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A Gaussian process approximation for two-color randomly reinforced urns*

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Abstract

The Pólya urn has been extensively studied and is widely applied in many disciplines. An important application is to use urn models to develop randomized treatment allocation schemes in clinical studies. As an extension of the Pólya urn, the randomly reinforced urn was recently proposed to optimize clinical trials in the sense that patients are assigned to the best treatment with probability converging to one. In this paper, we prove a Gaussian process approximation for the sequence of random compositions of a two-color randomly reinforced urn for both the cases with the equal and unequal reinforcement means. By using the Gaussian approximation, the law of the iterated logarithm and the functional central limit theorem in both the stable convergence sense and the almost-sure conditional convergence sense are established. Also as a consequence, we are able to to prove that a random limit of the normalized urn composition has no points masses under the only assumption of finite $(2+\epsilon)$-th moments.

Keywords: Reinforced urn model; Gaussian process; strong approximation; functional central limit theorem; Pólya urn; law of the iterated logarithm.
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1 Introduction

Asymptotic properties, including the strong consistency and asymptotic normality, of urn models and their applications are widely studied in recent years under various assumptions concerning the updating rules, for example, one may refer to Chauvin, Pouyanne and Sahnoun (2011), Bai, Hu and Rosenberger (2002), Bai and Hu (2005), Hu and Rosenberger (2006), Janson (2004, 2006), Zhang, Hu and Cheung (2006) etc. In this paper, we consider a kind of two-color urn model, called the randomly reinforced urn (RRU) model, which is a generalization of the original Pólya urn (cf. Eggenberger and Pólya (1923), Pólya (1931)). The main issue of this model different from most urn models in literature is that, as shown, the proportions of balls in the urn will not converge to a

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non-extreme constant and the numbers of different type balls may increase in different speeds. This issue makes its asymptotic properties quite different from the those of other urn models and difficult to study.

The RRU model is described as follows. Consider a two-color urn with the initial urn components $\boldsymbol{Y}_0 = (Y_{0,1}, Y_{0,2})$, where $Y_{0,k} > 0$ is the number of type k balls. The urn is sampled sequentially. Suppose the urn components are $\boldsymbol{Y}_m = (Y_{m,1}, Y_{m,2})$ after m samplings. At the $(m+1)$-th sampling, a ball of type k is drawn with a probability

$$p_{m+1,k} = \frac{Y_{m,k}}{|\boldsymbol{Y}_m|}, \quad \text{where } |\boldsymbol{Y}_m| = Y_{m,1} + Y_{m,2}.$$

And the sampled ball is replaced in the urn together with a nonnegative random number $U_{m+1,k}$ of balls of the same type k, generated from a distribution μ_k with mean $m_k > 0$. This is the model introduced and formally named the randomly reinforced urn in Muliere, Paganoni and Secchi (2006a). But it would appear in earlier literatures in different versions. For example, Durham and Yu (1990) proposed a similar model for sequential sampling in clinical trails. In our RRU setting, the numbers of balls take positive real values, not necessary integers. When $U_{m+1,1} = U_{m+1,2} = \alpha$ is a constant and a positive integer, a RRU is the original Pólya urn (cf. Eggenberger and Pólya (1923), Pólya (1931)) which is very popular in literatures. The RRU model is of fundamental importance in many areas of applications, for instance in economics (cf. Erev and Roth (1998), Beggs (2005), Hopkins and Posch (2005)), in information science (cf. Martin and Ho (2002)), in resampling theory etc. In clinical trial studies, the RRU model is utilized to define a response-adaptive design focusing to reduce the expected number of patients receiving inferior treatments (cf. Durham, Flournoy, Li (1998), Li, Durham and Flournoy (1996), Muliere, Paganoni and Secchi (2006a,b), Paganoni and Secchi (2007), May and Flournoy (2009) etc).

Suppose the reinforcement distributions μ_1 and μ_2 have bounded supports. In Muliere, Paganoni and Secchi (2006a), it is showed that the sequence $\{Z_n = \frac{Y_{n,1}}{|\boldsymbol{Y}_n|}\}$ of the random sample proportions in the urn converges to almost surely to a random limit $Z_\infty \in [0,1]$. When $\mu_1 = \mu_2$, Crimaldi (2009) proved a central limit theorem by showing the almost-sure conditional convergence to a Gaussian kernel of the sequence $\{\sqrt{n}(Z_n - Z_\infty)\}$. Aletti, May and Secchi (2009) extended Crimaldi's result to a general case where reinforcement means m_1 and m_2 are equal and proved that Z_∞ has no point masses in $[0,1]$ by using this kind of conditional central limit theorem. When the means m_1 and m_2 are different, the limit proportion Z_∞ of a RRU is showed to be a point mass either 1 and 0 by Beggs (2005), Hopkins and Posch (2005) and Muliere, Paganoni and Secchi (2006a) under the assumption that the supports of μ_1 and μ_2 are bounded from 0, and by Aletti, May and Secchi (2009) under the only assumption that μ_1 and μ_2 have bounded supports. May and Flournoy (2009) proved that the sequence $\left\{\frac{Y_{n,1}}{Y_{n,2}^{m_1/m_2}}\right\}$ converges almost surely to a random limit $\psi_\infty \in (0,\infty)$ both when $m_1 = m_2$ and $m_1 \neq m_2$. Berti et al. (2010, 2011) derived the almost-sure central limit theorems for a multi-color RRU. However, the reinforcement means are also assumed to be equal.

The purpose of this paper is to establish the Gaussian process approximation of the sequence $\{Z_n\}$ when $m_1 = m_2$ as well as the sequence $\left\{\frac{Y_{n,1}}{Y_{n,2}^{m_1/m_2}}\right\}$ when $m_1 \neq m_2$, under the assumption that μ_1 and μ_2 have only finite $(2+\epsilon)$-th moments. We will show that both these sequences can be approximated by a tail stochastic integral with respect to a Brownian motion mixed with a random variable. It is interesting that, as we will find, the mixed Gaussian process for approximating is nearly independent of the urn composition to be approximated. Our Gaussian process approximation enables us (i) to establish the law of the iterated logarithm; (ii) to establish the functional central limit theorem in both the stable convergence sense and the almost-sure conditional convergence sense; (iii) to prove that the limit ψ_∞ (resp. Z_∞) has no point masses in $[0,\infty]$ (resp. in $[0,1]$) when $m_1 \neq m_2$ (resp. when $m_1 = m_2$) under the assumption that μ_1 and μ_2 have only finite $(2+\epsilon)$-th moments. Another implication of our Gaussian approximation is that we are able to establish the central limit theorem in a simple way for the random number $N_{n,k}$ of draws, where $N_{n,k}$ is the number of type k balls being drawn in the first n samplings. In a response-adaptive design in clinical trials driven by a RRU model, $N_{n,k}$ is the number of patients allocated to treatment k, and its asymptotic behavior is of particular interest.

For the generalized Friedman urn models and immigrated urn models , Bai, Hu and Zhang (2002), Zhang and Hu (2009) and Zhang et al. (2011) established the Gaussian approximation for both the urn proportions $\frac{Y_{n,k}}{n}$ and the sampling proportion $\frac{N_{n,k}}{n}$. But the RRU which we consider here is not covered by their assumptions. The main reason is that the mean replacement matrix $diag(m_1, m_2)$ of a RRU is not irreducible and hence the limit of $\frac{Y_{n,k}}{n}$ and $\frac{N_{n,k}}{n}$ is not a constant in $(0,1)$.

The paper is organized as follows. The main approximation theorems with applications for the equal and unequal reinforcement mean cases are stated in Section 2 and Section 3, respectively, and the proofs of the approximations appear in the last section. Some remarks on unsolved problems are discussed in Section 4.

Notations. In the sequel of this paper if having not been specially mentioned, $(U_{l,1}, U_{l,2})$, $l = 1, 2, \dots$ are assumed to be independent identically distributed random vectors with finite second moments. Let $X_{m,k}$ be the result of the m-th drawing, i.e., $X_{m,k} = 1$ if the m-th drawn ball is of type k, and 0 otherwise. It is obvious that $N_{m,k} = \sum_{j=1}^m X_{m,k}$ and $X_{m,1} + X_{m,2} = 1$. Denote $\mathscr{F}_n = \sigma\big(U_{l,k}, X_{l,k}, Y_{l,k} : k = 1, 2; l = 1, \dots, n\big)$ to be the history σ-field generated by all the observations up to stage n, and $\mathscr{F}_\infty = \bigvee_n \mathscr{F}_n$. Further, for two positive sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n = O(b_n)$ if there is a constant C such that $a_n \leq C b_n$, $a_n \sim b_n$ if $\frac{a_n}{b_n} \to 1$, and $a_n \approx b_n$ if $a_n = O(b_n)$ and $b_n = O(a_n)$.

2 Equal reinforcement mean case

In this section, we consider the case of $m_1 = m_2 > 0$. Let $\sigma_k^2 = \mathsf{E}[({U_{1,k}}/{m_k})^2]$, $k = 1, 2$,

$$Z_n = \frac{Y_{n,1}}{Y_{n,1} + Y_{n,2}}, \quad Z_\infty(\omega) = \lim Z_n(\omega), \quad H(\omega) = \frac{\sigma_1^2}{Z_\infty} + \frac{\sigma_2^2}{1 - Z_\infty}.$$

To start, we shall assume $\mathsf{P}(Z_\infty = 0) = \mathsf{P}(Z_\infty = 1) = 0$, for otherwise H may have no definition. The almost sure convergence of Z_n is proved by May and Flournoy (2009) under the condition that the reinforcement distributions μ_1 and μ_2 have bounded supports, and by Zhang et al. (2014) under the condition that $\mathsf{E}[U_{1,k} \log^+ U_{1,k}] < \infty$, $k = 1, 2$. The following theorem is the main result on the Gaussian approximation.

Theorem 2.1. *Suppose $m_1 = m_2 > 0$, $EU_{1,1}^p < \infty$ and $EU_{1,2}^p < \infty$, where $2 \leq p < 4$. Then (possibly in an enlarged probability space) there is standard Brownian motion $B(y)$ such that*

$$Z_\infty - Z_n = Z_\infty(1 - Z_\infty) H \int_{nH}^\infty \frac{dB(y)}{y} + o(\lambda_n) \ a.s., \tag{2.1}$$

where

$$\lambda_n = \begin{cases} n^{-\frac{1}{2}} (\log\log n)^{\frac{1}{2}}, & \text{if } p = 2, \\ n^{\frac{1}{p} - 1} (\log n)^{\frac{1}{2}}, & \text{if } 2 < p < 4. \end{cases}$$

Furthermore, the Brownian motion $B(y)$ can be constructed with a filtration of σ-fields $\{\mathscr{G}_n\}$ and a non-decreasing sequence of stopping times $\{T_n\}$ satisfying the following properties:

Property (a) $\mathscr{F}_n \subset \mathscr{G}_n$, T_n *is* $\mathscr{G}_n$ *measurable;*

Property (b) $T_n = nH + o(n^{2/p})$ *a.s.;*

Property (c) *Conditionally on* $\mathscr{F}_n$, $B(T_n + y) - B(T_n)$, $y \geq 0$, *is also a standard Brownian motion.*

Remark 2.2. *Denote $W(t) = -t\int_t^\infty y^{-2}dB(y)$. By checking the covariance function, it is easily seen that $W(t), t > 0$ is a standard Brownian motion.*

Remark 2.3. *The process in (2.1) for approximating is a tail stochastic integral respective to the Brownian motion. It seems to be independent of Z_n. Actually, according to Property (b) nH can be replaced by T_n, and $\sqrt{T_n}\int_{T_n} y^{-1}dB(y)$ is indeed a normal random variable which is independent of Z_n. We will illustrate this interesting property in Corollary 2.6 in more details.*

We will prove Theorem 2.1 by first approximating $Z_n - Z_\infty$ to an infinite summation of a weighted martingale sequence and then approximating the martingale to a Brownian motion by applying the Skorokhod embedding method. The detail proof of Theorem 2.1 will be stated in Appendix A. In the sequel of this section, we give several corollaries as applications. Define

$$\widetilde{\sigma}(\omega) = \sqrt{Z_\infty(1 - Z_\infty)}\sqrt{(1 - Z_\infty)\sigma_1^2 + Z_\infty\sigma_2^2},$$

$$\widetilde{\sigma}_n(\omega) = \sqrt{Z_n(1 - Z_n)}\sqrt{(1 - Z_n)\sigma_1^2 + Z_n\sigma_2^2}.$$

The first corollary is the following law of the iterated logarithm.

Corollary 2.4. *Suppose $m_1 = m_2 > 0$, $\mathsf{E}U_{1,1}^2 < \infty$ and $\mathsf{E}U_{1,2}^2 < \infty$. Then*

$$\limsup_{n\to\infty} \frac{\sqrt{n}(Z_n - Z_\infty)}{\sqrt{2\log\log n}} = \widetilde{\sigma} \;\; a.s..$$

Proof. Write $\gamma(x) = \sqrt{x/(2\log\log x)}$, $G(x) = -\int_x^\infty y^{-1}dB(y)$. By (2.1), we need to show that

$$\limsup_{n\to\infty} \gamma(nH)G(nH) = \limsup_{T\to\infty} \gamma(T)|G(T)| = 1 \; a.s. \tag{2.2}$$

Note that $\gamma(x)G(x) = xG(x)/\sqrt{2x\log\log x}$, and that $xG(x)$ is also a standard Brownian motion. (2.2) follows from the law of the iterated logarithm of the Brownian motion. □

The next corollary is on the functional cental limit theorem.

Corollary 2.5. *Suppose $m_1 = m_2 > 0$, $\mathsf{E}U_{1,1}^p < \infty$ and $\mathsf{E}U_{1,2}^p < \infty$ for some $p > 2$. Define*

$$W_n(t) = t\sqrt{n}(Z_{[nt]} - Z_\infty), t > 0.$$

Then

$$W_n(\cdot) \xrightarrow{d} \widetilde{\sigma}B_\infty(\cdot) \;\; \textit{stably}, \tag{2.3}$$

in the Skorokhod topological space $D(0,\infty)$, where $B_\infty(t)$ is a standard Brownian motion which is independent of $\mathscr{F}_\infty$. In particular,

$$\lim_{n\to\infty} \mathsf{P}\left(\frac{\max_{0\leq l\leq n} l(Z_l - Z_n)}{\widetilde{\sigma}_n\sqrt{n}} \geq x\right) = e^{-2x^2}, \;\; x > 0. \tag{2.4}$$

Here the stable convergence in (2.3) means that for any bounded and (uniformly) continues function $f : D(0,\infty) \to (-\infty,\infty)$,

$$\mathsf{E}\left[f\big(W_n(\cdot)\big)I_E\right] \to \mathsf{E}\left[f\big(\widetilde{\sigma}B_\infty(\cdot)\big)I_E\right] \quad \textit{for any event } E.$$

Proof. (2.4) is due to fact that

$$\max_{0\le l\le n}\frac{l(Z_l-Z_n)}{\widetilde{\sigma}\sqrt{n}} = \sup_{0<t\le 1}\frac{W_n(t)-tW_n(1)}{\widetilde{\sigma}} \xrightarrow{d} \sup_{0<t\le 1}\left(B_\infty(t)-tB_\infty(1)\right)$$

and that $B_\infty(t){-}tB_\infty(1)$ is a Brownian bridge. For (2.3), note that $W(t) = -t\int_t^\infty y^{-1}dB(y)$ is also a standard Brownian motion. By (2.1),

$$n(Z_n-Z_\infty) - \frac{\widetilde{\sigma}}{\sqrt{H}}W(nH) = o(n^{1/2}) \ \ a.s.,$$

which implies that for any $T>0$,

$$\sup_{0<t\le T}\left|W_n(t)-\widetilde{\sigma}\frac{W(nHt)}{\sqrt{nH}}\right| \to o(1) \ a.s..$$

For the Brownian motion $W(\cdot)$, we have

$$\frac{W(n\cdot)}{\sqrt{n}} \xrightarrow{d} B_\infty(\cdot) \ \ \text{mixing},$$

i.e., for any given event E with $\mathsf{P}(E)>0$, the conditional distribution of $\frac{W(n\cdot)}{\sqrt{n}}$ converges to a Brownian motion. It follows that

$$\left(\frac{\widetilde{\sigma}}{\sqrt{H}}, H, \frac{W(n\cdot)}{\sqrt{n}}\right) \xrightarrow{d} \left(\frac{\widetilde{\sigma}}{\sqrt{H}}, H, B_\infty(\cdot)\right) \ \ \text{stably}.$$

Note that $\widetilde{\sigma}\frac{W(nH\cdot)}{\sqrt{nH}}$ is a continuous function of $\left(\frac{\widetilde{\sigma}}{\sqrt{H}}, H, \frac{W(n\cdot)}{\sqrt{n}}\right)$ of the form $f(r,h,x(\cdot)) = rx(\cdot h)$. It follows that

$$\widetilde{\sigma}\frac{W(nH\cdot)}{\sqrt{nH}} \xrightarrow{d} \widetilde{\sigma}\frac{B_\infty(\cdot H)}{\sqrt{H}} \overset{d}{=} \widetilde{\sigma}B_\infty(\cdot) \ \ \text{stably}.$$

The proof is now completed. □

Corollary 2.5 implies the central limit theorem for $\sqrt{n}(Z_n-Z_\infty)$. Aletti, May and Secchi (2009) proved a strong version of the central limit theorem. For every Borel set B, every ω, and $n=1,2,\ldots$, define

$$K_n(\omega,B) = \mathsf{P}\big(\sqrt{n}(Z_n-Z_\infty)\in B\big|\mathscr{F}_n\big)(\omega),$$

i.e., K_n is a version of the condition distribution of $\sqrt{n}(Z_n-Z_\infty)$ given $\mathscr{F}_n$. Aletti, May and Secchi (2009) showed that, if $m_1=m_2$ and the distributions of μ_1 and μ_2 have bounded supports, then for almost every ω, the sequence of probability distributions $K_n(\omega,\cdot)$ converges weakly to the normal distribution

$$N\big(0,\widetilde{\sigma}^2(\omega)\big).$$

We denote this kind of convergence by

$$\sqrt{n}(Z_n-Z_\infty)\Big|_{\mathscr{F}_n} \xrightarrow{d} N\big(0,\widetilde{\sigma}^2(\omega)\big) \ a.s.. \tag{2.5}$$

This kind of conditional central limit theorem was first established by Crimaldi (2009) for a special case that $\mu_1 = \mu_2$. Its generalizations to multi-color case can be found in Berti et al. (2010, 2011). Aletti, May and Secchi (2009) also showed that (2.5) implies that Z_∞ has no point masses in $(0,1)$. Our next corollary states that (2.5) and a type of conditional functional central limit theorems follow from the Gaussian approximation.

Corollary 2.6. *Suppose $m_1 = m_2 > 0$, $\mathsf{E}U_{1,1}^p < \infty$ and $\mathsf{E}U_{1,2}^p < \infty$ for some $p > 2$. Then*

$$\sup_{t\ge 1}\left|\sqrt{n}(Z_\infty - Z_{[nt]}) - \widetilde{\sigma}_n\sqrt{T_n}\int_{T_nt}^{\infty}\frac{dB(y)}{y}\right| = o(n^{-\epsilon}) \ \ a.s. \tag{2.6}$$

for some $\epsilon > 0$, and further, $\overline{B}_n(t) = -t\sqrt{T_n}\int_{T_nt}^{\infty} y^{-1}dB(y)$, $t \ge 1$, is a standard Brownian motion on $[1,\infty)$ which is independent of the history sigma field $\mathscr{F}_n$.

As a consequence,

$$W_n(\cdot)\big|_{\mathscr{F}_n} \xrightarrow{d} \widetilde{\sigma}(\omega)B_\infty(\cdot) \ \ a.s., \tag{2.7}$$

in the Skorokhod topological space $D[1,\infty)$, where $W_n(t)$ is defined as in Corollary 2.5, $B_\infty(t)$ is a standard Brownian motion which is independent of $\mathscr{F}_\infty$.

In particular, (2.5) holds, Z_∞ has no point masses in $(0,1)$, and there exists a sequence of standard normal random variables for which ζ_n is independent of $\mathscr{F}_n$ and

$$\sqrt{n}(Z_n - Z_\infty) = \widetilde{\sigma}_n\zeta_n + o(n^{-\epsilon}) \ \ a.s., \tag{2.8}$$

Proof. We first prove (2.6). By (2.1),

$$\sup_{t\ge 1}\left|\sqrt{n}(Z_\infty - Z_{[nt]}) - Z_\infty(1-Z_\infty)H\sqrt{n}\int_{ntH}^{\infty}\frac{dB(y)}{y}\right| = o(n^{-\epsilon}) \ a.s.$$

Let $H_n = \frac{\sigma_1^2}{Z_n} + \frac{\sigma_2^2}{1-Z_n}$. Note that $Z_n - Z_\infty = O(\sqrt{n^{-1}\log\log n})$ a.s. by Corollary 2.4, and $\frac{T_n}{n} = H + o(n^{2/p-1})$ a.s.. It follows that $\sqrt{n}Z_\infty(1-Z_\infty)H - \widetilde{\sigma}_n\sqrt{T_n} = o(n^{2/p-1/2})$ a.s.. Further,

$$n^{\frac{2}{p}-\frac{1}{2}}\sup_{t\ge 1}\left|\int_{ntH}^{\infty}\frac{dB(y)}{y}\right| = n^{\frac{2}{p}-\frac{1}{2}}O(\sqrt{n^{-1}\log\log n}) = o(n^{-\epsilon}) \ \ a.s.$$

It remains to show that

$$\sup_{t\ge 1}\left|\int_{ntH}^{T_nt}\frac{dB(y)}{y}\right| = o(n^{-\frac{1}{2}-\epsilon}) \ \ a.s.$$

Write $a_n = T_n - nH$. Note that $W(x) = -x\int_x^{\infty} y^{-1}dB(y)$ is a standard Brownian motion and

$$\int_{ntH}^{T_nt}\frac{dB(y)}{y} = \frac{W(nHt + a_nt) - W(nHt)}{T_nt} + \frac{W(ntH)}{ntH}\frac{nH - T_n}{T_n}.$$

The second term on the right hand of the above equality does not exceed

$$O(\sqrt{n^{-1}\log\log n})o(n^{\frac{2}{p}-1})$$

uniformly in $t \ge 1$ almost surely, by the law of the iterated logarithm. The first term does not exceed

$$O\left(\frac{\sqrt{a_nt(\log(nH) + \log\log(nHt))}}{T_nt}\right) = o(n^{\frac{1}{p}-1}(\log n)^{\frac{1}{2}})$$

uniformly in $t \ge 1$ almost surely, by the path properties of a Brownian motion (cf. Hanson and Russo (1983)). The proof of (2.6) is now completed.

Let $B_n(y) = B(T_n + y) - B(T_n)$. Then conditionally on $\mathscr{G}_n$, $B_n(y)$ is a standard Brownian motion. It is obvious that

$$\overline{B}_n(t) = -t\sqrt{T_n}\int_{T_n(t-1)}^{\infty} \frac{dB_n(y)}{T_n + y}.$$

Hence, conditionally on $\mathscr{G}_n$, $\overline{B}_n(t)$, $t \geq 1$, is a mean zero Gaussian process with covariance function

$$T_n ts \int_{T_n(t-1)}^{\infty} \frac{dy}{(T_n + y)^2} = s \quad \text{for } t \geq s \geq 1.$$

It follows that $\overline{B}_n(t)$, $t \geq 1$, is a standard Brownian motion and is independent of $\mathscr{G}_n$. So, it is independent of $\mathscr{F}_n$ because $\mathscr{F}_n \subset \mathscr{G}_n$. The proof of the main part of the corollary is now completed.

Now for (2.7), from (2.6) it follows that

$$dist\big(W_n(\cdot), \widetilde{\sigma}_n B_n(\cdot)\big) \to 0 \;\; a.s. \;\text{ in } D[1,\infty),$$

where $dist(\cdot,\cdot)$ is a metric in $D[1,\infty)$. Note that $\widetilde{\sigma}_n$ is $\mathscr{F}_n$-measurable and $\widetilde{\sigma}_n \to \widetilde{\sigma}$ a.s., and, conditionally on $\mathscr{F}_n$, $\widetilde{\sigma}_n B_n(\cdot)$ and $\widetilde{\sigma}_n B_\infty(\cdot)$ has the same distribution. Hence, (2.7) follows from (2.8) by noting the following fact that:

$$\begin{aligned} &\xi_n(\cdot)\big|_{\mathscr{F}_n} \xrightarrow{d} \widetilde{\sigma} B_\infty(\cdot) \; a.s. \text{ and } dist\big(\xi_n(\cdot), \eta_n(\cdot)\big) \to 0 \; a.s. \quad \text{in } D[1,\infty) \\ &\Longrightarrow \eta_n(\cdot)\big|_{\mathscr{F}_n} \xrightarrow{d} \widetilde{\sigma} B_\infty(\cdot) \; a.s. \quad \text{in } D[1,\infty). \end{aligned}$$

This fact follows from that for any bounded and uniformly continuous function $f : D[1,\infty) \to (-\infty,\infty)$,

$$\begin{aligned} &\limsup_{n\to\infty} \Big| \mathsf{E}\big[f(\eta_n) - f(\xi_n)\big|\mathscr{F}_n\big]\Big| \\ \leq &\mathsf{E}\Big[\limsup_{n\to\infty} |f(\eta_n) - f(\xi_n)|\Big| \bigvee_n \mathscr{F}_n\Big] = 0 \;\; a.s., \end{aligned}$$

due to Lemma A.2 of Crimaldi (2009).

Finally, (2.8) follows from (2.6) by letting $\zeta_n = \overline{B}_n(1)$, and (2.5) is a conclusion of (2.8) or (2.7). Aletti, May and Secchi (2009) showed that (2.5) implies Z_∞ having no point masses in $(0,1)$ by utilizing a metric of the weak convergence of probability measures with the limit distribution being absolutely continuous. Here we give a straightforward proof. Let $f(t) = e^{-t^2/2}$ be the characteristic function of a standard normal distribution. Firstly, note that (2.5) implies that for every $\bigvee_n \mathscr{F}_n$-measurable event E,

$$\lim_{n\to\infty} \mathsf{E}[e^{it\sqrt{n}(Z_n - Z_\infty)} I_E | \mathscr{F}_n] = f(\widetilde{\sigma}t) I_E \;\; a.s.$$

In fact, if let $I_n = \mathsf{E}[I_E|\mathscr{F}_n]$, then $I_n \to I_E$ a.s.. And hence

$$\begin{aligned} &\lim_{n\to\infty} \mathsf{E}[e^{it\sqrt{n}(Z_n - Z_\infty)} I_E | \mathscr{F}_n] = \lim_{n\to\infty} \mathsf{E}[e^{it\sqrt{n}(Z_n - Z_\infty)} I_n | \mathscr{F}_n] \\ &= \lim_{n\to\infty} \mathsf{E}[e^{it\sqrt{n}(Z_n - Z_\infty)} | \mathscr{F}_n] I_n = f(\widetilde{\sigma}t) I_E \;\; a.s., \end{aligned}$$

where in the fist equality we use the fact that

$$\eta_n \to 0 \;\; a.s. \;\text{ and } |\eta_n| \leq M \; a.s. \Longrightarrow \mathsf{E}[\eta_n|\mathscr{F}_n] \to 0 \;\; a.s..$$

This fact is due to Lemma A.2 of Crimaldi (2009). Next, choosing $E = \{Z_\infty = p\}$, $p \in (0,1)$, yields

$$\begin{aligned} &f(\widetilde{\sigma}t) I_E = \lim_{n\to\infty} \mathsf{E}[e^{it\sqrt{n}(Z_n - p)} I_E | \mathscr{F}_n] \\ &= \lim_{n\to\infty} e^{it\sqrt{n}(Z_n - p)} \mathsf{E}[I_E|\mathscr{F}_n] = \lim_{n\to\infty} e^{it\sqrt{n}(Z_n - p)} I_E \;\; a.s.. \end{aligned}$$

Hence, $|f(\widetilde{\sigma}t)|I_E = I_E$ a.s.. So $I_E = 0$ a.s. because $|f(\widetilde{\sigma}t)| < 1$ on E. The proof is now completed. □

From the above proof, we obtain the following corollary.

Corollary 2.7. *Suppose $\{Y_\infty, Y_n, n \geq 1\}$ is a sequence of random variables, $\{a_n\}$ is a sequence of constants with $a_n \to \infty$, and $\mathscr{G}_n$ is a filtration of σ-fields such that Y_n is $\mathscr{G}_n$-measurable. If for almost every ω, the distribution of $a_n(Y_n - Y_\infty)$ conditionally on $\mathscr{G}_n$ converges to a non-degenerate distribution, then Y_∞ has no point masses.*

Remark 2.8. *By (2.5), it can also be shown that for every event E,*

$$\begin{aligned}
&\limsup_{|t|\to\infty}\limsup_{n\to\infty}\left|\mathsf{E}\Big[\exp\big\{-itn^{\frac{1}{2}}Z_\infty\big\}I_E\Big]\right| \\
\leq &\limsup_{|t|\to\infty}\limsup_{n\to\infty}\mathsf{E}\left|\mathsf{E}\Big[\exp\big\{itn^{\frac{1}{2}}(Z_n - Z_\infty)\big\}I_E\big|\mathscr{F}_n\Big]\right| \\
= &\limsup_{|t|\to\infty}\limsup_{n\to\infty}\mathsf{E}\left|\mathsf{E}\big[f\big(t\widetilde{\sigma}_n\big)I_E\big|\mathscr{F}_n\big]\right| = \limsup_{|t|\to\infty}\mathsf{E}\Big[\big|f\big(t\widetilde{\sigma}_\infty\big)\big|\,I_E\Big] = 0.
\end{aligned}$$

Let $f_E(t)$ be the characteristic function of the conditional distribution of Z_∞ given E with $\mathsf{P}(E) > 0$. Then the above equality means that

$$\lim_{|t|\to\infty}\limsup_{n\to\infty}\left|f_E(tn^{\frac{1}{2}})\right| = 0.$$

Note that for any $t_0 \geq 1$, $n_0 \geq 1$ and $|s| \geq 2t_0n_0$, there exist a real number t with $|t| \geq t_0$ and an integer $n \geq n_0$ such that $s = tn^{1/2}$. We conclude that

$$\lim_{|s|\to\infty} f_E(s) = 0.$$

This is related to the Cramér condition. Obviously, if $E = \{Z_\infty = p\}$, $p \in (0,1)$, and $\mathsf{P}(E) > 0$, then $|f_E(t)| \equiv 1$ which is a contradiction.

The next corollary is the central limit theorem for the random number of draws.

Corollary 2.9. *Suppose $m_1 = m_2 > 0$, $\mathsf{E}U_{1,1}^p < \infty$ and $\mathsf{E}U_{1,2}^p < \infty$ for some $p > 2$. Then*

$$\sqrt{n}\left(\frac{N_{n,1}}{n} - Z_\infty\right) \xrightarrow{d} h(\omega)\cdot N(0,1) \ \ \textit{stably}, \tag{2.9}$$

where $h(\omega) = \sqrt{Z_\infty(1-Z_\infty)}\sqrt{(1-Z_\infty)(2\sigma_1^2-1) + Z_\infty(2\sigma_2^2-1)}$, and $N(0,1)$ is a standard normal random variable which is independent of $\mathscr{F}_\infty$.

Proof. We need to prove that

$$\begin{aligned}
&\sqrt{n}\left(\frac{N_{n,1}}{n} - Z_\infty\right) \\
\xrightarrow{d} &-\sqrt{Z_\infty}(1-Z_\infty)N_1\Big(0,\sigma_1^2-1\Big) \\
&\quad + \sqrt{1-Z_\infty}Z_\infty N_2\Big(0,\sigma_2^2-1\Big)\Big) + \widetilde{\sigma}\cdot N_3(0,1) \ \ \text{stably},
\end{aligned}$$

where $N_1\Big(0,\sigma_1^2-1\Big)$, $N_2\Big(0,\sigma_2^2-1\Big)$, $N_3(0,1)$ are three independent normal random variables which are independent of $\mathscr{F}_\infty$. Write

$$A_{n,k} = \frac{\sum_{l=1}^n X_{l,k}(U_{l,k}/m_k - 1)}{N_{n,k}}, \quad k = 1,2. \tag{2.10}$$

Then
$$A_{n,k} = O\big(\sqrt{n^{-1}\log\log n}\big)\ a.s. \ \text{ and } \ \frac{Y_{n,k}}{m_k N_{n,k}} = 1 + A_{n,k}.$$
By the Taylor expansion and (2.8), we have that
$$\begin{aligned}\frac{N_{n,1}}{n} - Z_\infty =& \frac{N_{n,1}}{N_{n,1}+N_{n,2}} - Z_\infty = \frac{\frac{Z_n}{1+A_{n,1}}}{\frac{Z_n}{1+A_{n,1}} + \frac{1-Z_n}{1+A_{n,2}}} - Z_\infty \\ =& - Z_n(1-Z_n)\big(A_{n,1}-A_{n,2}\big) + (Z_n - Z_\infty) + O\Big(\frac{\log\log n}{n}\Big) \\ =& - Z_n(1-Z_n)\big(A_{n,1}-A_{n,2}\big) + \widetilde{\sigma}_n\zeta_n + o(n^{-\frac{1}{2}}) \qquad (2.11) \\ =& - \frac{Z_n(1-Z_n)}{\sqrt{N_{n,1}}}\big(\sqrt{N_{n,1}}A_{n,1}\big) + \frac{Z_n(1-Z_n)}{\sqrt{N_{n,2}}}\big(\sqrt{N_{n,2}}A_{n,2}\big) \\ & + \widetilde{\sigma}_n\zeta_n + o(n^{-\frac{1}{2}})\ a.s.\end{aligned}$$
Note that ζ_n is a standard normal random variable which is independent of $\widetilde{\sigma}_n$, Z_n, $N_{n,k}$ and $A_{n,k}$, $k=1,2$. Also, $\widetilde{\sigma}_n \to \widetilde{\sigma}$ a.s., $\frac{Z_n(1-Z_n)\sqrt{n}}{\sqrt{N_{n,1}}} \to \sqrt{Z_\infty}(1-Z_\infty)$ a.s. and $\frac{Z_n(1-Z_n)\sqrt{n}}{\sqrt{N_{n,2}}} \to Z_\infty\sqrt{1-Z_\infty}$. The proof is completed if we have shown that
$$\big(\sqrt{N_{n,1}}A_{n,1}, \sqrt{N_{n,2}}A_{n,2}\big) \xrightarrow{d} \Big(N_1\big(0,\sigma_1^2-1\big), N_2\big(0,\sigma_2^2-1\big)\Big) \ \text{ mixing.} \qquad (2.12)$$
Note that $\sigma_k^2 - 1 = \mathsf{Var}\big(U_{1,k}/m_k\big)$, $k=1,2$. The above convergence follows from Theorem 4.1 of May and Flournoy (2009). ☐

The next corollary tells us that conditionally on $\mathscr{F}_n$, the distribution of $\sqrt{n}(\frac{N_{n,1}}{n} - Z_\infty)$ does not converge.

Corollary 2.10. *Suppose $m_1 = m_2 > 0$, $EU_{1,1}^p < \infty$ and $EU_{1,2}^p < \infty$ for some $p > 2$. Let E be an event such that, for $\omega \in E$, there is a distribution F_ω verifying*
$$\sqrt{n}\Big(\frac{N_{n,1}}{n} - Z_\infty\Big)\Big|_{\mathscr{F}_n} \xrightarrow{d} F_\omega. \qquad (2.13)$$
Then $\mathsf{P}(E) = 0$.

Proof. Recall (2.11). Let $\eta_n = \sqrt{nZ_n(1-Z_n)}(A_{n,1} - A_{n,2})$. Note that $Z_n \to Z_\infty$ a.s.,
$$\widetilde{\sigma}_n\xi_n\big|_{\mathscr{F}_n} \xrightarrow{d} \widetilde{\sigma}N(0,1) \ \ a.s.$$
and that Z_n, $A_{n,1}$ and $A_{n,2}$ are $\mathscr{F}_n$-measurable. By (2.11) and (2.13), there exists an event Ω_0 with $\mathsf{P}(\Omega_0) = 1$ such that $\sqrt{n}Z_n(1-Z_n)(A_{n,1} - A_{n,2})$ converges on $E \cap \Omega_0$. So there exists a random variable η such that
$$\eta_n(\omega) \to \eta(\omega) \ \ \forall \omega \in E \cap \Omega_0.$$
Suppose $\mathsf{P}(E) > 0$. Choose x such that $\mathsf{P}(\eta > x, E) > 0$. Then it follows that
$$\mathsf{P}(\eta_n \le x, \eta > x, E) \to \mathsf{P}(\eta \le x, \eta > x, E) = 0.$$
So $\mathsf{P}(\eta_n \le x | \eta > x, E) \to 0$. On the other hand, according to (2.12) we have
$$\eta_n = \frac{\sqrt{nZ_n(1-Z_n)}}{\sqrt{N_{n,1}}}\sqrt{N_{n,1}}A_{n,1} - \frac{\sqrt{nZ_n(1-Z_n)}}{\sqrt{N_{n,2}}}\sqrt{N_{n,2}}A_{n,2} \xrightarrow{d} N(0,1) \ \text{ mixing,}$$
where $N(0,1)$ is independent of $\mathscr{F}_\infty$. It follows that
$$\lim_n \mathsf{P}(\eta_n \le x | \eta > x, E) = \Phi(x) > 0.$$
We get a contradiction. The proof is completed. ☐

3 Unequal reinforcement mean case

Denote $\rho = \frac{m_1}{m_2}$, and

$$\psi_n = \frac{Y_{n,1}}{Y_{n,2}^{\rho}}, \ \ \psi_\infty = \lim_{n\to\infty} \psi_n.$$

It is obvious that $Z_n = \frac{\psi_n}{1+\psi_n}$ and $Z_\infty = \frac{\psi_\infty}{1+\psi_\infty}$ when $m_1 = m_2$. In this section, we consider the case of $m_1 \neq m_2$. Without loss of generality, we assume that $0 < m_1 < m_2$. May and Flournoy (2009) proved that the limit ψ_∞ exists almost surely with $\mathsf{P}(0 < \psi_\infty < \infty) = 1$ when the reinforcement distributions μ_1 and μ_2 have bounded supports. Durham and Yu (1990) proved a similar result, namely:

$$\frac{N_{n,1}}{N_{n,2}^{\rho}} \text{ converges almost surely to a finite limit } \eta_\infty.$$

It is easily seen that

$$\eta_\infty = \frac{m_2^{\rho}}{m_1}\psi_\infty \ \ a.s.$$

and

$$\lim_{n\to\infty} \frac{N_{n,1}}{n^{\rho}} = \frac{m_2^{\rho}}{m_1}\psi_\infty \ a.s., \ \ \lim_{n\to\infty} \frac{Y_{n,1}}{n^{\rho}} = m_2^{\rho}\psi_\infty \ a.s.$$

In a recent paper of Zhang et al. (2014), it is proved that the weakest condition for $\mathsf{P}(0 < \psi_\infty < \infty) = 1$ is that $\mathsf{E}[U_{1,k} \log^+ U_{1,k}] < \infty$, $k = 1, 2$, and a general multi-color RRU is consider. We state the convergence result as the following theorem.

Theorem 3.1. *Suppose $\mathsf{E}[U_{1,k} \log^+ U_{1,k}] < \infty$, $m_k > 0$, $k = 1, 2$. Then the limit ψ_∞ exists almost surely and $\mathsf{P}(0 < \psi_\infty < \infty) = 1$ both when $m_1 = m_2$ and $m_1 \neq m_2$.*

The following theorem is our main result on the Gaussian process approximation for ψ_n. From the Gaussian approximation we are able to show that ψ_∞ has no point masses in $(0, \infty)$. And accordingly, all the limits of the sequences $\{Y_{n,1}/Y_{n,2}^{\rho}\}$, $\{Y_{n,1}/n^{\rho}\}$, $\{Y_{n,1}/|\boldsymbol{Y}_n|^{\rho}\}$, $\{N_{n,1}/N_{n,2}^{\rho}\}$ and $\{N_{n,1}/n^{\rho}\}$ have no point masses in $[0, \infty]$.

Theorem 3.2. *Suppose $m_2 > m_1 > 0$, $\mathsf{E}U_{1,1}^p < \infty$ and $\mathsf{E}U_{1,2}^p < \infty$ for some $p > 2$. Denote $\sigma_k^2 = \mathsf{E}[(U_{1,k}/m_k)^2]$, $k = 1, 2$. Let $\delta_0 = \min\{\frac{1-\rho}{\rho}, \frac{1}{2} - \frac{1}{p}\}$. Then (possibly in an enlarged probability space) there is standard Brownian motion $B(y)$ such that for any $0 < \delta < \delta_0$,*

$$\psi_\infty - \psi_n = \frac{\sigma_1\sqrt{m_1}}{m_2^{\rho/2}} \int_{n^\rho/\psi_\infty}^{\infty} \frac{dB(y)}{y} + o(n^{-\frac{\rho(1+\delta)}{2}}) \tag{3.1}$$

$$= - n^{-\frac{\rho}{2}} \frac{\sigma_1\sqrt{m_1\psi_\infty}}{m_2^{\rho/2}} \frac{W\left(n^\rho/\psi_\infty\right)}{\sqrt{n^\rho/\psi_\infty}} + o(n^{-\frac{\rho(1+\delta)}{2}}) \ a.s., \tag{3.2}$$

where $W(x) = -x\int_x^\infty y^{-1} dB(y)$ is also a standard Brownian motion.

Furthermore, the Brownian motion $B(y)$ can be constructed with a filtration of σ-fields $\{\mathscr{G}_n\}$ and a non-decreasing sequence of stopping times $\{T_n\}$ satisfying Properties (a) and (c) in Theorem 2.1, and

Property (b′) $T_n = \frac{n^\rho}{\psi_\infty} + o(n^{\rho(1-\delta)})$ *a.s. for $0 < \delta < \delta_0$.*

The proof of this theorem will be given in the last section. Next, we state several corollaries. The first one is on the law of iterated logarithm and the central limit theorem for ψ_n.

Corollary 3.3. *Under the conditions in Theorem 3.2,*

$$\limsup_{n\to\infty} \frac{n^{\rho/2}\big(\psi_n - \psi_\infty\big)}{\sqrt{2\log\log n}} = \frac{\sigma_1\sqrt{m_1\psi_\infty}}{m_2^{\rho/2}} \;\; a.s. \tag{3.3}$$

and there exists a sequence $\{\zeta_n\}$ *of standard normal random variables for which* ζ_n *is independent of* $\mathscr{F}_n$ *and*

$$n^{\frac{\rho}{2}}\big(\psi_n - \psi_\infty\big) = \frac{\sigma_1\sqrt{m_1\psi_n}}{m_2^{\rho/2}}\zeta_n + o(n^{-\epsilon}) \; a.s. \text{ for some } \epsilon > 0. \tag{3.4}$$

Hence

$$n^{\frac{\rho}{2}}\big(\psi_n - \psi_\infty\big)\Big|_{\mathscr{F}_n} \xrightarrow{d} N\Big(0, \frac{\sigma_1^2 m_1}{m_2^{\rho}}\psi_\infty(\omega)\Big) \;\; a.s. \tag{3.5}$$

and ψ_∞ *has no point masses in* $(0,\infty)$.

Proof. (3.3) follows from (3.2) and the law of iterated logarithm of the Brownian motion. (3.4) can be proved in the same way as proving Corollary 2.4. □

Corollary 3.4. *Under the conditions in Theorem 3.2,*

$$n^{\frac{\rho}{2}}\big(\frac{N_{n,1}}{N_{n,2}^{\rho}} - \eta_\infty\big) \xrightarrow{d} N(0,1)\cdot\sqrt{\eta_\infty(2\sigma_1^2 - 1)} \;\; \textit{stably} \tag{3.6}$$

and

$$\begin{cases} n^{\frac{\rho}{2}}\big(\frac{N_{n,1}}{n^{\rho}} - \eta_\infty\big) \xrightarrow{d} N(0,1)\cdot\sqrt{\eta_\infty(2\sigma_1^2-1)} \;\; \textit{stably}, & \textit{if } \rho < 2/3, \\ n^{1-\rho}\big(\frac{N_{n,1}}{n^{\rho}} - \eta_\infty\big) \to -\rho\eta_\infty^2 \;\; a.s., & \textit{if } \rho > 2/3, \\ n^{\frac{\rho}{2}}\big(\frac{N_{n,1}}{n^{\rho}} - \eta_\infty + \rho\eta_\infty^2 n^{\rho-1}\big) \xrightarrow{d} N(0,1)\cdot\sqrt{\eta_\infty(2\sigma_1^2-1)} \;\; \textit{stably}, & \textit{if } \rho = 2/3, \end{cases} \tag{3.7}$$

$$n^{1-\rho}\big(1 - \frac{N_{n,2}}{n}\big) \to \eta_\infty \;\; a.s. \tag{3.8}$$

Proof. For (3.6), let $A_{n,k}$ be defined as in (2.10). Then

$$A_{n,2} = O\big(\sqrt{N_{n,2}^{-1}\log\log N_{n,2}}\big) = O\big(\sqrt{n^{-1}\log\log n}\big) = o(n^{-\frac{\rho}{2}-\epsilon}) \;\; a.s.,$$

$$A_{n,1} = O\big(\sqrt{N_{n,1}^{-1}\log\log N_{n,1}}\big) = O\big(\sqrt{n^{-\rho}\log\log n}\big) \;\; a.s..$$

Note that $\eta_\infty = \frac{\psi_\infty m_2^{\rho}}{m_1}$. It follows that

$$\begin{aligned} \frac{N_{n,1}}{N_{n,2}^{\rho}} - \eta_\infty =& -\eta_\infty + \psi_n\frac{m_2^{\rho}}{m_1}\frac{(1+A_{n,2})^{\rho}}{1+A_{n,1}} \\ =& -\eta_\infty + \psi_n\frac{m_2^{\rho}}{m_1}(1 - A_{n,1} + O(A_{n,1}^2))(1+O(A_{n,2})) \\ =& (\psi_n - \psi_\infty)\frac{m_2^{\rho}}{m_1} - \psi_n\frac{m_2^{\rho}}{m_1}A_{n,1} + o(n^{-\frac{\rho}{2}-\epsilon}) \\ =& n^{-\frac{\rho}{2}}\Big\{\sqrt{\frac{\psi_n m_2^{\rho}}{m_1}}\,\sigma_1\zeta_n - \psi_n\frac{m_2^{\rho}}{m_1}\sqrt{\frac{n^{\rho}}{N_{n,1}}}\big(\sqrt{N_{n,1}}A_{n,1}\big) + o(n^{-\epsilon})\Big\} \;\; a.s. \end{aligned}$$

The proof of (3.6) is completed by noting that $\frac{\psi_n m_2^{\rho}}{m_1} \to \eta_\infty$ a.s., $\frac{N_{n,1}}{n^{\rho}} \to \eta_\infty$ a.s., $\sqrt{N_{n,1}}A_{n,1} \xrightarrow{d} N(0,\sigma_1^2 - 1)$ mixing, and ζ_n is a standard normal random variable which is independent of $\mathscr{F}_n$.

For (3.7), it is sufficient to note that

$$\begin{aligned}\frac{N_{n,1}}{n^\rho} =&\frac{N_{n,1}}{N_{n,2}^\rho}\left(1-\frac{N_{n,1}}{n}\right)^\rho = \frac{N_{n,1}}{N_{n,2}^\rho}\left(1-\rho\frac{N_{n,1}}{n}+O\Big(\frac{N_{n,1}}{n}\Big)^2\right)\\ =&\frac{N_{n,1}}{N_{n,2}^\rho}-\rho\frac{N_{n,1}}{N_{n,2}^\rho}\frac{N_{n,1}}{n^\rho}n^{\rho-1}+O(n^{2(\rho-1)})\\ =&\frac{N_{n,1}}{N_{n,2}^\rho}-\rho\eta_\infty^2 n^{\rho-1}+o(n^{\rho-1})\ \ a.s.\end{aligned}$$

(3.8) is obvious because $1-\frac{N_{n,2}}{n}=\frac{N_{n,1}}{n}\sim\frac{\eta_\infty n^\rho}{n}$ a.s. The proof is now completed. □

Finally, we give the functional central limit theorem.

Corollary 3.5. *Define* $W_n(t)=n^{\frac{\rho}{2}}t^\rho(\psi_{[nt]}-\psi_\infty)$, $t>0$. *Then*

$$W_n(t)\xrightarrow{d}\sigma_1\sqrt{\eta_\infty}B_\infty(t^\rho)\ \ \textit{stably} \tag{3.9}$$

in the Skorokhod topological space $D(0,\infty)$, *where* $B_\infty(t)$ *is a standard Brownian motion which is independent of* $\mathscr{F}_\infty$. *In particular,*

$$\lim_{n\to\infty}\mathsf{P}\left(\frac{\max_{0\le l\le n}l^\rho(\psi_l-\psi_n)}{\sigma_1\sqrt{N_{n,1}}}\ge x\right)=e^{-2x^2},\ \ x>0. \tag{3.10}$$

Proof. The proof of (3.9) is similar to that of (2.3) by noting that $\eta_\infty=\frac{\psi_\infty m_1}{m_2^\rho}$. For (3.10), it is sufficient to see that

$$\max_{0\le l\le n}\frac{l^\rho(\psi_\infty-\psi_n)}{n^{\rho/2}}=\sup_{0<t\le 1}\left(W_n(t)-t^\rho W_n(1)\right)$$

and $N_{n,1}\sim\eta_\infty n^\rho$ a.s.. □

4 Concluding Remark

We approximated Z_n-Z_∞ and $\psi_n-\psi_\infty$ by a kind of Gaussian process $\int_t^\infty y^{-1}dB(y)$, which is a tail stochastic integral with respective to a Brownian motion, with time t stopping at a random variable nH or n/ψ_∞, where $H^2=(1+\psi_\infty)(\sigma_1^2/\psi_\infty+\sigma_2^2)$. But this does not mean that $\int_{nH}^\infty y^{-1}dB(y)$ and $\int_{n/\psi_\infty}^\infty y^{-1}dB(y)$ are Gaussian random variables and their distributions are unknown because the mixing distribution of ψ_∞ is unknown. For deriving the asymptotic distributions, the approximations (2.6) and (3.4) seem more powerful than (2.1) and (3.1) because the process for approximation is independent of other random variables considered. (2.1) and (3.1) are helpful for establishing the strong convergence such as the law of the iterated logarithm.

This paper only considers the two-color urn model. Berti et al. (2010, 2011) derived the almost-sure central limit theorems for the multi-color RRU. However, it is also assumed that the reinforcement means are equal. The strong convergence and asymptotic normality for a general multi-color reinforced urn model are studied recently by Zhang, et al. (2014). It is expected to approximate the urn components after being suitably normalized by a multi-dimensional Gaussian process. The Skorokhod embedding method used in this paper does not work for the multi-dimension case. Though strong approximations for multi-dimensional martingales can be found in literature, for example, Monrad and Philipp (1991), Eberlein (1986) and Zhang (2004), the martingales concerning to the reinforced urn model usually do not satisfied a condition that the asymptotic conditional variability is $\mathscr{F}_k$-measurable for some fixed k (cf. (A.5)), which is needed in the approximation theorems for multi-dimensional martingales. A new approach is needed for approximating the multi-color reinforced urn models.

A Proof of the main results

Recall $|\boldsymbol{Y}_n| = Y_{n,1} + Y_{n,2}$, $\mathsf{E}[X_{n,k}|\mathscr{F}_{n-1}] = \mathsf{P}(X_{n,k} = 1|\mathscr{F}_{n-1}) = \frac{Y_{n-1,k}}{|\boldsymbol{Y}_{n-1}|}$, $Z_n = \frac{Y_{n,1}}{|\boldsymbol{Y}_n|}$, $\psi_n = \frac{Y_{n,1}}{Y_{n,2}^{\rho}}$, $\rho = \frac{m_1}{m_2}$. Denote

$$Q_n = \frac{1}{m_1}\log Y_{n,1} - \frac{1}{m_2}\log Y_{n,2}.$$

Then

$$\begin{aligned}\Delta Q_n = Q_n - Q_{n-1} =& \frac{1}{m_1}X_{n,1}\log\left(1+\frac{U_{n,1}}{Y_{n-1,1}}\right) - \frac{1}{m_2}X_{n,2}\log\left(1+\frac{U_{n,2}}{Y_{n-1,2}}\right)\\ =&\left[X_{n,1}\frac{U_{n,1}/m_1}{Y_{n-1,1}} - X_{n,2}\frac{U_{n,2}/m_2}{Y_{n-1,2}}\right]\\ &+\left[-\frac{1}{m_1}X_{n,1}f\Big(\frac{U_{n,1}}{Y_{n-1,1}}\Big) + \frac{1}{m_2}X_{n,2}f\Big(\frac{U_{n,2}}{Y_{n-1,2}}\Big)\right]\\ :=&\Delta Q_n^{(1)} + \Delta Q_n^{(2)}, \end{aligned} \tag{A.1}$$

where $f(x) = x - \log(1+x)$ satisfying $0 \le f(x) \le x^2$ for $x \ge 0$.

Proof of Theorem 2.1. Let $g(x) = \frac{e^{m_1 x}}{1+e^{m_1 x}}$. Then $g(Q_n) = Z_n$ and $g'(Q_\infty) = m_1 Z_\infty(1 - Z_\infty)$. According to the Taylor expansion, it is sufficient to show that $B(t)$ and T_n can be constructed such that

$$Q_\infty - Q_n = \frac{H}{m_1}\int_{nH}^{\infty}\frac{dB(x)}{x} + o(\lambda_n) \ \ a.s.. \tag{A.2}$$

Recall $m_1 = m_2$. It is easily shown that $|\boldsymbol{Y}_n|/n \to m_1$ a.s.. According to Theorem 3.1, $Z_n \to Z_\infty = \frac{\psi_\infty}{1+\psi_\infty} \in (0,1)$ a.s., which implies that $Y_{n,k} \approx n$ a.s., $k = 1,2$. So, for $\Delta Q_n^{(2)}$ in (A.1) we have

$$\begin{aligned}\sum_{l=1}^{\infty}\lambda_l^{-1}\mathsf{E}[|\Delta Q_l^{(2)}||\mathscr{F}_{l-1}] \le& C\sum_{l=1}^{\infty}\lambda_l^{-1}\left(\mathsf{E}\Big[X_{l,1}\frac{U_{l,1}^2}{Y_{l-1,1}^2}\Big|\mathscr{F}_{l-1}\Big] + \mathsf{E}\Big[X_{l,2}\frac{U_{l,2}^2}{Y_{l-1,2}^2}\Big|\mathscr{F}_{l-1}\Big]\right)\\ =C\sum_{l=1}^{\infty}\lambda_l^{-1}&\left(\frac{Y_{l-1,1}}{|\boldsymbol{Y}_{n-1}|}\frac{\sigma_1^2}{Y_{l-1,1}^2} + \frac{Y_{l-1,2}}{|\boldsymbol{Y}_{n-1}|}\frac{\sigma_2^2}{Y_{l-1,2}^2}\right) \le C\sum_{l=1}^{\infty}\lambda_l^{-1}l^{-2} < \infty,\end{aligned}$$

which implies that $\sum_{l=n+1}^{\infty}|\Delta Q_l^{(2)}| = o(\lambda_n)$ a.s..

For $\Delta Q_n^{(1)}$, we use the truncation method. Let $\widetilde{U}_{n,k} = \frac{U_{n,k}}{m_k}$, $\overline{U}_{n,k} = \widetilde{U}_{n,k}I\{\widetilde{U}_{n,k} \le n^{\frac{1}{p}}\}$, $\overline{\sigma}_{n,k}^2 = \mathsf{E}\overline{U}_{n,k}^2$, $\overline{m}_{n,k} = \mathsf{E}[\overline{U}_{n,k}]$, $k = 1,2$, $\overline{m}_n = \overline{m}_{n,1} - \overline{m}_{n,2}$, and

$$\Delta M_n^{(1)} = m_1 n\left(X_{n,1}\frac{\overline{U}_{n,1}}{Y_{n-1,1}} - X_{n,2}\frac{\overline{U}_{n,2}}{Y_{n-1,2}}\right),$$

$$\Delta M_n = \Delta M_n^{(1)} - \mathsf{E}[\Delta M_n^{(1)}|\mathscr{F}_{n-1}] = \Delta M_n^{(1)} - \frac{m_1 n\overline{m}_n}{|\boldsymbol{Y}_{n-1}|}.$$

Then $\{\Delta M_n, \mathscr{F}_n\}$ is a sequence of martingale differences. Note that

$$\sum_{n=1}^{\infty}\mathsf{P}\Big(\frac{U_{n,k}}{m_k} > n^{\frac{1}{p}}\Big) \le \mathsf{E}\Big(\frac{U_{1,k}}{m_k}\Big)^p < \infty, \ \ k = 1,2.$$

From the Borel-Cantelli lemma, it follows that

$$\mathsf{P}(\Delta Q_n^{(1)} \ne \frac{1}{m_1 n}\Delta M_n \ \ i.o.) = 0.$$

Also,

$$\sum_{l=n+1}^{\infty} \frac{1}{m_1 l} |\mathsf{E}[\Delta M_l^{(1)}|\mathscr{F}_{l-1}]| = \sum_{l=n+1}^{\infty} \frac{|\overline{m}_l|}{|\mathbf{Y}_{l-1}|}$$
$$\leq C \sum_{l=n+1}^{\infty} \frac{1}{l} \sum_{k=1}^{2} \mathsf{E}[|\widetilde{U}_{1,k}| I\{\widetilde{U}_{1,k} > l^{\frac{1}{p}}\}] \leq C n^{\frac{1}{p}-1} \sum_{k=1}^{2} \mathsf{E}[|\widetilde{U}_{1,k}|^p] = o(\lambda_n).$$

Hence, we conclude that

$$\sum_{l=n+1}^{\infty} \Delta Q_l = \sum_{l=n+1}^{\infty} \frac{1}{m_1 l} \Delta M_l + o(\lambda_n)$$
$$= \frac{1}{m_1} \left(\sum_{l=n}^{\infty} \frac{1}{l(l+1)} M_l - \frac{M_n}{n} \right) + o(\lambda_n) \;\; a.s.$$

For the martingale $M_n = \sum_{l=1}^{n} \Delta M_l$, we have

$$\mathsf{E}[(\Delta M_n)^2|\mathscr{F}_{n-1}] = \Big(\frac{m_1 n}{|\mathbf{Y}_{n-1}|} \Big)^2 \Big(\frac{\mathsf{E}\overline{U}_{n,1}^2}{Z_{n-1}} + \frac{\mathsf{E}\overline{U}_{n,2}^2}{1 - Z_{n-1}} - \overline{m}_n^2 \Big), \tag{A.3}$$

$$\mathsf{E}[|\Delta M_n|^4|\mathscr{F}_{n-1}] \leq \sum_{k=1}^{2} \left(\frac{m_1 n}{Y_{n-1,k}} \right)^4 \mathsf{E}\overline{U}_{n,k}^4 \leq C(\omega) \sum_{k=1}^{2} \mathsf{E}\overline{U}_{n,k}^4. \tag{A.4}$$

By the Skorokhod embedding theorem (cf. Theorem A.1 of Hall and Heyde (1980,page 269)), (possibly in an enlarged probability space) there is a standard motion $B(x)$ with a filtration $\{\mathscr{G}_n\}$ and a sequence of nonnegative stopping times $\tau_1, \tau_2, \cdots$ with the following properties

(i) $M_n = B(T_n)$, where $T_n = \sum_{i=1}^{n} \tau_i$;

(ii) $\mathscr{F}_n \subset \mathscr{G}_n$, τ_n is $\mathscr{G}_n$ measurable, $\mathsf{E}[\tau_n|\mathscr{G}_{n-1}] = \mathsf{E}[(\Delta M_n)^2|\mathscr{F}_{n-1}]$ and $\mathsf{E}[\tau_n^r|\mathscr{G}_{n-1}] \leq C_r \mathsf{E}[(\Delta M_n)^{2r}|\mathscr{F}_{n-1}]$ for any $r \geq 1$;

(iii) Conditionally on $\mathscr{G}_n$, $B(T_n + x) - B(T_n)$, $x \geq 0$, is also a standard Brownian motion.

Now, we verify that the Brownian motion $B(x)$ and the stopping time T_n are desirable for Property (b) and (A.2). At first, we assume the following approximation for the conditional variance.

$$\mathsf{E}[(\Delta M_n))^2|\mathscr{F}_{n-1}] = H(\omega) + o(n^{\frac{2}{p}-1}) \;\; a.s. \tag{A.5}$$

From (A.5) it follows that

$$\sum_{i=1}^{n} \mathsf{E}[\tau_n|\mathscr{G}_{i-1}] = nH(\omega) + o(n^{\frac{2}{p}}) \; a.s.$$

On the other hand, by (ii) and (A.4) we have that

$$\sum_{n=1}^{\infty} \mathsf{E}\left[\Big(\frac{\tau_n}{n^{2/p}} \Big)^2 |\mathscr{G}_{n-1} \right] \leq C \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{2} \mathsf{E}\overline{U}_{n,k}^4}{n^{4/p}} \leq C \sum_{k=1}^{2} \mathsf{E}[(U_{1,k}/m_k)^p] < \infty.$$

By the law of large numbers for martingales, it follows that

$$\sum_{i=1}^{n} (\tau_i - \mathsf{E}[\tau_i|\mathscr{G}_{i-1}]) = o(n^{\frac{2}{p}}) \;\; a.s..$$

Hence

$$T_n = \sum_{i=1}^{n} \tau_i = nH + o(n^{\frac{2}{p}})\ a.s.$$

and Property (b) is verified. Then it follows from the path properties of a Browian motion (cf. Theorem 1.2.1 of Csörgő and Révész (1981)) that

$$B(T_n) - B(nH) = o\left(\sqrt{n^{\frac{2}{p}}\Big(\log\frac{n}{n^{2/p}} + \log\log n\Big)}\right) = o(n\lambda_n)\ \ a.s.$$

So

$$\begin{aligned}
\sum_{l=n+1}^{\infty} \Delta Q_l =& \frac{1}{m_1}\left(\sum_{l=n}^{\infty}\frac{1}{l(l+1)}B(T_l) - \frac{B(T_n)}{n}\right) + o(\lambda_n)\\
=& \frac{1}{m_1}\left(\sum_{l=n}^{\infty}\frac{1}{l(l+1)}B(lH) - \frac{B(nH)}{n}\right) + \sum_{l=n}^{\infty}\frac{1}{l(l+1)}o(l\lambda_l) + o(\lambda_n)\\
=& \frac{1}{m_1}\left(\int_n^{\infty}\frac{B(xH)}{x^2}dx - \frac{B(nH)}{n}\right) + o(\lambda_n)\\
=& \frac{1}{m_1}\left(H\int_{nH}^{\infty}\frac{B(x)}{x^2}dx - \frac{B(nH)}{n}\right) + o(\lambda_n)\\
=& \frac{H}{m_1}\int_{nH}^{\infty}\frac{dB(x)}{x} + o(\lambda_n)\ \ a.s..
\end{aligned}$$

Finally, we verify (A.5). Note that $\mathsf{E}[\overline{U}_{n,k}^2] \to \sigma_k^2$, $|\boldsymbol{Y}_n|/n \to m_1$ a.s., $k = 1,2$ and $Z_n \to Z_\infty$ a.s.. (A.5) is obvious for $p = 2$ by (A.3).

For $2 < p < 4$, we still have Corollary (2.4) due to the approximation for the case of $p = 2$. Hence

$$Z_n - Z_\infty = O(\sqrt{n^{-1}\log\log n}) = o(n^{\frac{2}{p}-1})\ \ a.s.$$

On the other hand,

$$\begin{aligned}
\frac{Y_{n,1}+Y_{n,2}}{m_1 n} - 1 =& \frac{\sum_{k=1}^{2}\sum_{i=1}^{n} X_{i,k}(U_{i,k} - \mathsf{E}[U_{i,k}])}{m_1 n}\\
=& O(\sqrt{n^{-1}\log\log n}) = o(n^{\frac{2}{p}-1}))\ a.s.
\end{aligned}$$

and

$$|\overline{m}_n| \le \sum_{k=1}^{2}\mathsf{E}[{}^{U_{1,k}}/_{m_k} I\{{}^{U_{1,k}}/_{m_k} > n^{\frac{1}{p}}\}] = o(n^{\frac{1}{p}-1}),$$

$$\sigma_k^2 - \mathsf{E}[\overline{U}_{n,k}^2] = \mathsf{E}[({}^{U_{1,k}}/_{m_k})^2 I\{{}^{U_{1,k}}/_{m_k} > n^{\frac{1}{p}}\}] = o(n^{\frac{2}{p}-1}),\ k = 1,2.$$

(A.5) follows by (A.3). The proof is now completed. □

Proof of Theorem 3.2. Let $g(x) = e^{m_1 x}$. Then $g(Q_n) = \psi_n$ and $g'(Q_\infty) = m_1\psi_\infty$. According to the Taylor expansion, it is sufficient to show that $B(t)$ and T_n can be constructed such that

$$Q_\infty - Q_n = \frac{\sigma_1}{\sqrt{m_1 m_2^\rho}\psi_\infty}\int_{n^\rho/\psi_\infty}^{\infty}\frac{dB(y)}{y^2} + o(n^{-\frac{\rho(1+\delta)}{2}})\ \ a.s.. \tag{A.6}$$

Recall (A.1) and note that $Y_{n-1,2} \sim m_2 n$, $Y_{n-1,1} \sim \psi_\infty Y_{n,2}^\rho \sim \psi_\infty(m_2 n)^\rho$ a.s.. It can be shown that for $0 < \delta_1 \le 1/2$ and $\delta_1 < (1-\rho)/\rho$,

$$\begin{aligned}
&\sum_{l=1}^{\infty} l^{\frac{\rho(1+\delta_1)}{2}}\mathsf{E}\Big[X_{l,k} f\Big(\frac{U_{l,k}}{Y_{l-1,k}}\Big)\Big|\mathscr{F}_{l-1}\Big]\\
\le& \sum_{l=1}^{\infty} l^{\frac{\rho(1+\delta_1)}{2}}\frac{Y_{l-1,k}}{|\boldsymbol{Y}_{l-1}|}\frac{\mathsf{E}U_{l,k}^2}{Y_{l-1,k}^2} \le c\sum_{l=1}^{\infty}\frac{l^{\rho(1+\delta_1)/2}}{l\cdot l^\rho} < \infty,
\end{aligned}$$

which implies $\sum_{l=n+1}^{\infty} X_{l,k} f\big(\frac{U_{l,k}}{Y_{l-1,k}}\big) = o(n^{-\frac{\rho(1+\delta_1)}{2}})$ a.s., $k=1,2$. Also, for the martingale differences $X_{l,2}\frac{U_{l,2}/m_2}{Y_{l-1,2}} - \frac{1}{|\mathbf{Y}_{l-1}|}$, we have

$$\begin{aligned}&\sum_{l=1}^{\infty}(l^{\frac{\rho(1+\delta_1)}{2}})^2 \mathsf{E}\Big[\Big(X_{l,2}\frac{U_{l,2}/m_2}{Y_{l-1,2}} - \frac{1}{|\mathbf{Y}_{l-1}|}\Big)^2\Big|\mathscr{F}_{l-1}\Big]\\ \le&\sum_{l=1}^{\infty}(l^{\frac{\rho(1+\delta_1)}{2}})^2 \frac{Y_{l-1,2}}{|\mathbf{Y}_{l-1}|}\frac{\mathsf{E}U_{l,2}^2/m_2^2}{Y_{l-1,2}^2} \le c\sum_{l=1}^{\infty}\frac{l^{\rho(1+\delta_1)}}{l^2} < \infty \ \ a.s.,\end{aligned}$$

which implies $\sum_{l=n+1}^{\infty}\big(X_{l,2}\frac{U_{l,2}/m_2}{Y_{l-1,2}} - \frac{1}{|\mathbf{Y}_{l-1}|}\big) = o(n^{-\frac{\rho(1+\delta_1)}{2}})$ a.s.. Similarly, we can show that $\sum_{l=n+1}^{\infty}\big(X_{l,1}\frac{U_{l,1}/m_2}{Y_{l-1,1}} - \frac{1}{|\mathbf{Y}_{l-1}|}\big) = o(n^{-\frac{\rho}{2}}\log n)$ a.s. It follows that

$$Q_\infty - Q_n = \sum_{l=n+1}^{\infty}\left(X_{l,1}\frac{U_{l,1}/m_1}{Y_{l-1,1}} - \frac{1}{|\mathbf{Y}_{l-1}|}\right) + o(n^{-\frac{\rho(1+\delta_1)}{2}}) \ a.s.$$

and

$$Q_\infty - Q_n = o(n^{-\frac{\rho}{2}}\log n) \ a.s. \tag{A.7}$$

Define

$$\Delta M_n = \frac{\sqrt{\rho m_2}(m_2 n)^{\rho/2} n^{\rho/2}}{\sigma_1}\left(X_{n,1}\frac{U_{n,1}/m_1}{Y_{n-1,1}} - \frac{1}{|\mathbf{Y}_{n-1}|}\right). \tag{A.8}$$

Then

$$Q_\infty - Q_n = \frac{\sigma_1}{\sqrt{\rho m_2 m_2^{\rho}}}\sum_{l=n+1}^{\infty}\frac{\Delta M_l}{l^\rho} + o(n^{-\frac{\rho(1+\delta_1)}{2}}) \ a.s. \tag{A.9}$$

and

$$\mathsf{E}[(\Delta M_n)^2|\mathscr{F}_{n-1}] = \frac{\rho m_2 (m_2 n)^\rho n^\rho}{|\mathbf{Y}_{n-1}|Y_{n-1,1}} - \frac{\rho m_2 (m_2 n)^\rho n^\rho}{\sigma_1^2|\mathbf{Y}_{n-1}|^2}.$$

Next, we first show that

$$\mathsf{E}[(\Delta M_n)^2|\mathscr{F}_{n-1}] = \frac{\rho}{\psi_\infty} n^{\rho-1}(1+o(n^{-\rho\delta_1})) \ \ a.s. \tag{A.10}$$

From (A.7) and the Taylor expansion, we conclude that

$$\frac{Y_{n,1}}{(Y_{n,2})^\rho} - \psi_\infty = e^{m_1 Q_n} - e^{m_1 Q_\infty} = o(n^{-\frac{\rho}{2}}\log n) \ \ a.s.$$

On the other hand,

$$\begin{aligned}\frac{Y_{n,2}}{m_2 n} =&1 - \frac{1}{m_1}\frac{Y_{n,1}}{n} + \sum_{k=1}^{2}\frac{\sum_{l=1}^{n} X_{l,k}(U_{l,k}/m_k - \mathsf{E}[U_{l,k}/m_k])}{n}\\ =&1 - \frac{1}{m_1}\frac{\psi_\infty (Y_{n,2})^\rho}{n} + o(n^{-\frac{\rho}{2}-1}\log n) + O(n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}})\\ =&1 - O(n^{\rho-1}) + o(n^{-\rho\delta_1}) = 1 + o(n^{-\rho\delta_1}) \ \ a.s.\end{aligned}$$

It follows that

$$\frac{|\mathbf{Y}_n|}{m_2 n} = \frac{Y_{n,2}}{m_2 n} + \frac{Y_{n,1}}{m_2 n} = 1 + o(n^{-\rho\delta_1}) \text{ and } \frac{Y_{n,1}}{(m_2 n)^\rho} - \psi_\infty = o(n^{-\rho\delta_1}) \ a.s.$$

(A.10) is verified. From (A.10), it follows that

$$\sum_{l=1}^{n}\mathsf{E}[(\Delta M_n)^2|\mathscr{F}_{n-1}] = \frac{n^\rho}{\psi_\infty}\big(1+o(n^{-\rho\delta_1})\big) \ \ a.s.$$

On the other hand, for $0 < \delta_2 < \frac{1}{2} - \frac{1}{p}$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\mathsf{E}[|\Delta M_n|^p|\mathscr{F}_{n-1}]}{(n^{\rho(1-\delta_2)})^{p/2}} \leq & C \sum_{n=1}^{\infty} \frac{n^{\rho p}}{n^{p(1-\delta_2)\rho/2}} \frac{Y_{n-1,1}}{Y_{n-1,1}^p |\mathbf{Y}_{n-1}|} \\ \leq & C \sum_{n=1}^{\infty} \frac{1}{n^{p(1-\delta_2)\rho/2}} \frac{1}{n^{1-\rho}} < \infty.\end{aligned}$$

So, similarly as in the proof of Theorem 2.1, by the Skorokhod embedding theorem, the standard motion $B(x)$, the filtration $\{\mathscr{G}_n\}$ and the stopping times $\{T_n\}$ can be constructed such that $M_n = B(T_n)$ and

$$T_n = \sum_{l=1}^{n} \mathsf{E}[(\Delta M_l)^2|\mathscr{F}_{l-1}] + o(n^{\rho(1-\delta_2)}) = \frac{1}{\psi_\infty} n^\rho + o(n^{\rho(1-\delta_1 \wedge \delta_2)}).$$

Denote $\delta_0 = \min\{(1-\rho)/\rho, 1/2 - 1/p\}$. It is remained to verify (3.1). By the Properties (b′) and the path properties of a Brownian motion, we have for any $0 < \delta < \delta_0$,

$$M_n - B\Big(\frac{n^\rho}{\psi_\infty}\Big) = o\big(n^{\frac{\rho(1-\delta)}{2}}\big) \ a.s.$$

Hence

$$\begin{aligned}\sum_{l=n+1}^{\infty} \frac{\Delta M_l}{l^\rho} &= \sum_{l=n}^{\infty} \left(\frac{1}{l^\rho} - \frac{1}{(l+1)^\rho}\right) M_l - \frac{M_n}{n^\rho} \\ &= \sum_{l=n}^{\infty} \left(\frac{1}{l^\rho} - \frac{1}{(l+1)^\rho}\right) B\Big(\frac{l^\rho}{\psi_\infty}\Big) - \frac{B(n^\rho/\psi_\infty)}{n^\rho} \\ &\quad + \sum_{l=n}^{\infty} \frac{o(l^{\rho(1-\delta)/2})}{l^{1+\rho}} + \frac{o(n^{\rho(1-\delta)/2})}{n^\rho} \\ &= \int_n^\infty \frac{\rho B(x^\rho/\psi_\infty)}{x^{1+\rho}} dx - \frac{B(n^\rho/\psi_\infty)}{n^\rho} + o(n^{-\frac{\rho(1+\delta)}{2}}) \\ &= \frac{1}{\psi_\infty} \int_{n^\rho/\psi_\infty}^\infty \frac{B(x)}{x^2} dx - \frac{B(n^\rho/\psi_\infty)}{n^\rho} + o(n^{-\frac{\rho(1+\delta)}{2}}) \\ &= \frac{1}{\psi_\infty} \int_{n^\rho/\psi_\infty}^\infty \frac{dB(x)}{x} dx + o(n^{-\frac{\rho(1+\delta)}{2}}) \ \ a.s.\end{aligned}$$

(A.6) is now proved by noting that (A.9), (A.7) and $\rho m_2 = m_1$. And hence (3.1) is verified. □

Remark A.1. *Using the truncation method as in the proof of Theorem 2.1, we can proved that (3.1) remains true under the assumption of only finite second moments if* $n^{-\frac{\rho(1+\delta)}{2}}$ *is replaced by* $n^{-\frac{\rho}{2}}(\log\log n)^{\frac{1}{2}}$. *This implies that the law of iterated logarithm (3.3) remains true when* $\mathsf{E}U_{1,1}^2 < \infty$ *and* $\mathsf{E}U_{1,2}^2 < \infty$.

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References

[1] Aletti, G., May, C. and Secchi, P. (2007). On the distribution of the limit proportion for a two-color, randomly reinforced urn with equal reinforcement distributions. *Adv. Appl. Probab.*, **39**: 690-707. MR-2357377

[2] Aletti, G., May, C. and Secchi, P. (2009). A central limit theorem, and related results, for a two-color randomly reinforced urn. *Adv. Appl. Probab.*, **41**: 829-844. MR-2571318

[3] Bai, Z. D. and Hu, F. (2005). Strong consistency and asymptotic normality for urn models. *Ann. Appl. Probab.*, **12**: 914-940.

[4] Bai, Z. D., Hu, F. and Rosenberger, W. F. (2002). Asymptotic properties of adaptive designs for clinical trials with delayed response. *Ann. Statist.*, **30**: 122-139. MR-1892658

[5] Bai, Z. D., Hu, F. and Zhang, L. X. (2002). Gaussian approximation theorems for urn models and their applications. *Ann. Appl.Probab.*, **12**: 1149-1173. MR-1936587

[6] Beggs, A. W. (2005). On the convergence of reinforcement learning. *J. Econom. Theory* , **122**(1): 1-36. MR-2131871

[7] Berti, P., Crimaldi, I., Pratelli, L. and Rigo, P. (2010) Central limit theorems for multicolor urns with dominated colors. *Stoch. Proc. Appl.*, **120**: 1473-1491. MR-2653262

[8] Berti, P., Crimaldi, I., Pratelli, L. and Rigo, P. (2011) A central limit theorem and its applications to multicolor randomly reinforced urns. *J. Appl. Probab.*, **48**(2): 527-546 MR-2840314

[9] Chauvin, B., Pouyanne, N. and Sahnoun, R. (2011). Limit distributions for large Pólya urns. *Ann. Appl. Probab.*, **21**(1): 1-32. MR-2759195

[10] Crimaldi, I. (2009). An almost sure conditional convergence result and an application to a generalized Pólya Urn. *International Mathematical Forum*, **4** (23): 1139-1156 MR-2524635

[11] Csörgő, M. and Révész, P. (1981). *Strong Approximations in Probability and Statistics*. Academic Press, New York. MR-0666546

[12] Durham, S. D., Flournoy, N. and Li, W. (1998). A sequential design for maximizing the probability of a favourable response. *Canad. J. Statist.*, **26** (3): 479-495. MR-1646698

[13] Durham, S. D. and Yu, K. F. (1990). Randomized play-the leader rules for sequential sampling from two populations. *Probability in Enginerring and Information Science*, **26** (4): 355-367.

[14] Eberlein, E. (1986). On strong invariance principles under dependence. *Ann. Probab.*, **14**: 260-270. MR-0815969

[15] Eggenberger, F. and Pólya, G. (1923). Uber die Statistik verketteter Vorgänge. *Zeitschrift Angew. Math. Mech.*, **3**: 279-289.

[16] Erev, I. and Roth, A. (1998). Predicting how people play games: reinforcement learning in experimental games with unique, mixed strategy equilibria. *Amer. Econ. Rev.*, **88**: 848-881.

[17] Hall, P. and Heyde, C. C. (1980). *Martingale Limit Theory and its Applications*. Academic Press, London. MR-0624435

[18] Hanson, D. L. and Russo, Ralph P. (1983). Some results on increments of the Wiener process with applications to lag sums of i.i.d.r.v. *Ann. Probab.*, **11**: 609-623. MR-0704547

[19] Hopkins, E. and Posch, M. (2005). Attainability of boundary points under reinforcement learning. *Games Econom. Behavior*, **53**: 110-125. MR-2173863

[20] Hu, F. and Rosenberger, W. F. (2006). *The Theory of Response-Adaptive Randomization in Clinical Trials*, John Wiley and Sons, Inc., New York. MR-2245329

[21] Hu, F. & Zhang, L. X. (2004). Asymptotic properties of doubly adaptive biased coin designs for multi-treatment clinical trials. *Ann. Statist.*, **32**: 268-301. MR-2051008

[22] Janson, S. (2004). Functional limit theorems for multitype branching processes and generalized Pólya urns. *Stochastic Process. Appl.*, **110**: 177-245. MR-2040966

[23] Janson, S. (2006). Limit theorems for triangular urn schemes. *Probab. Theory Relat. Fields*, **134**:417-452. MR-2226887

[24] Li, W., Durham, S. D. and Flournoy, N. (1996). Randomized polya urn designs. *Proceedings of the Biometric Section of the Statistical Association*: 166-170.

[25] May, C. and Flournoy, N. (2009). Asymptotics in response-adaptive designs generated by a two-color, randomly reinforced urn. *Ann. Statist.*, **37**(2): 1058-1078. MR-2502661

[26] Martin, C. F. and Ho, Y. C. (2002). Value of information in the Polya urn process. *Information Sciences*, **147**: 65-90. MR-1940746

[27] Monrad, D. and Philipp,W. (1991). Nearby variables with nearby conditional laws and a strong approximation theorem for Hilbert space valued martingales. *Probab. Theory Relat. Fields*, **88**: 381-404. MR-1100898

[28] Muliere, P., Paganoni, A. M. and Secchi, P. (2006a). A randomly reinforced urns. *J. Statisit. Plann. Inference*, **136**(6):1853-1874. MR-2255601

[29] Muliere, P., Paganoni, A. M. and Secchi, P. (2006b). Randomly reinforced urns for clinical trials with continuous responses. In *SIS-Proceedings of the XLIII Scientific Meeting*, 403- 414. Cleup, Padova.

[30] Paganoni, A. and Secchi, P. (2007). A numerical study for comparing two response-adaptive designs for continuous treatment effects. *Statstical Methods and Applications*, **16**: 321-346. MR-2413518

[31] Pólya, G. (1931). Sur quelques points de la théorie des probabilités. *Ann. Inst. Poincaré*, **1**: 117-161.

[32] Zhang, L. X. (2004). Strong approximations of martingale vectors and its applications in Markov-Chain adaptive designs. Acta Math. Appl. Sinica, English Series, **20**(2): 337–352. MR-2064011

[33] Zhang, L. X. and Hu, F. (2009). The Gaussian approximation for multi-color generalized Friedman's urn model. *Science in China, Ser. A*, **52** (6): 1305-1326. MR-2520576

[34] Zhang, L. X., Hu, F. and Cheung, S. H. (2006). Asymptotic theorems of sequential estimation-adjusted urn models for clinical trials. *Ann. Appl. Probab.*, **16**(1): 340-369 MR-2209345

[35] Zhang, L.X., Hu, F., Cheung, S.H. and Chan, W.S. (2011). Immigrated urn models – theoretical properties and applications. *Ann. Statist.*, **39**(1): 643-671. MR-2797859

[36] Zhang, L. X., Hu, F., Cheung, S. H. and Chan, W. S. (2014). Asymptotic properties of multi-color randomly reinforced Pólya urns. *Adv. Appl. Probab.*, **46**: 585-602. MR-3215547