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Large deviations for excursions of non-homogeneous Markov processes

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Abstract

In this paper, the large deviations at the trajectory level for ergodic Markov processes are studied. These processes take values in the non-negative quadrant of the two-dimensional lattice and are concentrated on step-wise functions. The rates of jumps towards the axes (downward jumps) depend on the position of the process – the higher the position, the greater the rate. The rates of jumps going in the same direction as the axes (upward jumps) are constants. Therefore the processes are ergodic. The large deviations are studied under equal scalings of both space and time. The scaled versions of the processes converge to 0. The main result is that the probabilities of excursions far from 0 tend to 0 exponentially fast with an exponent proportional to the square of the scaling parameter. The proportionality coefficient is an integral of a linear combination of path components. A rate function of the large deviation principle is calculated for continuous functions only.

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1 Introduction

There are different settings in the large deviation theory studying probabilities of rare events (see, for example, the books [3, 4, 5, 6, 7, 8]). This paper is devoted to investigations of the rare event probabilities for a specific class of ergodic Markov processes. The goal is to find the asymptotic behavior of logarithms of probabilities for excursions of the process far from equilibrium states. We apply the large deviation setting using equal contractions in time and in space. The large deviation principle in terms of paths of the process is obtained.

The basic random object studied is a continuous-time Markov ergodic process ξ with state space $\mathbb{Z}_+^2 := \{(z_1, z_2) : z_1 \geq 0, z_2 \geq 0\}$. Paths of ξ are piecewise constant functions. The jumps belong to the following set

$$\mathcal{Y} = \{(1,0), (0,1), (-1,0), (0,-1), (-1,-1)\}.$$

The probabilities of the jumps are such that they do not take the process outside of $\mathbb{Z}_+^2$. The intensities of the jumps depend on the value of ξ at the moment before the jump. If at a moment t the process value is equal to $\xi(t) = (z_1, z_2)$, then any increase (upward

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jump) of at least one of the components of (z_1, z_2) happens with a constant intensity. However, any decrease (downward jump) of at least one of the components of (z_1, z_2) happens with an intensity proportional to this co-ordinate. This property implies the ergodicity of the process ξ.

This study was inspired by the work [9], where ergodic properties of more complicated processes were studied. The goal of the authors of [9] was to describe market dynamics. Our goal is focused on some peculiarities of the large deviations for similar models and our version of the model is hardly proper for market investigations.

We consider the large deviations for the sequence $\xi_T(t) = (\frac{\xi(tT)}{T})_{T>0}$ of the processes on $t \in [0,1]$ with $\xi(0) = (0,0)$. The large deviation principle for ξ is established on a set of càdlàg functions X with a finite number of jumps, which includes all typical paths of ξ. The rate function is finite for a set F of continuous functions on $[0,1]$ such that any $\underline{f} \in F$ has positive co-ordinates except at $t = 0$, where $\underline{f}(0) = (0,0)$. When the processes ξ_T are localized in a small neighborhood of some function $\underline{f} \in F$, we say that the process ξ has an excursion far from equilibrium. We find that the rate function of $\underline{f} = (f_1, f_2)$ has the following integral form

$$I(\underline{f}) = \int_0^1 \big(c_1 f_1(t) + c_2 f_2(t) + c_3 \min\{f_1(t), f_2(t)\}\big) dt, \tag{1.1}$$

where constants c_1, c_2 and c_3 are parameters defining the process ξ (see exact definitions in section 2.2). A local principle of the large deviations proved in this paper implies that the probability of a long excursion in a small neighborhood $U(\underline{f})$ of a function $\underline{f} \in F$ is of order

$$e^{-T^2 I(\underline{f})}.$$

We use in this paper the uniform topology in F.

The proof is based on a comparison of the studied Markov process and a process with independent increments. A density of the Markov process with the respect to the process with independent increments (see (2.12)) gives the main contribution to the asymptotic

$$\ln \Pr\big(\xi_T(\cdot) \in U(f)\big) \sim -T^2 I(f).$$

Only the expression (2.13) of jump intensities in the process density creates an asymptotic of order T^2. The other parts of the density, (2.14) and (2.12), have asymptotics of order T.

The large deviation principle we obtained demonstrates some unusual features in contrast with known results for large deviations on processes in terms of paths. One of the features is that the rate function (1.1) does not depend on derivatives of the paths of the process. As a consequence, sharp peaks of a path make negligible contributions to the rate functions. Another peculiarity of this approach is that Cramèr transformation is not used. Although the method is applied to a very specific example of the Markov process, we believe that the method represented in this paper works in more general cases.

2 Results.

2.1 Notation.

Let $\xi(t) = (\xi_1(t),\ \xi_2(t)),\ t \in [0,\infty)$ be a Markov process with state space $\mathbb{Z}_+^2 := \{(z_1, z_2) :\ z_1 \ge 0,\ z_2 \ge 0\}$. The evolution of the process can be described in the following way. Let a state of the process at a moment $t \ge 0$ be $\xi(t) = \underline{z} = (z_1, z_2) \in \mathbb{Z}_+^2$. The state is not changed during a time $\tau_{\underline{z}}$, where $\tau_{\underline{z}}$ is a random variable distributed exponentially with a parameter $h(\underline{z})$. At the moment $t+\tau_{\underline{z}}$ the value of the process becomes equal to $\underline{z}+\underline{y}$, where $\underline{y}$ belongs to

$$\mathcal{Y}=\{(1,0),(0,1),(-1,0),(0,-1),(-1,-1)\}. \tag{2.1}$$

The intensity of the jumps is a sum

$$h(\underline{z}):=\lambda_{\underline{z}}(1,0)+\lambda_{\underline{z}}(0,1)+\lambda_{\underline{z}}(-1,0)+\lambda_{\underline{z}}(0,-1)+\lambda_{\underline{z}}(-1,-1), \tag{2.2}$$

where

$$\begin{aligned} &\lambda_{\underline{z}}(1,0):=\lambda(1,0), \quad \lambda_{\underline{z}}(0,1):=\lambda(0,1),\\ &\lambda_{\underline{z}}(-1,0):=z_1\lambda(-1,0), \quad \lambda_{\underline{z}}(0,-1):=z_2\lambda(0,-1),\\ &\lambda_{\underline{z}}(-1,-1):=\min\{z_1,z_2\}\lambda(-1,-1), \end{aligned} \tag{2.3}$$

and the constants $\lambda(\underline{y})$ at $\underline{y}\in\mathcal{Y}$ are positive. The probability of the jump $\underline{y}$ is

$$p_{\underline{z}}(\underline{y}):=\frac{\lambda_{\underline{z}}(\underline{y})}{h(\underline{z})}, \quad \underline{y}=(y_1,y_2)\in\mathcal{Y}. \tag{2.4}$$

2.2 The local large deviation principle.

In this section we study the local deviation principle for the measures (P_T) which are the distributions of the processes $(\xi_T(t)=\frac{1}{T}\xi(tT)),\ t\in[0,1]$. The support of the processes ξ_T is a subset of the set X of non-negative càdlàg functions

$$\underline{x}:\ [0,1]\to\mathbb{R}_+^2=\{(y_1,y_2)\in\mathbb{R}^2:\ y_1\geq 0, y_2\geq 0\},$$

which are right-continuous and have left limits everywhere, having finite numbers of jumps on $[0,1]$ and such that $\underline{x}(0)=(0,0)$ (definition of the càdlàg functions see, for example, in [1]). We introduce a *uniform* topology on X, which, in this case, is determined by the distance $d(\underline{x}_1,\underline{x}_2)$ between two functions $\underline{x}_1,\underline{x}_2\in X$ as follows

$$d(\underline{x}_1,\underline{x}_2)=\sup_{t\in[0,1]}\|\underline{x}_1(t)-\underline{x}_2(t)\|, \tag{2.5}$$

where $\|\cdot\|$ means the usual Euclidean norm in $\mathbb{R}^2$.

There is a weak convergence $P_T\Rightarrow\delta_{\underline{x}_0}$, where $\underline{x}_0(t)\equiv 0,\ t\in[0,1]$. Studying excursions far from $\underline{x}_0$ we consider the set $F\subset X$ of continuous functions $\underline{f}(t)=(f_1(t),f_2(t))$ satisfying the following properties:

F_1 $\underline{f}(0)=(0,0)$,

F_2 $f_1(t)>0$ and $f_2(t)>0$ for any $t>0$.

We have found the rate function for this class $F\subset X$ of continuous functions satisfying the conditions F_1 and F_2.

For brevity we shall use the notations $c_0=\lambda(1,0)+\lambda(0,1),\ c_1=\lambda(-1,0),\ c_2=\lambda(0,-1),\ c_3=\lambda(-1,-1)$. Thus we rewrite (2.2) as (see also (2.3))

$$h(\underline{z})\equiv h(z_1,z_2)=c_0+c_1z_1+c_2z_2+c_3\min\{z_1,z_2\}. \tag{2.6}$$

On the set X we define the following functional $I:\ X\to\mathbb{R}\cup\{\infty\}$

$$I(\underline{x}):=\begin{cases}\int_0^1\big(c_1x_1(t)+c_2x_2(t)+c_3\min\{x_1(t),x_2(t)\}\big)dt, & \text{if } \underline{x}\in F,\\ \infty, & \text{if } \underline{x}\notin F.\end{cases} \tag{2.7}$$

$I(\underline{x})$ is finite for all bounded continuous functions $\underline{x}\in F$. In the next theorem we prove the local large deviation principle with rate function $I(\underline{x})$.

Theorem 2.1. *For any* $\underline{f} \in F$

$$\lim_{\varepsilon\to 0}\lim_{T\to\infty}\frac{1}{T^2}\ln \mathbf{P}\big(\xi_T \in U_\varepsilon(\underline{f})\big) = -I(\underline{f}), \tag{2.8}$$

where (see (2.5))

$$U_\varepsilon(\underline{f}) = \{\underline{g} \in X :\ d(\underline{f},\underline{g}) < \varepsilon\}. \tag{2.9}$$

Proof of Theorem 2.1. Upper bound. We have to show that

$$L_+ := \limsup_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{T^2}\ln \mathbf{P}\big(\xi_T \in U_\varepsilon(\underline{f})\big) \le -I(\underline{f}). \tag{2.10}$$

In order to show this, consider a Markov process $\zeta(t) = (\zeta_1(t), \zeta_2(t)),\ t \in [0,T]$, with state space $\mathbb{Z}^2$ and its intensity of jumps equal to 1. The process $\zeta(t)$ is homogenous in time. At the moment of a jump the process ζ changes its value from $\underline{z} \in \mathbb{Z}^2$ to $\underline{z}+\underline{y}$ with uniform probabilities $1/5$ for $\underline{y} \in \mathcal{Y}$. This means that the process ζ is homogeneous in space, as well. The process ζ may take values outside $\mathbb{Z}^2_+$, moreover the process leaves $\mathbb{Z}^2_+$ with probability 1.

Let X_T be the set of all trajectories of the process ξ on the time interval $[0,T]$. The distribution of the process ξ is absolutely continuous with respect to ζ with density

$$\begin{aligned}
\mathcal{P}\left(\underline{u}(\cdot)\right) &= 5^{N_T(\underline{u})}\prod_{i=0}^{N_T(\underline{u})-1} h(\underline{u}(t_i))e^{-(h(\underline{u}(t_i))-1)\tau_{i+1}}p_{\underline{u}(t_i)}(\underline{u}(t_{i+1})-\underline{u}(t_i))\ \times \\
&\quad h(\underline{u}(t_{N_T(\underline{u})}))e^{-(h(\underline{u}(t_{N_T(\underline{u})}))-1)\tau_{N_T(\underline{u})+1}} \\
&= 5^{N_T(\underline{u})}\prod_{i=0}^{N_T(\underline{u})-1} e^{-(h(\underline{u}(t_i))-1)\tau_{i+1}}\lambda_{\underline{u}(t_i)}(\underline{u}(t_{i+1})-\underline{u}(t_i))\ \times \\
&\quad h(\underline{u}(t_{N_T(\underline{u})}))e^{-(h(\underline{u}(t_{N_T(\underline{u})}))-1)\tau_{N_T(\underline{u})+1}}
\end{aligned} \tag{2.11}$$

where $\underline{u}(\cdot) \in X_T$ with $N_T(\underline{u})$ jump moments $0 = t_0 < t_1 < \cdots < t_{N_T(\underline{u})} < t_{N_T(\underline{u})+1} = T$. For any $\underline{u}(\cdot) \notin X_T$, $\mathcal{P}\left(\underline{u}(\cdot)\right) = 0$. Hence

$$\mathbf{P}(\xi(\cdot) \in E) = e^T\mathbf{E}(e^{-A_T(\zeta)}e^{B_T(\zeta)+N_T(\zeta)\ln 5};\ \zeta(\cdot) \in E) \tag{2.12}$$

for any measurable set $E \subseteq X_T$, where for $\underline{u} \in E$

$$A_T(\underline{u}) := \sum_{i=0}^{N_T(\underline{u})} h(\underline{u}(t_i))\tau_{i+1} = \int_0^T h(\underline{u}(t))dt, \tag{2.13}$$

$$B_T(\underline{u}) := \sum_{i=0}^{N_T(\underline{u})-1} \ln\big(\lambda_{\underline{u}(t_i)}(\underline{u}(t_{i+1})-\underline{u}(t_i))\big) + \ln h\big(\underline{u}(t_{N_T(\underline{u})})\big). \tag{2.14}$$

We study the asymptotic behavior of the logarithm of the probability $\mathbf{P}\big(\xi_T(\cdot) \in U_\varepsilon(\underline{f})\big)$ for any $\underline{f} \in F$ using (2.12). The main contribution to this asymptotic comes from A_T. To prove this we consider the scaled processes $\zeta_T(s) = \frac{\zeta(sT)}{T}, s \in [0,1]$. Let $\underline{x}(s) = \frac{\underline{u}(sT)}{T}$ for $\underline{u} \in X_T$, then

$$\begin{aligned}
A_T(\underline{x}) := A_T(\underline{u}) &= T^2\int_0^1 \left[\frac{c_0}{T} + c_1\frac{u_1(sT)}{T} + c_2\frac{u_2(sT)}{T} + c_3\min\Big\{\frac{u_1(sT)}{T},\frac{u_2(sT)}{T}\Big\}\right]\mathrm{d}s \\
&= T^2\int_0^1 \left[\frac{c_0}{T} + c_1x_1(s) + c_2x_2(s) + c_3\min\{x_1(s),x_2(s)\}\right]\mathrm{d}s \\
&= T^2\left[\frac{c_0}{T} + I(\underline{x})\right].
\end{aligned}$$

Then for any ε there exists δ such that

$$T^2 I(\underline{f})(1-\delta) \le A_T(\underline{x}) \le T^2 I(\underline{f})(1+\delta) \tag{2.15}$$

for any $\underline{x} \in U_\varepsilon(\underline{f})$. Hence

$$L_+ \le -I(\underline{f}) + \limsup_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{T^2} \ln \mathbf{E}\big(e^{B_T(\zeta)+N_T(\zeta)\ln 5};\ \ \zeta_T(\cdot) \in U_\varepsilon(\underline{f})\big). \tag{2.16}$$

Next we show that the second term in (2.16) is equal to 0.

Let $\underline{y} \in U_\varepsilon(\underline{f})$ and $K_+ = K_+(\underline{y})$ be the number of jumps of $\underline{y}(\cdot) = (y_1(\cdot), y_2(\cdot))$ on the time interval $[0,1]$, such that the values of either y_1 or y_2 are increasing at the jump moments. Recall that the path $\underline{y}$ can increase by the increments $(1,0)$ or $(0,1)$.

Let $\varepsilon > 0$ be such that $f_i(1) - \varepsilon > 0, i = 1,2$. Then $y_i(1) > 0$, since $\underline{y} \in U_\varepsilon(\underline{f})$. Thus

$$K_+ - K_- > 0,$$

where K_- is the number of jumps on the time interval $[0,1]$, when the values of either y_1 or y_2 or both are decreasing at the jump moments. Note that $N_T(\underline{y}) = K_+ + K_-$, and hence

$$K_+ > \frac{1}{2} N_T(\underline{y}). \tag{2.17}$$

The next step of the proof is based on the following lemma.

Lemma 2.2. *For any $\underline{f} \in F$ there exist positive constants R_1 and R_2, which depend on $\underline{f}$, such that*

$$e^{C_T} := \mathbf{E}\big(e^{B_T(\zeta)+N_T(\zeta)\ln 5};\ \zeta_T(\cdot) \in U_\varepsilon(\underline{f})\big) \le \mathbf{E}\exp\Big\{\frac{N_T(\zeta)}{2}(\ln T + R_1) + \frac{1}{2}\ln(R_2 T)\Big\} \tag{2.18}$$

holds for small ε (see (2.16)).

Proof. Let $\underline{x}$ be some scaled trajectory of unscaled path $\underline{u} \in X_T$, $\underline{x}(s) = u(sT)/T, s \in [0,1]$ and $\{\tilde{s}_i\} \subset \{s_i\} = \{t_i/T\}$ be a subset of moments when the values x_1 or x_2 or both are decreasing. Remember that the number of such jumps is K_-. Thus (see (2.14) for the definition of B_T):

$$\begin{aligned}
B_T(\underline{u}) \quad &:= \quad B_T(\underline{x}) = \sum_{i=0}^{N_T(\underline{x})-1} \ln\big(\lambda_{T\underline{x}(s_i)}\left(T(\underline{x}(s_{i+1}) - \underline{x}(s_i))\right)\big) + \ln\left(h(T\underline{x}(t_{N_T(\underline{x})}))\right) \\
&\le \quad K_+ \ln c_0 + (K_- + 1)\ln\Big(T\big(c_0 + (\max c_i)\big(\sup_{t\in[0,1]} \max\{f_1(t), f_2(t)\} + \varepsilon\big)\big)\Big) \\
&\le \quad \frac{1}{2}(N_T(\underline{x}) + 1)(\ln T + C),
\end{aligned} \tag{2.19}$$

for some constant C that depends on $\underline{f}$. Choosing $R_1 = C + 2\ln 5,\ R_2 = e^C$ we obtain the proof of the lemma. ☐

To finish the proof of

$$\limsup_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{T^2} C_T = 0,$$

note that the random variable $N_S(\zeta)$ has Poisson distribution with a parameter S. Hence

$$\mathbf{E} e^{\theta N_S(\zeta)} = e^{S(e^\theta - 1)}.$$

Using (2.18) we obtain

$$e^{C_T} \le e^{T(e^{\frac{1}{2}(\ln T + R_1)} - 1)} R_2 T \le e^{T^{3/2} e^{\frac{R_1}{2}}} R_2 T,$$

which implies that

$$\limsup_{T\to\infty} \frac{1}{T^2} C_T \le \lim_{T\to\infty} \frac{1}{T^2} \left(T^{3/2} e^{\frac{R_1}{2}} + \ln(R_2 T) \right) = 0. \tag{2.20}$$

Therefore the proof of the upper bound (2.10) is completed.
Lower Bound. We have to prove the inequality

$$L_- := \liminf_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{T^2} \ln \mathbf{P}\big(\xi_T \in U_\varepsilon(\underline{f})\big) \ge -I(\underline{f}). \tag{2.21}$$

The probability of the event $\mathcal{U}(\underline{f}) := (\xi_T \in U_\varepsilon(\underline{f}))$ can be bounded below by the probability of a more restricted event $\mathcal{U}(\underline{f}, C) := (\xi_T \in U_\varepsilon(\underline{f}),\ N_T(\xi) \le CT)$. The value of the constant C depends on $\underline{f}$. Using the representation of the distribution of ξ in terms of the process ζ (see (2.12)), the inequalities (2.15) and that $B_T(\underline{x}) > N_T(\underline{x}) \ln(\tilde{c})$, where $\tilde{c} := \min c_i$, we obtain the lower bound

$$\begin{aligned} &\liminf_{T\to\infty} \frac{1}{T^2} \ln \mathbf{P}\big(\xi_T \in U_\varepsilon(\underline{f})\big) \\ &\ge -I(\underline{f})(1+\delta) + \liminf_{T\to\infty} \frac{1}{T^2} \ln \mathbf{E}\big(e^{N_T(\zeta)\ln(5\tilde{c})}; \zeta_T \in U_\varepsilon(\underline{f}),\ N_T(\zeta) \le CT\big). \end{aligned} \tag{2.22}$$

If $\ln(5\tilde{c}) > 0$, then $e^{N_T(\zeta)\ln(5\tilde{c})} > 1$ and the expectation in (2.22) is bounded below by the probability $\mathbf{P}(\mathcal{U}(\underline{f}, C))$. On the other hand, if $\ln(5\tilde{c}) < 0$, then the expectation is bounded below by $e^{CT\ln(5\tilde{c})}\mathbf{P}(\mathcal{U}(\underline{f}, C))$.

Recall that on the event $\mathcal{U}(\underline{f}, C)$, the values of the process $\zeta_T(t)$ are non-negative. The lower bound on $\ln \mathbf{P}(\mathcal{U}(\underline{f}, C))$ follows from the recent result in [2], (Theorems 3.1 and 3.3). Namely, there exists a constant $J > 0$ such that

$$\liminf_{T\to\infty} \frac{1}{T} \ln \mathbf{P}\big(\zeta_T \in U_\varepsilon(\underline{f}),\ N_T(\zeta_T) \le CT\big) \ge J > -\infty.$$

Thereby

$$\liminf_{T\to\infty} \frac{1}{T^2} \ln \mathbf{P}\big(\zeta_T \in U_\varepsilon(\underline{f}),\ N_T(\zeta_T) \le CT\big) = 0.$$

Even though the formula for the rate function (2.7) can be applied for discontinuous functions, in fact the rate function is infinite for such functions. That happens because $\mathbf{P}(\xi_T \in U_\varepsilon(\underline{x})) = 0$ in the uniform topology for any discontinuous function $\underline{x}$ if ε is small enough. □

Remark 2.3. *In [2], the large deviation principle is proved for real valued processes with independent increments. The result of Theorems 3.1 and 3.3 from [2] can be easily extended to finite-dimensional cases.*

2.3 A version for “integral" large deviation principle.

For any continuous function $\underline{f} = (f_1, f_2) \in F$ and any positive ε and M, consider the following sets:

$$\begin{aligned} B_{\underline{f},\varepsilon,M} &= \{\underline{x} = (x_1, x_2) \in X :\ f_i(t) - \varepsilon \le x_i(t) \le M,\ i = 1, 2,\ t \in [0, 1]\}, \\ B_{\underline{f},M} &= \{\underline{x} = (x_1, x_2) \in X :\ f_i(t) \le x_i(t) \le M,\ i = 1, 2,\ t \in [0, 1]\}. \end{aligned}$$

We will call them strips.

Theorem 2.4. *For any $\underline{f} \in F$ and any $M > \sup_{t\in[0,1]} \max\{f_1(t), f_2(t)\}$*

$$\lim_{\varepsilon\to 0} \lim_{T\to\infty} \frac{1}{T^2} \ln \mathbf{P}\left(\xi_T(\cdot) \in B_{\underline{f},\varepsilon,M}\right) = -\inf_{\underline{g}\in F\cap B_{\underline{f},M}} I(\underline{g}) = -I(\underline{f}). \tag{2.23}$$

Proof of Theorem 2.4. The upper bound follows from representation (2.12): for any ε there exists δ such that

$$\frac{1}{T^2}\ln \mathbf{P}(\xi_T(\cdot)\in B_{\underline{f},\varepsilon,M}) \le -\inf_{\underline{g}\in F\cap B_{\underline{f},M}} I(\underline{g})(1-\delta)+\sup_{\underline{g}\in F\cap B_{\underline{f},M}}\frac{1}{T^2}C_T,$$

see Lemma 2.2 for the definition of C_T. The proof of the relation

$$\limsup_{T\to\infty}\sup_{\underline{g}\in F\cap B_{\underline{f},M}}\frac{1}{T^2}C_T = o(1) \text{ as } \varepsilon\to 0$$

basically repeats the arguments of Section 2.2 replacing $\sup_{t\in[0,1]}\max\{f_1(t),f_2(t)\}$ by M in (2.19). This modification does not affect the principal inequality (2.20). This proves the upper bound.

The lower bound becomes obvious using the inequality

$$\mathbf{P}\big(\xi_T(\cdot)\in B_{\underline{f},\varepsilon,M}\big)\ge \mathbf{P}\big(\xi_T(\cdot)\in U_\varepsilon(\underline{f})\big),$$

and after that the usage of Theorem 2.1 completes the proof of the theorem. □

Remark 2.5. *Theorem 2.4 holds also if, in the definition of the strip, we substitute the upper bound M by a bound $(M_1,M_2)+\underline{g}$, where $\underline{g}$ is any continuous function on [0,1] with $\underline{g}(0)=(0,0)$ and M_1,M_2 are some positive constants. The lower bound is defined by a function $\underline{f}\in F$ such that*

$$\sup_{t\in[0,1]}\big(M_i+g_i(t)-f_i(t)\big)>0.$$

for any $i=1,2$.

We have not proven the large deviation principle in its complete form. There are some reasons for this. First, the rate function (2.7) is not compact. Second, in the topology we considered, exponential tightness does not hold. Moreover, the space X is not complete and it is not separable. Thus we stated the large deviation for some special sets, which we called strips. It seems that strips can be required in applications.

3 Conclusion

Notice that derivatives of $\underline{f}$ are not included in the expression for $I(\underline{f})$ (1.1). Such form of the rate function seems paradoxical. Indeed, let a continuous function $g_1 : [0,1]\to\mathbb{R}_+$ have a form of a high narrow peak such that $\int_0^1 g_1(t)\mathrm{d}t=\varepsilon_0$ is small, and let $\underline{g}=(g_1,0)$. The difference of the rate functions $I(\underline{f}+\underline{g})$ and $I(\underline{f})$, for $\underline{f}\in F$, is small and equal to $c_1\varepsilon_0$, but $\sup_t\{g_1(t)-f_1(t)\}$ can be very large. An explanation of this paradox is that the probability that the process ξ_T belongs to a “neighborhood" of $\underline{g}$ is of order

$$e^{-T\ln(T)C}, \tag{3.1}$$

where C is a constant that depends on $\underline{g}$. The asymptotic (3.1) is not proved in this paper. The word “neighborhood" is in quotation marks because (3.1) has to be proved in different settings (it will be done in another paper). This means that the probability of ξ_T being away from zero for a long time is much smaller than the probability of a high ejection during a short period.

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