On a dyadic approximation of predictable processes of finite variation

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Abstract

We show that any càdlàg predictable process of finite variation is an a.s. limit of *elementary* predictable processes; it follows that predictable stopping times can be approximated 'from below' by predictable stopping times which take finitely many values. We then obtain as corollaries two classical theorems: predictable stopping times are announceable, and an increasing process is predictable iff it is natural.

We recall that a process $S = (S_t)_t$ is called *of class D* if the family of random variables $(S_\tau)_\tau$, where τ ranges through all stopping times, is uniformly integrable. If $S = (S_t)_{t\in[0,1]}$ is a submartingale of class D, then it has a unique Doob-Meyer decomposition $S = M + A$, where M is a uniformly integrable martingale and A is a predictable increasing integrable process starting from zero, called the compensator of S. One can give constructive proofs of the existence of the Doob-Meyer decomposition by taking limits of the discrete time Doob decompositions $(M_t^n + A_t^n)_{t\in\mathcal{D}_n}$ of the sampled process $(S_t)_{t\in\mathcal{D}_n}$ relative to refining partitions $(\mathcal{D}_n)_n$. Indeed in [9] the compensator is obtained as the limit of the A^n's in the $\sigma(L^1, L^\infty)$-topology; more simply, even if in general these discrete time approximations do not converge in probability to A for all t, one can always build some forward convex combinations $\mathcal{A}^n$ of the A^n such that $\limsup_n \mathcal{A}_t^n = A_t$ a.s. for all t, as was shown in [1]. It follows that $\mathcal{A}_t^n \to A_t$ a.s. as $n \to \infty$ along a subsequence which a priori depends on both t and ω; it is then natural to ask whether there is such a subsequence $(n_k)_k$ which works simultaneously for all (t, ω), so that $\mathcal{A}_t^{n_k}$ converges to A_t a.s. for all t as $k \to \infty$; in this paper we show that this is indeed the case, in particular proving that any predictable increasing process A is a pointwise limit of predictable increasing processes $\mathcal{A}^n$ of the form

$$\mathcal{A}^n = \mathbb{1}_{\{0\}} A_0 + \sum_{k=1}^{2^n} \mathbb{1}_{(\frac{k-1}{2^n}, \frac{k}{2^n}]} \mathcal{A}_{\frac{k}{2^n}}^n \ .$$

From this 'dyadic' approximation, by time change it easily follows that predictable stopping times can be approximated 'from below' by predictable stopping times which take finitely many values; this being a predictable analogue of the simple fact that any stopping time is the limit of a decreasing sequence of stopping times, each taking values in a finite set.

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We notice how these results can be used to provide an alternate derivation of the following well known theorems: predictable stopping times are announceable, and an increasing process is predictable iff it is natural (we show this passing to the limit the analogous discrete time statement).

To prove the convergence results of [1] and our main theorem, use is made of the existence of a sequence of stopping times which exhaust the jumps of a càdlàg adapted process. We prove this classic result without using the deep debut and section theorems, by showing explicitly that the jumps times, and the 'first-approach time', of a càdlàg (predictable) adapted process are (predictable) stopping times; our proofs are elementary, and hold even if the filtration does not satisfy the usual conditions. Moreover, we show how a simple variant of this result can be used to characterize the continuity of local martingales and of the compensator of special semimartingales.

The rest of the paper is organized as follows: in Section 1 we introduce some definitions and conventions, and we state our results on the approximation of predictable processes of finite variation and of predictable stopping times, and we prove the second one. In Section 2 we discuss the equivalent characterizations of predictable stopping times. In Section 3 we state and prove some classical results on predictable stopping times. In Section 4 we prove our previously-stated main result on the convergence of the dyadic approximations. In Section 5 we show that an increasing process is predictable iff it is natural. Finally, in Section 6 we derive some corollaries about special semimartingales.

1   The main results

In this Section, after introducing some definitions and conventions, we state our results on the approximation of predictable processes of integrable variation and of predictable stopping times, and we prove the second one.

In this article we will consider a fixed filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and we assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,\infty]}$ satisfies the usual conditions of right continuity and saturatedness. By convention, the $\inf$ of an empty set will be ∞. Inequalities are meant in the weak sense, so $t_n \uparrow t$ means $t_n \leq t_{n+1} \leq t$ and $t_n \to t$, 'increasing' means 'non-decreasing' etc. We will say that a process A is increasing (resp. of finite variation) if it is adapted and $(A_t(\omega))_t$ is increasing (resp. of finite variation) for $\mathbb{P}$ a.e. ω. A process X is called integrable if $\sup_t |X_t| \in L^1(\mathbb{P})$. A property of a process S (integrability, martingality, boundedness etc.) is said to hold locally if there is a sequence of stopping times $\tau_n \uparrow \infty$ s.t., for each $n \in \mathbb{N}$, $S^{\tau_n} \mathbb{1}_{\{\tau_n > 0\}}$ satisfies the property. All local martingales we will deal with are assumed to be càdlàg. Given a càdlàg process X, we set $X_{0-} := X_0$ and $X_- := (X_{t-})_t$, and define $\Delta X_t := X_t - X_{t-}$ for $t \in [0,\infty)$ and $\Delta X_t = 0$ for $t = \infty$. We will call a process predictable if it is measurable with respect to the sigma algebra generated on $[0,\infty) \times \Omega$ by the càg adapted processes; a stopping time τ will be called predictable if $\mathbb{1}_{[\tau,\infty)}$ is predictable. We denote with $\mathcal{D}_n$ the set $\{k/2^n : k = 0, \ldots, 2^n\}$ of dyadics of order n in $[0,1]$, and with $\mathcal{D} = \cup_{n \in \mathbb{N}} \mathcal{D}^n$ the set of all dyadics in $[0,1]$.

To state and prove our main theorem we need the following non-standard definitions: we will say that a process B is $\mathcal{D}_n$-*predictable* if it is of the form

$$B = \mathbb{1}_{\{0\}} B_0 + \sum_{s \in \mathcal{D}_n \setminus \{0\}} \mathbb{1}_{(s-2^{-n}, s]} B_s \, , \tag{1.1}$$

where B_0 is $\mathcal{F}_0$-measurable and B_s is $\mathcal{F}_{s-2^{-n}}$-measurable for every $s \in \mathcal{D}_n \setminus \{0\}$; given $a \in (0,1]$, we define $\mathcal{D}_k(a) := \max\{s \in \mathcal{D}_k : s < a\}$.

Theorem 1.1. *If $A = (A_t)_{t \in [0,1]}$ is a càdlàg predictable process with finite variation,*

there exist a subsequence $(N_n)_n$ of $(N)_{N \in \mathbb{N}}$ and, for each $n \in \mathbb{N}$, a $\mathcal{D}_{N_n}$-predictable process $\mathcal{A}^n$ such that $\exists \lim_n \mathcal{A}_t^n = A_t$ a.s. for all $t \in [0,1]$ and $\mathcal{A}_0^n = A_0$. If A is increasing then each $\mathcal{A}^n$ can be chosen to be increasing, and if A has integrable variation then $(\mathcal{A}^n)_n$ can be chosen so that $|var(\mathcal{A}^n)_1| \leq h$ for all $n \in \mathbb{N}$ for some $h \in L^1(\mathbb{P})$.

Because of the relationship between increasing adapted processes and time-changes, it is now easy to prove the following.

Theorem 1.2. *If τ is a predictable stopping time, one can construct for each $n \in \mathbb{N}$ a predictable stopping time σ_n with values in a finite set and such that $\sigma_n \to \tau$, $\sigma_n = 0$ on $\{\tau = 0\}$ and, if $\omega \in \{\tau > 0\}$, there exists $n_0(\omega)$ s.t. $\sigma_n(\omega) < \tau(\omega)$ for all $n \geq n_0(\omega)$.*

Proof. As $[0,1]$ is homeomorphic to $[0,\infty]$, we can assume w.l.o.g. that τ has values in $[0,1]$. Apply Theorem 1.1 to $A := \mathbb{1}_{[\tau,1]}$ to obtain N_n and increasing $\mathcal{A}^n$. Define $\sigma_n := \inf\{t \in [0,1] : \mathcal{A}_t^n \geq 1/2\} \wedge 1$, and notice that since $\mathcal{A}^n$ is $\mathcal{D}_{N_n}$-predictable, trivially σ_n is a predictable stopping time with values in $\mathcal{D}_{N_n}$, and $\sigma_n = 0$ on $\{\tau = 0\}$ since $\mathcal{A}_0^n = A_0$. Since $\lim_n \mathcal{A}_\tau^n = A_\tau = 1$ for $\mathbb{P}$ a.e. ω, there exists $n_0 = n_0(\omega)$ s.t. $\mathcal{A}_\tau^n > 1/2$ for all $n \geq n_0$; so on $\{\tau > 0\}$, since $\mathcal{A}^n$ is constant on the interval $(\mathcal{D}_n(\tau), \mathcal{D}_n(\tau) + 2^{-n}]$ which contains τ, necessarily $\sigma_n \leq \mathcal{D}_{N_n}(\tau) < \tau$ holds for all $n \geq n_0$.

Moreover if $\varepsilon > 0$, $\lim_n \mathcal{A}_{\tau-\varepsilon}^n = A_{\tau-\varepsilon} = 0$ a.s. on $\{\tau - \varepsilon > 0\}$, and so there exists $n_1 = n_1(\omega)$ s.t. $\mathcal{A}_{\tau-\varepsilon}^n \leq 1/4$ for all $n \geq n_1$; it follows that a.s. $\liminf_n \sigma_n \geq \tau - \varepsilon$, and so a.s. $\tau \leq \liminf_n \sigma_n \leq \limsup_n \sigma_n \leq \tau$ on $\{\tau > 0\}$. Now just re-define σ_n as $\max\{0, \tau - 1/n\}$ on the null set where either $\lim_k \sigma_k = \tau$ fails or $\sigma_n \geq \tau > 0$. $\square$

Notice that in the previous proof the re-defined σ_n's are still predictable, because any measurable process indistinguishable from zero is predictable when the filtration satisfies the usual conditions (see [10, Lemma 13.8]).

2 Predictable, fair and announceable stopping times

In this section we notice that Theorem 1.2 immediately implies that predictable stopping times are announceable, and then discuss other proofs of this important result which are found in the literature.

If τ_n is an increasing sequence of stopping times converging to τ and such that $\tau_n < \tau$ on $\{\tau > 0\}$ for all n, we will say that τ_n *announces* τ; a stopping time τ for which such an announcing sequence exists is called *announceable*. Trivially announceable stopping times are predictable: if τ_n announces τ, the process $\mathbb{1}_{[\tau,\infty)}$ is the pointwise limit of the càg adapted processes $\mathbb{1}_{\{0\}}\mathbb{1}_{\{\tau_n=0\}} + \mathbb{1}_{(\tau_n,\infty)}$; we now show that the opposite holds too, and can even be strengthened.

Corollary 2.1. *Any predictable stopping time can be announced by a sequence of predictable stopping times.*

Proof. If $(\sigma_n)_n$ are as in Theorem 1.2 then $\inf_{k \geq n} \sigma_k$ is attained, and so the increasing sequence of stopping times $\tau_n := \inf_{k \geq n} \sigma_k$ satisfies $\mathbb{1}_{[\tau_n,\infty)} = \inf_{k \geq n} \mathbb{1}_{[\sigma_k,\infty)}$; thus, each τ_n is a predictable stopping time. Since trivially $\lim_n \tau_n = \tau$, $\tau_n < \tau$ on $\{\tau > 0\}$ and $\tau_n = 0$ otherwise, the thesis follows. $\square$

Notice that one could alternatively first prove Corollary 2.1, and then easily derive from it Theorem 1.2. Indeed, let $(\tau_n)_n$ be predictable stopping times announcing τ and $(\tau_n^k)_k$ be a decreasing sequence of predictable stopping times, each taking values in a finite set, s.t. $\lim_k \tau_n^k = \tau_n$ and $\tau_n^k = 0$ on $\{\tau_n = 0\}$. Then, if $(k_n)_n$ is a subsequence s.t. $\mathbb{P}(\tau_n^{k_n} \geq \tau > 0) < 1/2^n$, Theorem 1.2 follows taking first $\sigma_n := \tau_n^{k_n}$, and then re-defining σ_n as $\max\{0, \tau - 1/n\}$ on the null set where $\tau_m^{k_m} \geq \tau > 0$ happens for infinitely many m's.

Corollary 2.1 is typically proved using yet another useful condition equivalent to being announceable; following [10], we will say that a stopping time τ is *fair* if $\mathbb{E}[M_\tau] = \mathbb{E}[M_{\tau-}]$ holds for every bounded martingale M (where we set $M_\infty := M_{\infty-} := \lim_{t\to\infty} M_t$). Since by the optional sampling theorem $M_{\tau_n} = \mathbb{E}[M_\tau | F_{\tau_n}]$ holds for every uniformly integrable martingale M, taking expectations and passing to the limit shows that announceable stopping times are fair (more, it shows that if τ is announceable and M is a uniformly integrable martingale then $M_{\tau-} \in L^1(\mathbb{P})$ and $\mathbb{E}[M_{\tau-}] = \mathbb{E}[M_\tau]$). The opposite implication is also true, so being predictable, being fair and being announceable are equivalent conditions for a stopping time. For a proof of the fact that fair stopping times are announceable we refer to [10, Chapter 6, Theorem 12.6], as we have nothing to add to this implication. Knowing that fair stopping times are announceable, to conclude the proof of all the equivalences typically one shows directly that predictable stopping times are fair (in the present paper, this follows instead from Corollary 2.1 since, as explained above, announceable stopping times are fair). This fact is often derived as a consequence of the (difficult) section theorem for predictable sets; one can however find in [10, Chapter 6, Theorem 12.6] a direct proof which does not use the section theorems themselves, but does involve ideas from their proofs, which are essentially based on Choquet's capacity theorem. Another possibility is to proceed as [7] and give a proof of the Doob-Meyer decomposition which shows inter-alia that increasing predictable processes are natural; applying this to the predictable process $A := \mathbb{1}_{[\tau,1]}$ shows that predictable stopping times are fair (in particular in this paper, instead of proving Theorem 1.2 directly, we could see it as an immediate corollary of Theorem 5.2). Yet another way is to proceed as in [6]; this proof, although intuitive, uses the theory of integration with respect to general predictable bounded integrands, as well as the Bichteler-Dellacherie theorem and Jacod's countable expansion theorem.

3 Predictable stopping times

In this section we prove that a number of commonly used hitting times are (predictable) stopping times when the underlying process is càdlàg and adapted (predictable), and we use this to show that one can exactly exhaust the jumps of a càdlàg adapted (predictable) process with sequence of (predictable) stopping times. To precisely state this fact, we recall that $[\![\sigma]\!] := \{(t,\omega) \in [0,\infty) \times \Omega : t = \sigma(\omega)\}$ denotes the *graph of a stopping time* σ. If X is càdlàg we say that a sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}}$ *exactly exhausts the jumps of X* in $B \subseteq \mathbb{R}$ if $\{\Delta X \in B\} = \cup_n [\![\sigma_n]\!]$ and $[\![\sigma_n]\!] \cap [\![\sigma_m]\!] = \emptyset$ whenever $n \neq m$ (i.e. $\sigma_n \neq \sigma_m$ on $\{\sigma_m < \infty\}$ for $n \neq m$). We will say that $(\sigma_n)_{n\in\mathbb{N}}$ is *strictly increasing to ∞* if $\sigma_n < \sigma_{n+1}$ on $\{\sigma_n < \infty\}$ and $\sigma_i \to \infty$ as $i \to \infty$. Given $B \subseteq \mathbb{R}$ we set $d(x,B) := \inf\{|x-y| : y \in B\}$.

Theorem 3.1. *If $X = (X_t)_{t\geq 0}$ is a càdlàg adapted process, C_k are closed in $\mathbb{R}$ and $0 \notin F = \cup_k C_k$ then there exist a sequence of stopping times $(\sigma_n)_n$ which exactly exhausts the jumps of X in F, and if X is predictable then each σ_n can be chosen to be predictable. Moreover, if $d(0,F) > 0$ then $(\sigma_n)_n$ can be chosen to be strictly increasing to ∞.*

The rest of this section is devoted to giving an *elementary* proof of the classical Theorem 3.1 and of the following lemma (which we only use in Section 6); thus, the reader interested in new results may safely decide to jump directly to Section 4.

Lemma 3.2. *If X is càdlàg adapted and C is closed, the first-approach time*

$$\sigma := \inf\{t \geq 0 : X_t \in C \text{ or } X_{t-} \in C\} \tag{3.1}$$

is a stopping time, and it is predictable if X is predictable.

We remark that all the results in this section hold, with exactly the same proof, if the process X has values not in $\mathbb{R}$ but in a generic topological vector space Y which supports a translation invariant distance d that generates the topology, in which case we assume that Y is endowed with the Borel sigma algebra $\mathcal{B}(Y)$. Notice that the function $d(\cdot, B) := \inf\{d(x, y) : y \in B\}$ is Lipschitz (with constant 1), so it is Borel measurable. We will use without further mention the fact that if B is closed and $y_k \to y$ then $d(y_k, B) \to 0$ implies $y \in B$.

Proof of Lemma 3.2. Let $t_n \downarrow t$ be s.t. either $X_{t_n} \in C$ or $X_{t_n-} \in C$. If $t_n = t$ then either $X_t \in C$ or $X_{t-} \in C$, and if $t_n \downarrow t$ and $t_n > t$ for all n then, whether $X_{t_n} \in C$ or $X_{t_n-} \in C$, necessarily $X_t \in C$; thus, the infimum in (3.1) is attained. From this and the compactness of $[0, t]$ it follows that $\{\sigma \le t\} = L$, where

$$\{X_t \in C\} \cup \bigcap_{n \in \mathbb{N}} \bigcup_{q \in \mathbb{Q} \cap [0,t)} \{d(X_q, C) < 1/n\} =: L \quad \text{belongs to } \mathcal{F}_t; \tag{3.2}$$

thus, σ is a stopping time.

Now suppose that X is predictable. Since the infimum in (3.1) is attained, if $t \le \sigma$ then $\sigma = t$ iff either $X_t \in C$ or $X_{t-} \in C$; in other words

$$\mathbb{1}_{\{\sigma\}} = \mathbb{1}_{[0,\sigma]}\Big(\mathbb{1}_C(X) \vee \mathbb{1}_C(X_-)\Big). \tag{3.3}$$

The càg processes $\mathbb{1}_{[0,\sigma]}$ and X_- are adapted and thus predictable, so (3.3) implies that $\mathbb{1}_{\{\sigma\}}$ is predictable and so also $\mathbb{1}_{[\sigma,\infty)} = 1 - \mathbb{1}_{[0,\sigma]} + \mathbb{1}_{\{\sigma\}}$ is predictable. $\qquad\square$

Although we could, similarly to [2, Chapter 1, Proposition 1.32], prove part of Theorem 3.1 using Lemma 3.2 (making use of the concept of the sigma algebra $\mathcal{F}_{\tau-}$), we find it more natural to study directly the jumps times of X as follows.

Lemma 3.3. *If τ is a stopping time, X is càdlàg adapted, C_n are closed sets and $F = \cup_n C_n$ satisfies $d(0, F) > 0$ then*

$$\sigma := \inf\{t > \tau : \Delta X_t \in F\} \tag{3.4}$$

is a stopping time s.t. $\sigma > \tau$ on $\{\tau < \infty\}$, and σ is predictable if X is predictable.

Proof. We will use the fact that, since X is càdlàg, for any compact interval J the set $\{t \in J : d(\Delta X_t, 0) \ge d(0, F)\}$ is finite, so the set $D := \{t > \tau : \Delta X_t \in F\}$ is discrete; in particular, the inf defining σ is attained, so $\sigma > \tau$ on $\{\tau < \infty\}$. That σ is a stopping time follows from the identity $\{\sigma \le t\} = L \in \mathcal{F}_t$, where

$$L := \bigcup_{q \in \mathbb{Q} \cap (0,1)} \Big(\{\tau \le qt\} \cap \Big(\bigcup_{n \ge 1} \bigcap_{k \ge 1} \bigcup_{(u,s) \in A_k^q(t)} \{d(X_s - X_u, C_n) < \frac{1}{k}\}\Big)\Big),$$

and $A_k^q(t)$ is the countable set

$$A_k^q(t) := \{(at, bt) : a, b \in \mathbb{Q} \cap (q, 1], a < b < a + 1/k\}.$$

To prove $\{\sigma \le t\} = L$, consider that, since D is discrete, $\sigma \le t$ iff $t > \tau$ and there exist $s \in (\tau, t]$ and $n \in \mathbb{N}$ s.t. $\Delta X_s \in C_n$. Thus, $\sigma \le t$ iff there exist $q \in \mathbb{Q} \cap (0, 1)$, $n \in \mathbb{N}$, and sequences $u_k \uparrow s$ and $s_k \downarrow s$ s.t. $tq \ge \tau$, $u_k < s$, $(u_k, s_k) \in A_k^q(t)$ and

$$d(X_{s_k} - X_{u_k}, C_n) < 1/k \quad \text{for all } k. \tag{3.5}$$

This shows that $\{\sigma \leq t\} \subseteq L$, and that to prove the opposite inequality given $q \in \mathbb{Q} \cap (0,1)$ and $(u_k, s_k) \in A_k^q(t)$ such that $tq \geq \tau$ and (3.5) hold, we only need to show that we can replace (u_k, s_k) with some $(\hat{u}_k, \hat{s}_k)$ satisfying the same properties and additionally s.t. $\hat{u}_k < s$, $\hat{u}_k \uparrow s$ and $\hat{s}_k \downarrow s$ for some s. This is easily done: by compactness there exists a subsequence $(n_k)_k$ s.t. u_{n_k} (resp. s_{n_k}) is converging to some u (resp s) and w.l.o.g. the convergence is monotone; since $u_k < s_k < u_k + 1/k$ necessarily $u = s$, and since (3.5) implies $\liminf_k d(X_{s_k} - X_{u_k}, 0) \geq d(0, F) > 0$ necessarily s_{n_k} must be decreasing and u_{n_k} increasing and s.t. $u_{n_k} < s$ (otherwise $\exists \lim_k X_{s_k} - X_{u_k} = 0$); thus, we can choose $(\hat{u}_k, \hat{s}_k) := (u_{n_k}, s_{n_k}) \in A_{n_k}^q(t) \subseteq A_k^q(t)$ as it also satisfies (3.5).

Now suppose that X is predictable and notice that, since D is discrete, if $\tau < t \leq \sigma$ then $\sigma = t$ iff $\Delta X_t \in F$; in other words $\mathbb{1}_{\{\sigma\}} = \mathbb{1}_{(\tau, \sigma]} \mathbb{1}_F(\Delta X)$, which implies that σ is predictable (just as (3.3) does in Lemma 3.2). $\qquad\square$

Proof of Theorem 3.1. If $d(0, F) > 0$, let $\sigma_{-1} := 0$ and define recursively $(\sigma_k)_{k \in \mathbb{N}}$ by setting $\sigma_{k+1} := \inf\{t > \sigma_k : \Delta X_t \in F\}$. By Lemma 3.3 each σ_k is a stopping time, and a predictable one if X is predictable. Since $\sigma_n < \sigma_{n+1}$ on $\{\sigma_n < \infty\}$, and since for any compact interval J the set $\{t \in J : d(\Delta X_t, 0) \geq d(0, F)\}$ is finite, $(\sigma_k)_{k \in \mathbb{N}}$ exhausts the jumps of X in F and it is strictly increasing to ∞.

For general F s.t. $0 \notin F$, we reduce to the previous case by using the annullus $D_n := \{y : d(y, 0) \in (2^n, 2^{n+1}]\}$. Notice that D_n can be written as the union of countably many closed sets, and so also can $F \cap D_n$. Now, given $n \in \mathbb{Z}$, set $\sigma_{-1}^n := 0$ and define recursively $(\sigma_k^n)_{k \in \mathbb{N}}$ by setting $\sigma_{k+1}^n := \inf\{t > \sigma_k^n : \Delta X_t \in F \cap D_n\}$, so that

$$\{\Delta X \in F\} = \cup_n \{\Delta X \in F \cap D_n\} = \cup_{k,n} [\![\sigma_k^n]\!].$$

Moreover, since $\sigma_{k+1}^n > \sigma_k^n$ on $\{\sigma_k^n < \infty\}$, and since D_n and D_m are disjoint for $n \neq m$ and $\Delta X_{\sigma_k^n} \in F \cap D_n$ on $\{\sigma_k^n < \infty\}$ for every k, it follows that $\sigma_j^i \neq \sigma_k^n$ on $\{\sigma_k^n < \infty\}$ if $(i,j) \neq (n,k)$, so enumerating the countable family $(\sigma_k^n)_{k,n \in \mathbb{Z}}$ we get a sequence which exactly exhausts the jumps of X in F. $\qquad\square$

4 How to approximate the compensator

In this section we prove Theorem 1.1; to do this, we revisit the proof of the existence of the Doob-Meyer decomposition given [1], and we strengthen it as to obtain that $\mathcal{A}_t^n \to A_t$ a.s. for all t along a subsequence. For didactical reasons we prefer to present below the whole proof, rather than explaining how to modify the one given in [1].

To obtain convergence at a given stopping time, we will use the following lemma, which is reminiscent of [5, Lemma A.2], and whose point is that the subsequences (in the assumption and in the conclusion) are not allowed to depend on ω.

Lemma 4.1. *Let $f, g, (f^n)_n, (g^n)_n$ be random variables in $L^1(\mathbb{P})$ that satisfy*

$$0 \leq f^n \leq g^n, \quad g^n \to g \text{ in } L^1(\mathbb{P}), \quad \lim_{n \to \infty} \mathbb{E}[f^n] = \mathbb{E}[f].$$

Assume moreover that for every subsequence $(n_i)_i$

$$\limsup_{i \to \infty} f^{n_i}(\omega) = f(\omega) \quad \text{for } \mathbb{P} \text{ a.e. } \omega. \tag{4.1}$$

Then, there exists $h \in L^1(\mathbb{P})$ and a subsequence $(n_i)_i$ such that $f^{n_i} \leq h$ for all i and $(f^{n_i})_i$ converges almost surely to f as $i \to \infty$.

Proof. Passing to a subsequence (without relabeling) we get that $||g^n - g||_{L^1(\mathbb{P})} \leq 2^{-n}$, thus the random variable $h := g + \sum_n |g^n - g|$ is integrable and dominates the sequence

$(f^n)_n$. By the dominated convergence theorem and (4.1) it follows that

$$\lim_{n \to \infty} \mathbb{E}\left[\sup_{m \geq n} f^m\right] = \mathbb{E}[f]. \tag{4.2}$$

The assumption $\mathbb{E}[f] = \lim_n \mathbb{E}[f^n]$ and (4.2) imply that $h^n := f^n - \sup_{m \geq n} f^m$ converges to 0 in $\mathbb{L}^1$ (since $h^n \leq 0$). We can then extract a further subsequence (not relabeled) such that h^n converges to 0 $\mathbb{P}$ a.s. Thus, thanks to the monotonicity of $(\sup_{m \geq n} f^m)_n$, also $(f^n)_n$ also converges a.s., and then (4.1) implies that its limit is f. $\qquad\square$

Proof of Theorem 1.1. The identity $var(A)_t = var(A)_{t-} + |A_t - A_{t-}|$ shows that $var(A)$ is predictable (since $var(A)_-$ and A_- are adapted and càg). Thus $A^{\pm} := (var(A) \pm A)/2$ are predictable increasing and satisfy $A = A^+ - A^-$, so we can assume w.l.o.g. that A is increasing. Moreover, by passing to an equivalent measure we can assume w.l.o.g. that A is integrable.

If $A_0 = 0$ set $S := A$, which trivially is a submartingale of class D. Let $(M^n_t + A^n_t)_{t \in \mathcal{D}_n}$ be the discrete time Doob decomposition of the sampled process $(S_t)_{t \in \mathcal{D}_n}$, and extend M^n and A^n to $[0,1]$ setting

$$M^n_t := \mathbb{E}[M^n_1 | \mathcal{F}_t] \quad \text{and} \quad A^n_t := A^n_{k/2^n} \text{ for } t \in ((k-1)/2^n, k/2^n];$$

then it follows from [1, Lemma 2.1 and 2.2] that there exist $\hat{M} \in L^1(\mathbb{P})$ and convex weights $\lambda^n_n, \ldots, \lambda^n_{N_n}$ such that $\mathcal{M}^n := \lambda^n_n M^n + \ldots + \lambda^n_{N_n} M^{N_n}$ satisfies $\mathcal{M}^n_1 \to \hat{M}$ in L^1. Now define

$$M_t := \mathbb{E}[\hat{M} | \mathcal{F}_t], \quad B := S - M, \quad \mathcal{A}^n := \lambda^n_n A^n + \ldots + \lambda^n_{N_n} A^{N_n}. \tag{4.3}$$

We take of course the càdlàg versions of the martingales M^n and M; in particular, B is càdlàg. We now want to show that, a.s. for all $t \in [0,1]$, $\exists \lim_i \mathcal{A}^{n_i}_t = B_t$ for some subsequence $(n_i)_i$ (which does not depend on t nor ω); this would show[1] that B is predictable, so $S = M + B$ would be a Doob-Meyer decomposition of $S = 0 + A$, and thus $B = A$ by the uniqueness of the decomposition (which follows from [3, Lemma 22.11]).

Since $\mathcal{M}^n_1 \to \hat{M} = M_1$ in $L^1(\mathbb{P})$, by Jensen inequality and the optional sampling theorem we get that, for every $[0,1]$-valued stopping time τ, $S_\tau - \mathcal{M}^n_\tau$ converges to $S_\tau - M_\tau = B_\tau$ in L^1; in particular, since $\mathcal{A}^n_t = S_t - \mathcal{M}^n_t$ holds for $t \in \mathcal{D}_n$, we get that $\mathcal{A}^n_t \to B_t$ in L^1 for all $t \in \mathcal{D}$. Passing to a subsequence (without relabeling), we can also obtain that $\|\mathcal{A}^n_1 - B_1\|_{L^1(\mathbb{P})} \leq 2^{-n}$ and $\mathcal{A}^n_t \to B_t$ a.s. for all $t \in \mathcal{D}$. It follows that B is a.s. increasing on $\mathcal{D}$, and so by right-continuity also on $[0,1]$, and the random variable $h := B_1 + \sum_n |\mathcal{A}^n_1 - B_1|$ is integrable and dominates the sequence $(\mathcal{A}^n_1)_n$.

We remark that, since the equality $\mathcal{A}^n_t = S_t - \mathcal{M}^n_t$ generally fails if $t \notin \mathcal{D}_n$, it is unclear for now if, given a $[0,1]$-valued stopping time τ, we can also get $\mathcal{A}^n_\tau \to B_\tau$ a.s.; we will now explain how to obtain this by passing to a subsequence. We only need to show that $\exists \lim_i \mathbb{E}[\mathcal{A}^{n_i}_\tau] = \mathbb{E}[B_\tau]$ and $\limsup_i \mathcal{A}^{n_i}_\tau = B_\tau$ a.s. for every subsequence $(n_i)_i$; indeed, applying Lemma 4.1 to $f^n = \mathcal{A}^n_\tau$, $f = B_\tau, g^n = \mathcal{A}^n_1$ and $g = B_1$ would then yield a subsequence $(\tilde{n}_i)_i$ such that $\lim_i \mathcal{A}^{\tilde{n}_i}_\tau = B_\tau$ a.s.. Take then an arbitrary subsequence $(n_i)_i$, and recall that $\mathcal{A}^n, B$ are increasing and $\mathcal{A}^n_t \to B_t$ a.s. and in L^1 for all $t \in \mathcal{D}$. It follows that $\limsup_i \mathcal{A}^{n_i}_\tau \leq B_\tau$, and that applying Fatou's lemma to $(\mathcal{A}^n_1 - \mathcal{A}^{n_i}_\tau)_i$ gives

$$\liminf_i \mathbb{E}[A^{n_i}_\tau] \leq \liminf_i \mathbb{E}[\mathcal{A}^{n_i}_\tau] \leq \limsup_i \mathbb{E}[\mathcal{A}^{n_i}_\tau] \leq \mathbb{E}[\limsup_i \mathcal{A}^{n_i}_\tau] \leq \mathbb{E}[B_\tau].$$

---

[1] Because $\mathcal{A}^n$ is adapted and càg, and any measurable process indistinguishable from zero is predictable when the filtration satisfies the usual conditions (see [10, Lemma 13.8]).

Thus, to conclude the existence of $(\tilde{n}_i)_i$ such that $\lim_i \mathcal{A}_\tau^{\tilde{n}_i} = B_\tau$ a.s. it is enough to show that $\exists \lim_n \mathbb{E}[A_\tau^n] = \mathbb{E}[B_\tau]$. This is easy: since S is of class D, if $\theta_n := \min\{t \in \mathcal{D}_n : t \geq \tau\}$ then $\theta_n \downarrow \tau$ and $A_\tau^n = A_{\theta_n}^n$ so we get

$$\mathbb{E}[A_\tau^n] = \mathbb{E}[A_{\theta_n}^n] = \mathbb{E}[S_{\theta_n}] - \mathbb{E}[M_0] \to \mathbb{E}[S_\tau] - \mathbb{E}[M_0] = \mathbb{E}[B_\tau].$$

Now use Theorem 3.1 to obtain $[0,1] \cup \{\infty\}$-valued[2] stopping times $(\sigma_k)_k$ which exactly exhaust all the jumps of B (i.e. the jumps of B in $\mathbb{R} \setminus \{0\}$), and set $\tau_k := 1 \wedge \sigma_k$. As shown above, there exists a subsequence $(n_i)_i$ such that $\lim_i \mathcal{A}_{\tau_k}^{n_i} = B_{\tau_k}$ a.s. for $k = 1$. By the same token, passing to further subsequences (without relabeling) and using a diagonal procedure, we can find a subsequence $(n_i)_i$ such that $\lim_i \mathcal{A}_{\tau_k}^{n_i} = B_{\tau_k}$ a.s. simultaneously for all k. Since $\mathcal{A}^n, B$ are increasing, B is càdlàg and $\mathcal{A}_t^n \to B_t$ for all $t \in \mathcal{D}$, necessarily $\exists \lim_n \mathcal{A}_t^n = B_t$ if B is continuous at t; since $\exists \lim_i \mathcal{A}_{\tau_k}^{n_i} = B_{\tau_k}$ for all k and $(\tau_k)_k$ exhausts all the jumps of B, it follows that $\exists \lim_i \mathcal{A}_t^{n_i} = B_t$ a.s. for all $t \in [0,1]$. Since $\mathcal{A}_0^n = 0$, this concludes the proof in the case $A_0 = 0$; in the general case, apply the above to $\tilde{A} = A - A_0 \mathbb{1}_{[0,\infty)}$ to obtain some $\tilde{\mathcal{A}}^n$ and $\tilde{h}$, and then set $\mathcal{A}^n := A_0 \mathbb{1}_{[0,\infty)} + \tilde{\mathcal{A}}^n$ and $h := |A_0| + \tilde{h}$. $\qquad\square$

5 Predictable processes are natural, and vice versa

In this section we prove in continuous time that an increasing process is predictable iff it is natural by passing to the limit the analogous discrete time statement, making use of Theorem 1.1.

In discrete time, we will call *increasing process* an increasing sequence of integrable random variables $A = (A_n)_{n \in \mathbb{N}}$ such that $A_0 = 0$. An increasing process is called predictable if A_{n+1} is $\mathcal{F}_n$-measurable for every $n \geq 0$, and is called natural if, for every bounded martingale $M = (M_n)_{n \in \mathbb{N}}$, we have $\mathbb{E}[M_n A_n] = \mathbb{E}[\sum_{k=1}^n M_{k-1}(A_k - A_{k-1})]$ for every $n \geq 0$. In this setting it is trivial to prove that an increasing processes is predictable iff it is natural (see e.g. [4, Chapter 1, Proposition 4.3]).

When working on the time interval $[0,1]$, a càdlàg increasing integrable process A s.t. $A_0 = 0$ is called *natural* if, for every bounded martingale M, $\mathbb{E}[M_1 A_1] = \mathbb{E}[\int_0^1 M_{s-} dA_s]$. Notice that in some books an equivalent definition is used: in discrete time it is trivial to prove that every increasing process A satisfies $\mathbb{E}[M_n A_n] = \mathbb{E}[\sum_{k=1}^n M_k(A_k - A_{k-1})]$ for all bounded martingales M and $n \in \mathbb{N}$, and that taking continuous time limits one immediately obtains that every càdlàg increasing process A satisfies $\mathbb{E}[M_1 A_1] = \mathbb{E}[\int_0^1 M_s dA_s]$ (see e.g. [4, Chapter 1, Lemma 4.7]); thus A is natural iff $\mathbb{E}[\int_0^1 \Delta M_s dA_s] = 0$. To pass to the continuous time limit, we will need the following approximation lemma. Given functions f, g and a partition $\pi = \{0 = t_0 \leq t_1 \leq \ldots \leq t_{n+1} = 1\}$, set

$$f^\pi := \sum_{i=0}^n f(t_i) \mathbb{1}_{[t_i, t_{i+1})} + f(1) \mathbb{1}_{\{1\}}, \quad \sum_\pi f \Delta g := \sum_{i=0}^n f(t_i)(g(t_{i+1}) - g(t_i)).$$

Lemma 5.1. *Given $f, g, g_n : [0,1] \to \mathbb{R}$, with g, g_n increasing and f, g càdlàg, let D be a dense subset of $[0,1]$ and $(\pi^n)_{n \in \mathbb{N}}$ be partitions of $[0,1]$ which satisfy $\pi_n \subseteq \pi_{n+1}$ and $\cup_n \pi_n = D \supseteq \{\Delta f \neq 0\}$. Then $g_k(t) \to g(t)$ for all $t \in D$ implies that*

$$\sum_{\pi_k} f \Delta g_k \to \int_{(0,1]} f(s-) dg(s) =: \int f_- dg \quad \text{as } k \to \infty. \tag{5.1}$$

Proof. Since f is right continuous at $0 \in D$, for any $k \in \mathbb{N}$ the set

$$A_k := \{t \in (0,1] : \limsup_n \sup_{[0,t)} |f^{\pi_n} - f| < 1/k\}$$

is non-empty. Its supremum $\bar{t}$ is attained, since D is dense and $\exists f(\bar{t}-)$, and cannot be < 1: otherwise, whether $\Delta f(\bar{t}) = 0$ or $\bar{t} \in \pi_n$ for big enough n, the right continuity of f would imply the existence of a $s > \bar{t}$ in A. Thus $A_k = [0,1]$ for all k, so f^{π_n} converges uniformly to f. In particular $\int f_-^{\pi_n} dg \to \int f_- dg$ as $n \to \infty$, and since the sequence $(g_k(1))_k$ is converging and thus is bounded, the inequality

$$|\sum_{\pi_k} (f^{\pi_n} - f) \Delta g_k| \leq \sup_s |(f^{\pi_n} - f)(s)| \sup_k g_k(1)$$

shows that $\sum_{\pi_k} (f^{\pi_n} - f) \Delta g_k \to 0$ as $n \to \infty$, uniformly in k. Thus, to conclude the proof it is enough to show (5.1) when f is replaced by f^{π_n}. Since, for $k \geq n$,

$$\int f_-^{\pi_n} dg = \sum_{\pi_n} f \Delta g \quad \text{and} \quad \sum_{\pi_k} f^{\pi_n} \Delta g_k = \sum_{\pi_n} f^{\pi_n} \Delta g_k = \sum_{\pi_n} f \Delta g_k \,,$$

we need to show that $\sum_{\pi_n} f \Delta g_k \to \sum_{\pi_n} f \Delta g$ as $k \to \infty$, which is trivially true since $g_k(t) \to g(t)$ at every $t \in D \supseteq \pi_n$. □

We will call *optional partition* an increasing finite or infinite sequence of stopping times π which is *pointwise finite on compacts*, meaning that $\pi = (\sigma_n)_{n \in I}, I \subseteq \mathbb{N}, \sigma_n \leq \sigma_{n+1}$ for all n and $\{n : \sigma_n(\omega) \leq t\}$ is a finite set for any ω and $t < \infty$. Notice that, working on the time index $[0,1]$, Theorem 3.1 will tell us that the jumps of size at least $1/n$ (i.e. the jumps in $(-\infty, -1/n] \cup [1/n, \infty)$) of a càdlàg process X form an optional partition of $[0,1] \cup \{\infty\}$-valued stopping times. If $\pi = (\sigma_n)_n$, we will denote write $\pi(\omega)$ for the sequence of reals $(\sigma_n(\omega))_n$. Given two finite optional partitions $\pi = (\sigma_n)_{n=0}^N, \hat{\pi} = (\hat{\sigma}_j)_{j=0}^J$, a convenient way to construct a finite *increasing* family of stopping times $\pi \cup \hat{\pi}$ which satisfies $(\pi \cup \hat{\pi})(\omega) = \pi(\omega) \cup \hat{\pi}(\omega)$ for all ω is to define $\pi \cup \hat{\pi}$ to be the ordered[3] family of stopping times

$$\sigma_1 \wedge \hat{\sigma}_j, \quad \sigma_n, \quad \sigma_n \vee (\hat{\sigma}_j \wedge \sigma_{n+1}), \quad \sigma_N, \quad \sigma_N \vee \hat{\sigma}_j,$$

where $j = 0, \ldots, J$, $n = 0, \ldots, N-1$. If $\pi = (\tau_j)_{j=0}^J$ is a finite partition and N, B are càdlàg processes, define

$$\sum_\pi N_- \Delta B := \sum_{j=1}^J N_{\tau_{j-1}} (B_{\tau_j} - B_{\tau_{j-1}}),$$

which satisfies $(\sum_\pi N_- \Delta B)(\omega) = \sum_{\pi(\omega)} N_-(\omega) \Delta B(\omega)$. Given a finite optional partition $\pi = (\sigma_n)_{n=0}^N$, we will say that a process B is π-*predictable* if B_{σ_n} is $\mathcal{F}_{\sigma_{n-1}}$-measurable, B_0 is $\mathcal{F}_0$-measurable and

$$B = \mathbb{1}_{\{0\}} B_0 + \sum_{n=1}^N \mathbb{1}_{(\sigma_{n-1}, \sigma_n]} B_{\sigma_n} \,. \tag{5.2}$$

Notice that, if α, β, γ are stopping times and A is a $\mathcal{F}_\alpha$ measurable random variable then $C := A\mathbb{1}_{\{\alpha \leq \gamma\}}$ is $\mathcal{F}_\alpha \cap \mathcal{F}_\gamma \subseteq \mathcal{F}_{\alpha \vee (\beta \wedge \gamma)}$-measurable and

$$A\mathbb{1}_{(\alpha, \gamma]} = A\mathbb{1}_{(\alpha, \alpha \vee (\beta \wedge \gamma)]} + C\mathbb{1}_{(\alpha \vee (\beta \wedge \gamma), \gamma]};$$

so, if B is π-predictable, it is trivially $(\pi \cup \hat{\pi})$-predictable for any finite optional partition $\hat{\pi}$ (this is why we defined $\pi \cup \hat{\pi}$ as above).

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[3]One can indeed order this family!

Theorem 5.2. *A càdlàg increasing integrable process A s.t. $A_0 = 0$ is natural iff it is predictable.*

Proof. If M is a martingale bounded by a constant C and $A = (A_t)_{t \in [0,1]}$ is predictable, let $h, \mathcal{A}^n, N_n$ be as in Theorem 1.1, let $(\sigma_i)_i$ be the optional partition of the jumps of M of size at least $1/n$, and set $\hat{\pi}^n := (1 \wedge \sigma_i)_i$. Let $\pi_k^n = (\tau_i)_i$ be the finite optional partition $\mathcal{D}_{N_n} \cup (1 \wedge \sigma_i)_{i=0,\dots,k}$. Since $\mathcal{A}^n$ is $\mathcal{D}_{N_n}$-predictable, it is π_k^n-predictable, and thus $(\mathcal{A}_{\tau_i}^n)_i$ is a $(\mathcal{F}_{\tau_i})_i$-predictable increasing process, and so also a $(\mathcal{F}_{\tau_i})_i$-natural process. Thus $\mathbb{E}[M_1 \mathcal{A}_1^n] = \mathbb{E}[\sum_{\pi_k^n} M_- \Delta \mathcal{A}^n]$ holds since $(M_{\tau_i}, \mathcal{F}_{\tau_i})_i$ is a bounded martingale.

Now, fix a generic $\omega \in \Omega$ and set $\pi^n(\omega) := \mathcal{D}_{N_n} \cup \hat{\pi}^n(\omega)$; since $(\sigma_i)_i$ is pointwise finite on compacts, there exists $k_0 = k_0(\omega)$ s.t. $\pi_k^n(\omega) = \pi^n(\omega)$ if $k \geq k_0$. Thus

$$\Big(\sum_{\pi_k^n} M_- \Delta \mathcal{A}^n \Big)(\omega) \overset{k \to \infty}{\longrightarrow} \Big(\sum_{\pi^n} M_- \Delta \mathcal{A}^n \Big)(\omega) := \sum_{\pi^n(\omega)} M_-(\omega) \Delta \mathcal{A}^n(\omega),$$

the sum on the RHS being well defined, as it is finite for each ω. Since $(\sum_{\pi_k^n} M_- \Delta \mathcal{A}^n)_k$ is dominated by $C \mathcal{A}_1^n$, it converges also in $L^1(\mathbb{P})$, so

$$\mathbb{E}[M_1 \mathcal{A}_1^n] = \mathbb{E}[\sum_{\pi^n} M_- \Delta \mathcal{A}^n]. \tag{5.3}$$

We can now apply Lemma 5.1 and Theorem 1.1 and obtain that $\sum_{\pi^n} M_- \Delta \mathcal{A}^n$ (resp. $\mathcal{A}_1^n$) converges $\mathbb{P}$ a.s. to $\int_0^1 M_{s-} dA_s$ (resp. $\mathcal{A}_1$); since it is dominated by Ch (resp. h), we can pass (5.3) to the limit and obtain that A is natural.

Assume now that A is natural, and let $A = M + B$ be its Doob-Meyer decomposition; the càdlàg increasing integrable process B is predictable, thus natural, and now $A = B$ follows from the uniqueness of the Doob-Meyer decomposition of a submartingale of class D into a martingale plus a natural process, which is easy to prove (it follows from [4, Chapter 1, Theorem 4.10]). $\qquad \square$

6 Consequences for special semimartingales

In this section we show how some well known facts about special semimartingales and predictable processes can be derived as simple consequences of Theorem 3.1 applied to the set $F = (0, \infty)$; in particular, we characterize which special semimartingales S have a continuous compensator.

We will often use without explicit mention the following trivial consequence of the optional sampling theorem: if τ is an announceable stopping time and $S = M + A$, where M is a uniformly integrable martingale and A is càdlàg increasing integrable and s.t. $A_0 = 0$, then $M_{\tau-} \in L^1(\mathbb{P})$ and $\mathbb{E}[\Delta M_\tau] = 0$, so $S_{\tau-} \in L^1(\mathbb{P})$ and $\mathbb{E}[\Delta S_\tau] = \mathbb{E}[\Delta A_\tau]$. Also, we will use without further notice the fact that predictability is preserved by stopping (this follows from $X^\tau \mathbb{1}_{\{\tau > 0\}} = X \mathbb{1}_{(0,\tau]} + X_\tau \mathbb{1}_{(\tau, \infty)}$).

Theorem 6.1. *Almost every path of a predictable local martingale M is continuous.*

Proof. By localization we can assume that M is a uniformly integrable martingale. Theorem 3.1 provides us with a sequence $(\sigma_n)_n$ of predictable stopping times which exactly exhausts the positive jumps of M (i.e. the jumps of M in $(0, \infty)$), and Corollary 2.1 tells us that $(\sigma_n)_n$ are announceable. It follows that $\mathbb{E}[M_{\sigma_n}] = \mathbb{E}[M_{\sigma_n-}]$ and so, since by definition we have $\Delta M_{\sigma_n} > 0$ on $\{\sigma_n < \infty\}$ and $\Delta M_{\sigma_n} = 0$ on $\{\sigma_n = \infty\}$, necessarily each $\{\sigma_n < \infty\}$ has probability zero. It follows that $\{\sup_t \Delta M_t > 0\} = \cup_n \{\sigma_n < \infty\}$ has probability zero, and analogously so does $\{\inf_t \Delta M_t < 0\} = \{\sup_t \Delta(-M)_t > 0\}$, so M is a.s. continuous. $\qquad \square$

We will say that $M + A$ is a (canonical) semimartingale decomposition of a process S if M is a local martingale, A is a càdlàg adapted (resp. predictable) process of finite variation s.t. $A_0 = 0$ and $S = M + A$. A process S admitting a (canonical) semimartingale decomposition is called a (special) semimartingale. Recall that the canonical semimartingale decomposition is unique (for a proof see e.g. [3, Lemma 22.11]), and that the process A is called the *compensator* of S; M is called the (local) martingale part of S.

We now need the following theorem, whose elementary proof (which we provide below for convenience of the reader) is essentially standard; the only unconventional choice is to prove it using Lemma 3.2. The advantage of this approach is that it is much easier to show that σ defined in (3.1) is a stopping time than showing that $\sigma := \inf\{t \geq 0 : |X_t| \geq K\}$ is one (see e.g. [11, Chapter 2, Lemma 75.1]); of course it would be even easier to use $\tau := \inf\{t \geq 0 : |X_t| > K\}$ instead, but the problem is that it is unclear whether τ is a *predictable* stopping time when X is predictable.

Theorem 6.2. *Any càdlàg predictable process X is locally bounded.*

Proof. Given X càdlàg predictable, let $C_k := (-\infty, k] \cup [k, \infty)$ and

$$\sigma_k := \inf\{t \geq 0 : X_t \in C_k \text{ or } X_{t-} \in C_k\}. \tag{6.1}$$

Lemma 3.2 and Corollary 2.1 show that σ_k is an announceable stopping time. Trivially $\sigma_k \leq \sigma_{k+1}$; since X is càdlàg, each of its paths is bounded on compacts, so $\sigma_k \to \infty$. Let $(\tau_k^n)_n$ be a sequence of stopping times announcing σ_k, and $(n_k)_k$ be a subsequence s.t. $\mathbb{P}(\tau_k^{n_k} + 1/2^k \leq \sigma_k < \infty) < 1/2^k$, so that a.s. $\tau_k^{n_k} + 1/2^k \leq \sigma_k < \infty$ holds for at most finitely many k's, and thus the increasing sequence of stopping times $\varrho_i := \inf_{k \geq i} \tau_k^{n_k}$ converges to $\lim_k \sigma_k = \infty$. Since $|X^{\varrho_k}| \mathbb{1}_{\{\varrho_k > 0\}} \leq k$ holds because $\varrho_k \leq \tau_k^{n_k} < \sigma_k$ on $\{\sigma_k > 0\}$, X is locally bounded. $\square$

Here an immediate and useful consequence of Theorem 6.2.

Corollary 6.3. *If A is càdlàg predictable and of finite variation then its variation is locally bounded.*

From Corollary 6.3 it follows that if S is a special semimartingale then one can write S as $M + A$ for a local martingale M and a càdlàg adapted process A of locally integrable variation (the vice versa is also true, and is given by the Doob-Meyer decomposition). Moreover, the optional sampling theorem implies that any local martingale is locally integrable (see [8, Chapter 3, Theorem 38]), thus in *any* decomposition of a special semimartingale S as $M + A$, where A is a process of *finite* variation and M is a local martingale, the process A is of locally integrable variation. The next important characterization of special semimartingales is also a consequence of Corollary 6.3. For its simple proof we refer to [8, Chapter 3, Theorem 32]); we remark that the proof implicitly makes use of the uniqueness of the canonical decomposition to obtain the existence of a canonical decomposition of S on $[0, \infty)$ from the ones on $[0, \sigma_n]$.

Corollary 6.4. *A semimartingale S is special iff the process $X_t := \sup_{s \leq t} |\Delta S_s|$ is locally integrable (or equivalently if $S_t^* := \sup_{s \leq t} |S_s|$ is locally integrable).*

It follows from Corollary 6.4 that any continuous semimartingale is special, and then Theorem 6.1 implies that its local martingale part and compensator are continuous processes. More generally, one can characterize which special semimartingales S have a continuous compensator.

Theorem 6.5. *If $S = M + A$ is the canonical decomposition of the special semi-martingale S, then A is a.s. continuous iff, for all announceable stopping times τ, $\mathbb{E}[\Delta S_\tau^{\sigma_n}] = 0$ holds for one (and thus all) sequences of stopping times $\sigma_n \uparrow \infty$ s.t. $\mathbb{1}_{\{\sigma_n > 0\}}(\sup_{t \le \sigma_n} |M_t| + var(A)_{\sigma_n}) \in L^1(\mathbb{P})$. In particular, if $S = M + A$ is the Doob-Meyer decomposition of a submartingale S of class D, then A is a.s. continuous iff $\mathbb{E}[\Delta S_\tau] = 0$ for all announceable stopping times τ.*

Proof. One implication is obvious. For the opposite one, assume by localization that $\sup_t |M_t|$ and $var(A)$ are integrable, and let $(\tau_n)_n$ be a sequence of predictable stopping times which exactly exhausts the positive jumps of A. Since by Corollary 2.1 predictable stopping times are announceable, we obtain that $0 = \mathbb{E}[\Delta S_{\tau_n}] = \mathbb{E}[\Delta A_{\tau_n}]$. Since $\Delta A_{\tau_n} \ge 0$, it follows that $\Delta A_{\tau_n} = 0$ a.s. for all n, so $\tau_n = \infty$ a.s. and $\mathbb{P}(\{\sup_t \Delta A_t > 0\}) = \sum_n \mathbb{P}(\{\tau_n < \infty\}) = 0$. Analogously $\mathbb{P}(\{\Delta \sup_t (-A)_t > 0\}) = 0$, so A has a.s. continuous paths. $\square$

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