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The monotonicity of f-vectors of random polytopes*

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Abstract

Let K be a compact convex body in $\mathbb{R}^d$, let K_n be the convex hull of n points chosen uniformly and independently in K, and let $f_i(K_n)$ denote the number of i-dimensional faces of K_n. We show that for planar convex sets, $E[f_0(K_n)]$ is increasing in n. In dimension $d \geqslant 3$ we prove that if $\lim_{n\to\infty} \frac{E[f_{d-1}(K_n)]}{An^c} = 1$ for some constants A and $c > 0$ then the function $n \mapsto E[f_{d-1}(K_n)]$ is increasing for n large enough. In particular, the number of facets of the convex hull of n random points distributed uniformly and independently in a smooth compact convex body is asymptotically increasing. Our proof relies on a *random sampling* argument.

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1 Introduction

What does a random polytope, that is, the convex hull of a finite set of random points in $\mathbb{R}^d$, look like? This question goes back to Sylvester's *four point problem*, which asked for the probability that four points chosen at random be in convex position. There are, of course, many ways to distribute points randomly, and as with Sylvester's problem, the choice of the distribution drastically influences the answer.

Random polytopes. In this paper, we consider a *random polytope* K_n obtained as the convex hull of n points distributed uniformly and independently in a *convex body* $K \subset \mathbb{R}^d$, i.e. a compact convex set with nonempty interior. This model arises naturally in applications areas such as computational geometry [7, 11], convex geometry and stochastic geometry [15] or functional analysis [6, 10]. A natural question is to understand the behavior of the $\boldsymbol{f}$*-vector* of K_n, that is, $\boldsymbol{f}(K_n) = (f_0(K_n), \ldots, f_{d-1}(K_n))$ where $f_i(K_n)$ counts the number of i-dimensional faces, and the behaviour of the *volume* $V(K_n)$. Bounding $f_i(K_n)$ is related, for example, to the analysis of the computational complexity of algorithms in computational geometry.

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Exact formulas for the expectation of $\boldsymbol{f}(K_n)$ and $V(K_n)$ are only known for convex polygons [2, Theorem 5]. In general dimensions, however, the asymptotic behavior, as n goes to infinity, is well understood; the general picture that emerges is a dichotomy between the case where K is a polytope when

$$\begin{aligned} E[f_i(K_n)] &= c_{d,i,K} \log^{d-1} n + o(\log^{d-1} n), \\ E[V(K) - V(K_n)] &= c_{d,K} n^{-1} \log^{d-1} n + o(n^{-1} \log^{d-1} n), \end{aligned} \tag{1.1}$$

and the case where K is a smooth convex body (i.e. with boundary of differentiability class C^2 and with positive Gaussian curvature) when

$$\begin{aligned} E[f_i(K_n)] &= c'_{d,i,K} n^{\frac{d-1}{d+1}} + o(n^{\frac{d-1}{d+1}}), \\ E[V(K) - V(K_n)] &= c_{d,K} n^{-\frac{2}{d+1}} + o(n^{-\frac{2}{d+1}}). \end{aligned} \tag{1.2}$$

(Here $c_{d,i,K}$ and $c'_{d,i,K}$ are constants depending only on i, d and certain geometric functionals of K.) The results for $E[V(K_n)]$ follow from the corresponding results for $f_0(K_n)$ via Efron's formula [8]. Numerous papers are devoted to such estimates and we refer to Reitzner [14] for a recent survey.

Monotonicity of f-vectors. In spite of numerous papers devoted to the study of random polytopes, the natural question whether these functionals are monotone remained open in general.

Concerning the monotonicity of $E[V(K_n)]$ with respect to n, the elementary inequality

$$E[V(K_n)] \leqslant E[V(K_{n+1})] \tag{1.3}$$

for any fixed K follows immediately from the monotonicity of the volume. Yet the monotonicity (at first sight similarly obvious) with respect to K, i.e. the inequality that $K \subset L$ implies

$$E[V(K_n)] \leqslant E[V(L_n)]$$

surprisingly turned out to be false in general. This problem was posed by Meckes [9], see also [14], and a counterexample for $n = d + 1$ was recently given by Rademacher [12].

A "tantalizing problem" in stochastic geometry, to quote Vu [16, Section 8], is whether $f_0(K_n)$ is monotone in n, that is, whether similar to Equation (1.3):

$$E[f_0(K_n)] \leqslant E[f_0(K_{n+1})].$$

This is not a trivial question as the convex hull of $K_n \cup \{x\}$ has fewer vertices than K_n if x lies outside K_n and sees more than d of its vertices. Some bounds are known for the expected number of points of K_n seen by a random point x chosen uniformly in $K \backslash K_n$ [16, Corollary 8.3] but they do not suffice to prove that $E[f_0(K_n)]$ is monotone.

It is known that $E[f_0(K_n)]$ is always bounded from below by an increasing function of n, namely $c(d) \log^{d-1} n$ where $c(d)$ depends only on the dimension: this follows, via Efron's formula [8], from a similar lower bound on the expected volume of K_n due to Bárány and Larman [1, Theorem 2]. While this is encouraging, it does not exclude the possibility of small oscillations preventing monotonicity. In fact, Bárány and Larman [1, Theorem 5] also showed that for any functions s and ℓ such that $\lim_{n\to\infty} s(n) = 0$ and

$\lim_{n\to\infty} \ell(n) = \infty$ there exists[1] a compact convex domain $K \subset \mathbb{R}^d$ and a sequence $(n_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$:

$$E[f_0(K_{n_{2i}})] > s(n_{2i}) n_{2i}^{\frac{d-1}{d+1}} \quad \text{and} \quad E[f_0(K_{n_{2i+1}})] < \ell(n_{2i+1}) \log^{d-1}(n_{2i+1}).$$

When general convex sets are considered, there may thus be more to this monotonicity question than meets the eye.

Results. This paper present two contributions on the monotonicity of the $\boldsymbol{f}$-vector of K_n. First, we show that for any *planar* convex body K the expectation of $f_0(K_n)$ is an increasing function of n. This is based on an explicit representation of the expectation $E[f_0(K_n)]$.

Theorem 1.1. *Assume K is a planar convex body. For all integers n,*

$$E[f_i(K_{n+1})] > E[f_i(K_n)],\ i = 0, 1.$$

Second, we show that if K is a compact convex set in $\mathbb{R}^d$ with a C^2 boundary then the expectation of $f_{d-1}(K_n)$ is increasing for n large enough (where "large enough"depends on K); in particular, it follows from Euler's polytope formula and the simplicity of K_n that for smooth compact convex bodies K in $\mathbb{R}^3$ also the expectation of $f_0(K_n)$ becomes monotone in n for n large enough.

Theorem 1.2. *Assume $K \subset \mathbb{R}^d$ is a smooth convex body. Then there is an integer n_K such that for all $n \geqslant n_K$*

$$E[f_i(K_{n+1})] > E[f_i(K_n)],\ i = d-2, d-1.$$

Our result is in fact more general and applies to convex hulls of points i.i.d. from any "sufficiently generic" distribution (see Section 3) and follows from a simple and elegant random sampling technique introduced by Clarkson [4] to analyze *non-random* geometric structures in discrete and computational geometry.

2 Monotonicity for convex domains in the plane

Assume K is a planar convex body of volume one. Given a unit vector $u \in \mathbb{S}^1$, each halfspace $\{x \in \mathbb{R}^2 :\ x \cdot u \leqslant p\}$ cuts off from K a set $\{x \in K :\ x \cdot u \leqslant p\}$ of volume $s \in [0,1]$. On the other hand, given $u \in \mathbb{S}^1, s \in (0,1)$ there is a unique line $\{x \in \mathbb{R}^2 :\ x \cdot u = p\}$ and a corresponding halfspace $\{x \in \mathbb{R}^2 :\ x \cdot u \leqslant p\}$ which cuts off from K a set of volume precisely s. Denote by $L(s,u)$ the square of the length of this unique chord.

Using this notation, Buchta and Reitzner [2] showed that the expectation of the number of points on the convex hull of n points chosen randomly uniformly in K can be computed with the following formula.

Lemma 2.1 (Buchta,Reitzner)**.**

$$E[f_0(K_n)] = \frac{1}{6} n(n-1) \int_{\mathbb{S}^1} \int_0^1 s^{n-2} L(s,u)\, ds du$$

Here du denotes integration with respect to Lebesgue measure on $\mathbb{S}^1$. By a change of variables we obtain

$$E[f_0(K_n)] = \frac{1}{6}(n-1) \int_{\mathbb{S}^1} \int_0^1 t^{-\frac{1}{n}} L(t^{\frac{1}{n}}, u)\, dt du = \frac{1}{6} \int_{\mathbb{S}^1} \mathcal{I}_n(u) du$$

[1] More precisely, *most* (in the Baire sense) compact convex sets exhibit this behavior.

with $\mathcal{I}_n(u) = (n-1)\int_0^1 t^{-\frac{1}{n}} L(t^{\frac{1}{n}}, u)\, dt$. Observe that $L(1,u) := \lim_{s\to 1} L(s,u) = 0$ for almost all $u \in \mathbb{S}^1$. Also observe that for fixed $u \in \mathbb{S}^1$ the partial derivative $\frac{\partial}{\partial s} L(s,u)$ exists for almost all $s \in (0,1)$. This is a consequence of the a.e. differentiability of a convex function. In the following we write $L(\cdot) = L(\cdot, u)$, $\frac{\partial}{\partial s} L(\cdot, u) = L'(\cdot)$. Integration by parts in t gives

$$\mathcal{I}_n(u) = -\int_0^1 L'(t^{\frac{1}{n}})\, dt \tag{2.1}$$

for almost all $u \in \mathbb{S}^1$. Finally, the convexity of K induces the following lemma.

Lemma 2.2. *Given a value u, the derivative $s \mapsto L'(s)$ is a non-increasing function.*

Proof. Fix $u \in \mathbb{S}^1$. We denote $l(p) = l(p,u)$ the length of the chord $K \cap \{x \in \mathbb{R}^2 :\ x \cdot u = p\}$, and define

$$\underline{p} = \inf\{p :\ l(p) > 0\},\ \overline{p} = \sup\{p :\ l(p) > 0\}$$

which is the support function of K in direction $-u$, resp. u. Moreover, $L(s(p)) = l(p)^2$ where

$$s(p) = vol(\{x \in K :\ x \cdot u \leqslant p\})$$

is the volume of the part of K on the 'left' of the chord of length $l(p)$. For $p \in [\underline{p}, \overline{p}]$ the volume $s(p)$ is a monotone and therefore injective function of p with inverse $p(s)$, and we have $\frac{d}{dp} s(p) = l(p)$.

First observe that because K is a convex body, the chord length $l(p)$ is concave as a function of $p \in [\underline{p}, \overline{p}]$. Thus its derivative $\frac{d}{dp} l(p)$ is non-increasing.

Hence

$$L'(s) = 2l(p(s))\frac{d}{ds} l(p(s)) = 2\frac{d}{ds} l(p(s))\ (\frac{d}{dp} s(p))|_{p=p(s)} = 2\frac{d}{dp} l(p)|_{p=p(s)}.$$

Since the derivative $\frac{d}{dp} l(p)$ is non-increasing and $p(s)$ is monotone, we see that $\frac{d}{ds} L(s,u)$ is non-increasing in s. □

This allows us to conclude that the expectancy of the number of points on the convex hull is increasing.

Proof of Theorem 1.1. According to Lemma 2.2, $L'(s)$ is decreasing. Since for all $t \in [0,1]$, $t^{\frac{1}{n}} < t^{\frac{1}{n+1}}$, this implies that $L'(t^{\frac{1}{n}}) \geqslant L'(t^{\frac{1}{n+1}})$. Combined with equation (2.1) we have

$$\mathcal{I}_n(u) \leqslant \mathcal{I}_{n+1}(u)$$

which proves $E[f_0(K_n)] \leqslant E[f_0(K_{n+1})]$. In the planar case, the number of edges is also an increasing function because $f_0(K_n) = f_1(K_n)$. □

3 Random sampling

We denote by $\sharp S$ the cardinality of a finite set S and we let $\mathbb{1}_X$ denote the characteristic function of event X: $\mathbb{1}_{p\in F}$ is 1 if $p \in F$ and 0 otherwise. Let S be a finite set of points in $\mathbb{R}^d$ and let $k \geqslant 0$ be an integer. A *k-set of S* is a subset $\{p_1, p_2, \ldots p_d\} \subseteq S$ that spans a hyperplane bounding an open halfspace that contains exactly k points from S; we say that the k-set *cuts off* these k points. In particular, if S is a set of points in general position, its 0-sets are the facets of its convex hull, which we denote $CH(S)$. For any finite subset S of D we let $s_k(S)$ denote the number[2] of k-sets of S.

[2] If the hyperplane separates S in two subsets of k elements ($2k + d = n$) the k-set is counted twice.

Generic distribution assumption. Let $\mathcal{P}$ denote a probability distribution on $\mathbb{R}^d$; we assume throughout this section that $\mathcal{P}$ is such that d points chosen independently from $\mathcal{P}$ are generically affinely independent.

Determining the order of magnitude of the maximum number of k-sets determined by a set of n points in $\mathbb{R}^d$ has been an important open problem in discrete and computational geometry over the last decades. In the case $d = 2$, Clarkson [4] gave an elegant proof that $s_{\leqslant k}(S) = O(nk)$ for any set S of n points in the plane, where:

$$s_{\leqslant k}(S) = s_0(S) + s_1(S) + \ldots + s_k(S).$$

(See also Clarkson and Shor [5] and Chazelle [3, Appendix A.2].) Although the statement holds for any fixed point set, and not only in expectation, Clarkson's argument is probabilistic and will be our main ingredient. It goes as follows. Let R be a subset of S of size r, chosen randomly and uniformly among all such subsets. An i-set of S becomes a 0-set in R if R contains the two points defining the i-set but none of the i points it cuts off. This happens approximately with probability $p^2(1-p)^i$, where $p = \frac{r}{n}$ (see [5] for exact computations). It follows that:

$$E[s_0(R)] \geqslant \sum_{0 \leqslant i \leqslant k} p^2(1-p)^i s_i(S) \geqslant p^2(1-p)^k s_{\leqslant k}(S).$$

Since $E[s_0(R)]$ cannot exceed $\sharp R = r$, we have $s_{\leqslant k}(S) \leqslant \frac{n}{p(1-p)^k}$ which, for $p = 1/k$, yields the announced bound $s_{\leqslant k}(S) = O(nk)$. A similar random sampling argument yields the following inequalities.

Lemma 3.1. *Let $s_k(n)$ denote the expected number of k-sets in a set of n points chosen independently from $\mathcal{P}$. We have*

$$s_0(n) = s_0(n-1) + \frac{d\, s_0(n) - s_1(n)}{n} \tag{3.1}$$

and for any integer $1 \leqslant r \leqslant n$

$$s_0(r) \geqslant \frac{\binom{n-d}{r-d}}{\binom{n}{r}} s_0(n) + \frac{\binom{n-d-1}{r-d}}{\binom{n}{r}} s_1(n). \tag{3.2}$$

Proof. Let S be a set of n points chosen, independently, from $\mathcal{P}$ and let $q \in S$. The 0-sets of S that are not 0-sets of $S \backslash \{q\}$ are precisely those that contain q. Conversely, the 0-sets of $S \backslash \{q\}$ that are not 0-sets of S are precisely those 1-sets of S that cut off q. We can thus write:

$$s_0(S) = s_0(S \backslash \{q\}) + \sum_{F \text{ facet of } CH(S)} \mathbb{1}_{q \in F} - \sharp\text{1-sets cutting off } q.$$

Note that the equality remains true in the degenerate cases where several points of S are identical or in a non generic position if we count the facets of the convex hull of S with multiplicities in the sum and in $s_0(n)$. Summing the previous identity over all points q of S, we obtain

$$ns_0(S) = \left(\sum_{q \in S} s_0(S \backslash \{q\}) \right) + \left(\sum_{F \text{ facet of } CH(S)} \sum_{q \in S} \mathbb{1}_{q \in F} \right) - s_1(S),$$

and since a facet of $CH(S)$ generically has d vertices,

$$ns_0(S) = \left(\sum_{q\in S} s_0(S\backslash\{q\})\right) + ds_0(S) - s_1(S) \quad \text{almost surely}.$$

Taking n points chosen randomly and independently from $\mathcal{P}$, then deleting one of these points, chosen randomly with equiprobability, is the same as taking $n-1$ chosen randomly and independently from $\mathcal{P}$. We can thus average over all choices of S and obtain

$$ns_0(n) = ns_0(n-1) + d\, s_0(n) - s_1(n),$$

which implies Equality (3.1).

Now, let $r \leqslant n$ and let R be a random subset of S of size r, chosen uniformly among all such subsets. For $k \leqslant \frac{r}{2}$, a k-set of S is a 0-set of R if R contains that k-set and none of the k points it cuts off. For each fixed k-set A of S, there are therefore $\binom{n-d-k}{r-d}$ distinct choices of R in which A is a 0-set. Counting the expected number of 0-sets and 1-sets of S that remain/become a 0-set in R, and ignoring those 0-sets of R that were k-sets of S for $k \geqslant 2$, we obtain:

$$E[s_0(R)] \geqslant \frac{\binom{n-d}{r-d}}{\binom{n}{r}} s_0(S) + \frac{\binom{n-d-1}{r-d}}{\binom{n}{r}} s_1(S).$$

Recall that the expectation is taken over all choices of a subset R of r elements of the fixed point set S. We can now average over all choices of a set S of n points taken, independently, from $\mathcal{P}$, and obtain Inequality (3.2). □

Letting $p = \frac{r-d}{n-d}$, Inequality (3.2) yields the, perhaps more intuitive, inequality:

$$s_0(r) \geqslant p^d s_0(n) + p^d(1-p)s_1(n).$$

We can now prove our main result.

Theorem 3.2. *Let Z_n denote the convex hull of n points chosen independently from $\mathcal{P}$. If $E[f_{d-1}(Z_n)] \sim An^c$ when $n \to \infty$, for some A, $c > 0$, then there exists an integer n_0 such that for any $n \geqslant n_0$ we have $E[f_{d-1}(Z_{n+1})] > E[f_{d-1}(Z_n)]$.*

Proof of Theorem 3.2. Let $\mathcal{P}$ be a probability distribution on $\mathbb{R}^d$ and let $s_k(n)$ denote the expected number of k-sets in a set of n points chosen independently from $\mathcal{P}$. Recall that s_0 counts the expected number of facets in the convex hull of n points chosen independently and uniformly from $\mathcal{P}$. By Equality (3.1), for s_0 to be increasing it suffices that $s_1(n)$ be bounded from above by $ds_0(n)$.

Let $\psi(r,n) = \frac{s_0(r)}{s_0(n)}$. Substituting into Inequality (3.2),

$$\psi(r,n)s_0(n) \geqslant \frac{\binom{n-d}{r-d}}{\binom{n}{r}} s_0(n) + \frac{\binom{n-d-1}{r-d}}{\binom{n}{r}} s_1(n),$$

which rewrites as

$$s_1(n) \leqslant \frac{\binom{n-d}{r-d}}{\binom{n-d-1}{r-d}} \left(\psi(r,n)\frac{\binom{n}{r}}{\binom{n-d}{r-d}} - 1\right) s_0(n) \tag{3.3}$$

We let $q = \frac{n-d}{r-d}$. Developing the binomial expressions:

$$\frac{\binom{n-d}{r-d}}{\binom{n-d-1}{r-d}} = \frac{n-d}{n-r} = \frac{q}{q-1} \qquad \text{and} \qquad \frac{\binom{n}{r}}{\binom{n-d}{r-d}} = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdots \frac{n-d+1}{r-d+1}$$

And for $0 \leqslant k < d$ we have $\frac{n-k}{r-k} < \frac{n-d}{r-d} = q$. Thus:

$$s_1(n) \leqslant q \frac{\psi(r,n)\frac{n}{r}q^{d-1} - 1}{q-1} s_0(n)$$

Assume now that we know a function g such that $s_0(n) \sim g(n)$. Then for any $\frac{1}{2} > \epsilon > 0$, there is $N_\epsilon \in \mathbb{N}$ such that for all $n > r > N_\epsilon$ we have:

$$\psi(r,n) = \frac{s_0(r)}{s_0(n)} < \left(\frac{1+\epsilon}{1-\epsilon}\right) \frac{g(r)}{g(n)} < (1+4\epsilon)\frac{g(r)}{g(n)}$$

which gives:

$$s_1(n) \leqslant q \frac{\frac{g(r)/r}{g(n)/n} q^{d-1} - 1}{q-1} s_0(n) + 4\epsilon \left(\frac{g(r)/r}{g(n)/n}\right) \frac{q^d}{q-1} s_0(n). \tag{3.4}$$

In the case where $s_0(n) \sim An^c$, we have:

$$\frac{g(r)/r}{g(n)/n} = \left(\frac{n}{r}\right)^{1-c} < q^{1-c}$$

And plugging back in Equation (3.4), we get:

$$s_1(n) \leqslant q \frac{q^{d-c} - 1}{q-1} s_0(n) + 4\epsilon \frac{q^{d+1}}{q-1} s_0(n) \tag{3.5}$$

The expression $q\frac{q^{d-c}-1}{q-1}$ converges toward $d-c$ when q approaches 1. Thus, there exists ϵ_d such that for all $1 < q < 1 + \epsilon_d$:

$$q\frac{q^{d-c} - 1}{q-1} < d - \frac{c}{2}$$

The second term of Equation (3.5) is bounded for all $1 + \frac{\epsilon_d}{2} < q < 1 + \epsilon_d$ by:

$$4\epsilon \frac{q^{d+1}}{q-1} s_0(n) < 8\epsilon \frac{(1+\epsilon_d)^{d+1}}{\epsilon_d} s_0(n)$$

Finally, let $\epsilon = \frac{c\epsilon_d}{32(1+\epsilon_d)^{d+1}}$. For all r such that:

$$N_\epsilon < r < n \qquad \text{and} \qquad 1 + \frac{\epsilon_d}{2} < \frac{n-d}{r-d} < \epsilon_d \tag{3.6}$$

we have:

$$s_1(n) < (d - \frac{c}{4}) s_0(n)$$

With Equality (3.1) this implies that $s_0(n) > s_0(n-1)$.

It remains to check that for n large enough, there always exists r satisfying Condition (3.6). We can rewrite condition (3.6) as:

$$d + \frac{n-d}{1+\epsilon_d} < r < d + \frac{n-d}{1+\epsilon_d/2}$$

In particular, there exists an integer r satisfying this condition as soon as $\frac{n-d}{1+\epsilon_d/2} - \frac{n-d}{1+\epsilon_d} > 1$. Thus, as soon as $n > max(N_\epsilon, d + \frac{1}{\frac{1}{1+\epsilon_d/2} - \frac{1}{1+\epsilon_d}})$, Condition (3.6) is satisfied, which concludes the proof. □

Now, from equations (1.1) and (1.2) we can see that Theorem 3.2 holds for random polytopes K_n when K is smooth, but not when K is a polytope. This proves that for smooth K the expectation $E[f_{d-1}(K_n)]$ is asymptotically increasing, i.e. the first part of Theorem 1.2.

The genericity assumption on $\mathcal{P}$ implies that the convex hull Z_n of n points chosen independently from $\mathcal{P}$ is almost surely simplicial. Thus, in Z_n any $(d-1)$-face is almost surely incident to exactly d faces of dimension $d-2$ and $f_{d-2}(Z_n) = \frac{d}{2} f_{d-1}(Z_n)$. Theorem 3.2 therefore implies that $f_{d-2}(Z_n)$ is asymptotically increasing, i.e. the second part of Theorem 1.2.. For $d = 3$, with Euler's relation this further implies that $f_0(Z_n)$ is asymptotically increasing.

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