta_0)\, b^3, \end{aligned} \tag{3.26}$$

and

$$Ee^{\theta_0 b^2 \xi} \quad \leq \quad 1 + E(\theta_0 b^2 \xi)^2 e/2 = 1 + O(1)(1+\theta_0)b^3. \tag{3.27}$$

Therefore, by a similar argument as in the proof of (3.15) in Lemma 3.3 and noting

$$\bar{\varphi}_k(b) := Ee^{\theta_0 b^2 W_k}\big(1 - e^{bT_k}\big)I_{T_k<0} \leq Ee^{\theta_0 b^2 W_k} \leq \exp\big\{O(1)k(1+\theta_0)b^3\big\},$$

we have

$$\begin{aligned} Ee^{bT_n + \theta_0 b^2 W_n} &= \bar{f}^n(b) + \sum_{k=1}^{n-1} \bar{f}^k(b)\,\bar{\varphi}_{n-k}(b) \\ &\leq \Big(\sum_{k=1}^{n} e^{kb^2/2}\Big)\exp\{O(1)n(1+\theta_0)b^3\} \\ &\leq 4b^{-2}\exp\{nb^2/2 + O(1)n(1+\theta_0)b^3\}, \end{aligned} \tag{3.28}$$

where we have used the fact that, for sufficient small b,

$$\sum_{k=1}^{n} e^{-b^2k/2} \leq \frac{1}{1-e^{-b^2/2}} \leq 4b^{-2}.$$

Now, for any $0 < \theta_0 \leq (\sqrt{n}/x)^{1-\alpha}$ and $b = x/\sqrt{n}$, we have

$$\begin{aligned} R_n &\leq P(\max_{1\leq k\leq n} \bar{S}_k \geq x\sqrt{n},\ \ \bar{V}_n^2 > n(1+\varepsilon_n/x^2)) \\ &\leq P(b\max_{1\leq k\leq n} \bar{S}_k + \theta_0 b^2 \bar{V}_n^2 > x^2 + \theta_0(x^2+\varepsilon_n)) \\ &\leq P(b\max_{1\leq k\leq n} (\bar{S}_k - E\bar{S}_k) + \theta_0 b^2(\bar{V}_n^2 - E\bar{V}_n^2) > x^2 + \theta_0\varepsilon_n') \\ &\leq \exp\{-(x^2+\theta_0\varepsilon_n')\}Ee^{bT_n+\theta_0 b^2 W_n} \\ &\leq 4b^{-2}\exp\big\{-x^2/2 - \theta_0\epsilon_n + O(1)(1+\theta_0)x^3/\sqrt{n}\big\} \\ &\leq O(1)\,b^{-2}\exp\big\{-x^2/2 - \theta_0\epsilon_n/2\big\}, \end{aligned} \tag{3.29}$$

uniformly in the range $1 \leq x \leq \epsilon_n n^{1/6}$, where we have used the fact that

$$\begin{aligned} \varepsilon_n' &:= \varepsilon_n + \theta_0 b^2(n - E\bar{V}_n^2) - nb|EXI(|X| \leq (\sqrt{n}/x)^\alpha)| \\ &= \varepsilon_n + \theta_0 nb^2 EX^2 I(|X| > (\sqrt{n}/x)^\alpha) - nb|EXI(|X| > (\sqrt{n}/x)^\alpha)| \\ &= \varepsilon_n + o(1)nb^3 = \varepsilon_n + o(1)x^3/\sqrt{n}. \end{aligned} \tag{3.30}$$

Since we may choose $\epsilon_n \to 0$ sufficient slow so that $\epsilon_n \geq (x/\sqrt{n})^{-(1-\alpha)/2}$, by recalling $0 < \alpha < 1$ and taking $\theta_0 = (\sqrt{n}/x)^{1-\alpha}$ in (3.29), we obtain

$$R_n \quad \leq \quad 4b^{-2}\exp\{-x^2/2 - b^{-(1-\alpha)/2}\} = o(1)\Big[1-\Phi(x)\Big],$$

uniformly in the range $1 \leq x \leq \epsilon_n n^{1/6}$. This proves (3.25) and also completes the proof of Proposition 2.2. □

4 Proof of Theorem 1.2

By (4.2) in Shao(1997) and the fact $\max_{1\le k\le n} S_k \ge S_n$, it suffices to show that

$$\lim_{n\to\infty} x_n^{-2} \log P\Big(\max_{1\le k\le n} S_k \ge x_n V_n\Big) \le -\frac{1}{2}.$$

Put

$$l(x) = EX^2 I(|X| \le x), \qquad b = \inf\{x \ge 1 : l(x) > 0\},$$
$$z_n = \inf\Big\{s : s \ge b+1, \frac{l(s)}{s^2} \le \frac{x_n^2}{n}\Big\}.$$

Then $z_n \to \infty$ and $nl(z_n) = x_n^2 z_n^2$ for n large enough. For any $0 < \varepsilon < 1/2$, we have

$$\begin{aligned}
&P\Big(\max_{1\le k\le n} S_k \ge x_n V_n\Big) \\
&\le P\Big(\max_{1\le k\le n} S_k' \ge (1-\varepsilon) x_n V_n\Big) \\
&\quad + P\Big(\sum_{i=1}^n |X_i| I(|X_i| > z_n) \ge \varepsilon x_n V_n,\ V_n > 0\Big) + P(X=0)^n \\
&:= J_1 + J_2 + P(X=0)^n,
\end{aligned}$$

where $S_k' = \sum_{i=1}^k X_i I(|X_i| \le z_n)$. To see this, it sufficies to note that

$$\begin{aligned}
\max_{1\le k\le n} S_k &= \max_{1\le k\le n} \Big(S_k' + \sum_{i=1}^k X_i I(|X_i| > z_n)\Big) \\
&\le \max_{1\le k\le n} S_k' + \sum_{i=1}^n |X_i| I(|X_i| > z_n).
\end{aligned}$$

As in Shao(1997), we have

$$J_2 \le \Big(\frac{o(1)}{\varepsilon}\Big)^{\varepsilon^2 x_n^2}.$$

As for J_1, we have

$$\begin{aligned}
J_1 &\le P\Big(\max_{1\le k\le n} S_k' \ge (1-\varepsilon) x_n V_n'\Big) \\
&\le P\Big(\max_{1\le k\le n} S_k' \ge (1-\varepsilon)^2 x_n \sqrt{nl(z_n)}\Big) + P(V_n'^2 \le (1-\varepsilon) nl(z_n)) \\
&:= J_3 + J_4,
\end{aligned}$$

where $V_n'^2 = \sum_{i=1}^k X_i^2 I(|X_i| \le z_n)$. It follows from Shao(1997) again

$$J_4 \le \exp\{-x_n^2 + o(x_n^2)\}.$$

Since $EX^2 I(|X| \le x)$ is slowly varying as $x \to \infty$,

$$|ES_k'| \le nE|X| I(|X| \ge z_n) = o(nl(z_n)/z_n), \quad k = 1, \cdots, n.$$

Then by a Lévy inequality, we have

$$\begin{aligned} J_3 &\leq P\Big(\max_{1\leq k\leq n}(S'_k - ES'_k) \geq (1-\varepsilon)^2 x_n \sqrt{nl(z_n)} - o(nl(z_n)/z_n)\Big) \\ &\leq 2P\Big(S'_n - ES'_n \geq [(1-\varepsilon)^2 - o(1)]x_n\sqrt{nl(z_n)}\Big) \\ &\leq 2P\Big(S'_n \geq [(1-\varepsilon)^2 - o(1)]x_n\sqrt{nl(z_n)}\Big) \\ &= \exp\Big(-\big((1-\varepsilon)^2 - 1/2\big)x_n^2 + o(x_n^2)\Big) \end{aligned}$$

where the last equality is from Shao(1997).

Now (4.1) follows from the above inequalities and the arbitrariness of ε. The proof of Theorem 1.1 is complete. □

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