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HARDY OPERATORS AND COMMUTATORS ON GENERALIZED CENTRAL FUNCTION SPACES

LE TRUNG NGHIA

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Abstract. In this article, we study the boundedness of operators of Hardy type on generalized central function spaces, such as the generalized central Hardy space $\mathbf{HA}_{\varphi}^{p,r}(\mathbb{R}^n)$, the generalized central Morrey space $\dot{\mathbf{M}}_{\varphi}^{p,r}(\mathbb{R}^n)$, and the generalized central Campanato space $\mathrm{C\dot{M}O}_{\varphi}^{p,r}(\mathbb{R}^n)$, with $p \in (1,\infty)$, and $\varphi(t) : (0,\infty) \to (0,\infty)$. We first show that $\mathbf{HA}_{\varphi}^{p',r'}(\mathbb{R}^n)$ is the predual of $\mathrm{C\dot{M}O}_{\varphi}^{p,r}(\mathbb{R}^n)$. After that, we investigate the boundedness of operators of Hardy type on those spaces. By duality, we obtain the boundedness characterization of function $b \in \mathrm{C\dot{M}O}_{\varphi}^{p,r}(\mathbb{R}^n)$ via the $\dot{\mathbf{M}}_{\varphi}^{p,r}(\mathbb{R}^n)$-boundedness of commutator $[b, \mathcal{H}^*]$.

1. Introduction and main results

Firstly, we introduce a singular solution outside Sobolev spaces and its description via central Morrey and Campanato spaces.

Problem setting. We consider the semilinear elliptic equation on the unit ball $B := B_1(0) \subset \mathbb{R}^n$ (with $n \geq 3$)

$$-\Delta u - \frac{\mu}{|x|^2} u = u^q \quad \text{in } B, \quad u > 0, \quad u \in H^1_{\rm loc}(B \setminus \{0\}) \tag{1.1}$$

with

- $\mu \in \left(0, \left(\frac{n-2}{2}\right)^2\right)$ (subcritical Hardy potential),
- $q > 1$ subcritical,
- $u = 0$ on ∂B (in weak sense).

Explicit singular solution. A classical singular approximate solution is given by

$$u(x) = C|x|^{-\gamma}, \quad \text{where } \gamma = \frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 - \mu}. \tag{1.2}$$

This function is

- Not in $H^1(B)$ because
$$\int_B |\nabla u|^2 dx \sim \int_0^1 r^{n-1-2(\gamma+1)} dr = \infty$$
- In $H^1_{\rm loc}(B \setminus \{0\})$, with singularity at the origin.

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Why the solution is independent of q. Let us verify whether $u(x) = C|x|^{-\gamma}$ solves the full nonlinear equation

$$-\Delta u - \frac{\mu}{|x|^2} u = u^q. \tag{1.3}$$

We compute

$$\Delta(|x|^{-\gamma}) = \gamma(\gamma + 2 - n)|x|^{-\gamma-2},$$

so that

$$-\Delta u - \frac{\mu}{|x|^2} u = C\left[\gamma(\gamma + 2 - n) - \mu\right]|x|^{-\gamma-2}.$$

We compare this with $u^q = C^q|x|^{-\gamma q}$, and for equality we must have

$$-\gamma - 2 = -\gamma q \quad \Rightarrow \quad \gamma(q-1) = 2 \quad \Rightarrow \quad \gamma = \frac{2}{q-1}.$$

Hence, the singular function $u(x) = |x|^{-\gamma}$ solves the full nonlinear equation only when $\gamma = \frac{2}{q-1}$. Thus, $u(x) = |x|^{-\gamma}$ is a solution to the *nonlinear* problem only when

$$\frac{2}{q-1} = \frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 - \mu}.$$

But in our example, we fixed:

$$\gamma = \frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 - \mu},$$

which only matches $\frac{2}{q-1}$ for a specific q. Therefore, u is not a solution to the nonlinear equation for general q, but it serves as a model to study the singularity and local behavior, independently of the nonlinearity. It serves as a model to study the singular behavior of more general solutions, particularly near the origin.

Membership in central Morrey spaces. We test whether $u \in \mathcal{L}^{p,\lambda}(B)$:

$$\|u\|_{\mathcal{L}^{p,\lambda}} := \sup_{r<1} r^{-\lambda}\Big(\int_{B(0,r)} |u(x)|^p dx\Big)^{1/p} < \infty. \tag{1.4}$$

Let $u(x) = |x|^{-\gamma}$, then

$$\int_{B(0,r)} |x|^{-p\gamma} dx \sim \int_0^r \rho^{n-1-p\gamma} d\rho = r^{n-p\gamma} \quad \Rightarrow \quad \|u\|_{\mathcal{L}^{p,\lambda}} \sim r^{-\lambda + \frac{n}{p} - \gamma}.$$

So $u \in \mathcal{L}^{p,\lambda}$ for

$$\lambda < \frac{n}{p} - \gamma. \tag{1.5}$$

Oscillation near zero: Campanato spaces. Consider the Campanato seminorm

$$\|f\|^2_{\mathcal{L}^{2,\lambda}_{\rm Camp}} := \sup_{r<1} r^{-\lambda} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^2 dx. \tag{1.6}$$

For $f(x) = |x|^{-\gamma}$, the mean oscillation behaves like

$$r^{-\lambda} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^2 dx \sim r^{n-2\gamma-\lambda}. \tag{1.7}$$

So $f \in \mathcal{L}^{2,\lambda}_{\rm Camp}(B)$ for

$$\lambda < n - 2\gamma. \tag{1.8}$$

In particular, since BMO corresponds to $\lambda = n$, the function $f(x) = |x|^{-\gamma}$ is not in BMO, but lies in a Campanato space with smaller λ.

In summary. the function $u(x) = |x|^{-\gamma}$

- is not in the Sobolev space $H^1(B)$,
- is in the central Morrey space $\mathcal{L}^{p,\lambda}(B)$ for suitable λ,
- is in the Campanato space $\mathcal{L}^{2,\lambda}(B)$ for $\lambda < n - 2\gamma$,
- is not in BMO,

- serves as a barrier function that captures the singularity of the Hardy potential, independent of the nonlinearity.

This demonstrates how generalized central spaces such as Morrey and Campanato capture the behavior of singular solutions to elliptic PDEs with Hardy-type potentials, even when the nonlinearity is not presented.

Building on this observation, our studies are the following twofold. First, we study some generalized central function spaces, such as $\dot{\mathbf{M}}^{p,r}_{\varphi}(\mathbb{R}^n)$, $\dot{\mathrm{CMO}}^{p,r}_{\varphi}(\mathbb{R}^n)$, and $\mathbf{HA}^{p,r}_{\varphi}(\mathbb{R}^n)$, where $p \in (1,\infty)$. Through the paper, we always assume that $\varphi(t)$ is non-increasing on $(0,\infty)$, and $t^{\frac{n}{p}}\varphi(t)$ is nondecreasing on $(0,\infty)$. Then, we demonstrate that $\mathbf{HA}^{p',r'}_{\varphi}(\mathbb{R}^n)$ is the predual of $\dot{\mathrm{CMO}}^{p,r}_{\varphi}(\mathbb{R}^n)$. Second, we investigate the boundedness of operators of Hardy type on those spaces. By duality, we obtain the boundedness characterization of function b in $\dot{\mathrm{CMO}}^{p,r}_{\varphi}(\mathbb{R}^n)$ by means of the boundedness of commutators $[b,\mathcal{H}]$ and $[b,\mathcal{H}^*]$ in the above central function spaces.

Notation: For $q \in (1,\infty)$, we denote q' the conjugate exponent, $\frac{1}{q}+\frac{1}{q'}=1$. With $|\Omega|$ we denote the Lebesgue measure of a measurable set Ω in $\mathbb{R}^n$, and B_t is the ball centered at $0 \in \mathbb{R}^n$ with radius t. As usual, we denote a constant by C, which may depend on p, n and is probably different at different occurrences. Finally, we denote $A \lesssim B$ if there exists a constant $C > 0$ such that $A \le CB$.

The Hardy operator is defined by

$$\mathcal{H}(f)(x) = \frac{1}{\nu_n |x|^n} \int_{|y|<|x|} f(y)\,dy, \quad x \in \mathbb{R}^n \setminus \{0\}\,, \tag{1.9}$$

and its dual form is

$$\mathcal{H}^*(f)(x) = \frac{1}{\nu_n} \int_{|y|\ge|x|} \frac{f(y)}{|y|^n} dy, \quad x \in \mathbb{R}^n \setminus \{0\}\,, \tag{1.10}$$

where $\nu_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the volume of unit ball in $\mathbb{R}^n$. In the pioneering work, when $n=1$, Hardy [17] established the integral inequality

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)^p dx \le \Big(\frac{p}{p-1}\Big)^p \int_0^\infty f(x)^p\,dx \tag{1.11}$$

for all non-negative $f \in L^p(\mathbb{R}_+)$, with $1 < p < \infty$. Note that the constant $\frac{p}{p-1}$ is sharp.

By considering two-sided averages of f instead of one-sided, (1.11) can be equivalently formulated as

$$\|\mathcal{H}(f)\|_{L^p(\mathbb{R})} \le \frac{p}{p-1}\|f\|_{L^p(\mathbb{R})}\,. \tag{1.12}$$

Christ-Grafakos [4] extended (1.12) to n-dimension. Furthermore, a sharp bound of weak type (p,p) of $\mathcal{H}$ was obtained by the authors in [12]. Specifically, for any $1 \le p \le \infty$ we have

$$\|\mathcal{H}(f)\|_{L^{p,\infty}} \le \|f\|_{L^{p,r}}$$

for all $f \in L^p(\mathbb{R}^n)$. In addition,

$$\|\mathcal{H}\|_{L^p \to L^{p,\infty}} = 1\,.$$

It is known that the inequalities of Hardy type play important roles in many areas of mathematics such as analysis, probability and partial differential equations (see, e.g., [1, 2, 4, 11, 16, 18, 21, 23] and the references therein). For example, a slight modification of (1.11) by setting $F(x) = \int_0^x f(t)\,dt$ provides us

$$\int_0^\infty \frac{F(x)^p}{x^p}\,dx \le \Big(\frac{p}{p-1}\Big)^p \int_0^\infty F'(x)^p\,dx\,.$$

The analogue of this inequality in $\mathbb{R}^n$ for $n > 1$ is

$$\int_{\mathbb{R}^n} \Big|\frac{f(x)}{x}\Big|^p \le \Big(\frac{p}{n-p}\Big)^p \int_{\mathbb{R}^n} |\nabla f(x)|^p\,dx \tag{1.13}$$

where ∇f is the gradient of f as usual; this holds for all $f \in \mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ if $n < p < \infty$, and for all $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ if $1 \le p < n$. The constant is sharp and equality can only be attained by functions $f = 0$ a.e.

Since the Hardy operators are centrosymmetric, the function spaces, which are characterized by the boundedness of $\mathcal{H}$ and $\mathcal{H}^*$ are central ones. For example, Shi-Lu, [25] established the boundedness of $\mathcal{H}$ and $\mathcal{H}^*$ in the central Morrey spaces $\dot{\mathbf{M}}_{\varphi}^{p,\lambda}(\mathbb{R}^n)$ (see Definition 1.2).

Theorem 1.1 (Shi-Lu [25])**.** *Let $1 < p < \infty$ and $\lambda \in (0, \frac{n}{p})$. Then $\mathcal{H}$ (resp. $\mathcal{H}^*$) is a bounded operator from $\dot{\mathbf{M}}_{\varphi}^{p,\lambda}(\mathbb{R}^n) \to \dot{\mathbf{M}}_{\varphi}^{p,\lambda}(\mathbb{R}^n)$.*

Moreover, the boundedness characterization of operators of Hardy type in the homogeneous Herz spaces has been studied by the authors in [13]. Inspired by the above results, we would like to study the boundedness of operators of Hardy type in generalized central function spaces. Therefore, it is convenient for us to introduce the notions of those spaces.

Definition 1.2. A real-valued function f is said to belong to the generalized central Morrey space $\dot{\mathbf{M}}_{\varphi}^{p,r}(\mathbb{R}^n)$ provided the following norm is finite:

$$\|f\|_{\dot{\mathbf{M}}_{\varphi}^{p,r}} = \sup_{B_t} \frac{\|f\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)}\,,$$

where the supremum is taken over all the balls B_t in $\mathbb{R}^n$.

Remark 1.3. A canonical example is $\varphi(t) = t^{-\lambda}$, $\lambda \in (0, \frac{n}{p})$. In this case, we denote $\dot{\mathbf{M}}_{\varphi}^{p,r}(\mathbb{R}^n)$ by $\dot{\mathbf{M}}^{p,\lambda}(\mathbb{R}^n)$.

Next, let us define the φ-central Campanato space $\dot{\mathrm{CMO}}_{\varphi}^{p,r}(\mathbb{R}^n)$.

Definition 1.4. A function $f \in L^p_{\rm loc}(\mathbb{R}^n)$ is said to belong to $\dot{\mathrm{CMO}}_{\varphi}^{p,r}(\mathbb{R}^n)$ if

$$\|f\|_{\dot{\mathrm{CMO}}_{\varphi}^{p,r}} := \sup_{t>0} \frac{\|f - f_{B_t}\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} < \infty\,,$$

with $f_B = \frac{1}{|B|}\int_B f(y)\,dy$, for set B in $\mathbb{R}^n$.

Remark 1.5. When $\varphi(t) \equiv 1$, we denote $\dot{\mathrm{CMO}}_{\varphi}^{p,r}(\mathbb{R}^n)$ by $\dot{\mathrm{CMO}}^{p,r}(\mathbb{R}^n)$ for short. And, if $\varphi(t) = t^{-\lambda}$, $\lambda \in (0, \frac{n}{p}]$, we denote $\dot{\mathrm{CMO}}_{\varphi}^{p,r}(\mathbb{R}^n)$ by $\dot{\mathrm{CMO}}^{p,\lambda}(\mathbb{R}^n)$.

Remark 1.6. If there exists a constant $D_0 \in (0,1)$ such that $\varphi(2t) \le D_0\varphi(t)$ for all $t > 0$, then by using the same argument as in [30], we also obtain

$$\dot{\mathbf{M}}_{\varphi}^{p,r}(\mathbb{R}^n) = \dot{\mathrm{CMO}}_{\varphi}^{p,r}(\mathbb{R}^n)\,. \tag{1.14}$$

In particular, we have $\dot{\mathbf{M}}^{p,\lambda}(\mathbb{R}^n) = \dot{\mathrm{CMO}}^{p,\lambda}(\mathbb{R}^n)$, with $\lambda \in (0, \frac{n}{p}]$.

Remark 1.7. Obviously, for $1 \le p_1 < p_2$ we have

$$\dot{\mathrm{CMO}}_{\varphi}^{p_2,r}(\mathbb{R}^n) \subset \dot{\mathrm{CMO}}_{\varphi}^{p_1,r}(\mathbb{R}^n)\,. \tag{1.15}$$

Moreover, it is known that

$$\mathrm{BMO}(\mathbb{R}^n) \subsetneq \dot{\mathrm{CMO}}^{p_2,r}(\mathbb{R}^n) \subsetneq \dot{\mathrm{CMO}}^{p_1,r}(\mathbb{R}^n)\,. \tag{1.16}$$

We emphasize that $\dot{\mathrm{CMO}}^{p,r}(\mathbb{R}^n)$ depends on p. Therefore, there is no analogy of the famous John-Nirenberg inequality of $\mathrm{BMO}(\mathbb{R}^n)$ for the space $\dot{\mathrm{CMO}}^{p,r}(\mathbb{R}^n)$.

Our last interested central function space is the generalized central Hardy space. To define this space, we first point out the definition of a central $(1, q, \varphi)$-atom.

Definition 1.8. Let $1 < p \le \infty$, and $\varphi(t) : (0,\infty) \to (0,\infty)$. A function $a(x)$ is called a central $(1, p, r, \varphi)$-atom, if there exists a ball B_t in $\mathbb{R}^n$ such that

(i) $\operatorname{supp}(a) \subset B_t$,
(ii) $\int_{B_t} a(x)\,dx = 0$,
(iii) $\|a\|_{L^{p,r}} \le \frac{1}{|B_t|^{1/q'}\varphi(t)}$.

Now, we are ready to define $\mathbf{HA}_{\varphi}^{p,r}(\mathbb{R}^n)$ (see definition $\mathbf{H}^{p,\varphi}(\mathbb{R}^n)$ by Zorko [30]).

Definition 1.9. Let $1 < p < \infty$, and let $\varphi(t) : (0,\infty) \to (0,\infty)$. We denote, by $\mathbf{HA}^{p,r}_{\varphi}(\mathbb{R}^n)$, the family of distributions h that, in the sense of distributions, can be written as

$$h = \sum_{j=0}^{\infty} \lambda_j a_j \,,$$

where a_j, $j \geq 0$ are central $(1, p, r, \varphi)$-atoms, and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$.

It is clear that $\mathbf{HA}^{p,r}_{\varphi}(\mathbb{R}^n)$ is a vector space. In addition, we denote

$$\|h\|_{\mathbf{HA}^{p,r}_{\varphi}} = \inf \Big\{ \sum_{j=0}^{\infty} |\lambda_j| \Big\},$$

where the infimum is taken over all possible decompositions of h as above.

Then $\big(\mathbf{HA}^{p,r}_{\varphi}(\mathbb{R}^n), \|\cdot\|_{\mathbf{HA}^{p,r}_{\varphi}}\big)$ becomes a normed space.

Such a space of this type has been studied by the authors in [3, 15, 14] and in the references cited therein when $\varphi(t) \equiv 1$. In fact, Chen–Lau, [3] studied a theory of Hardy spaces $\mathbf{HA}^{p,r}_{\varphi}(\mathbb{R})$ associated with the Beurling algebras $\mathbf{A}^p$, $1 < p < \infty$, the space consisting of functions f on $\mathbb{R}^n$ for which

$$\|f\|_{\mathbf{A}^p} = \sum_{k=0}^{\infty} 2^{\frac{kn}{p'}} \|f\chi_k\|_{L^p} < \infty \,,$$

where χ_k is the characteristic function on the set $\big\{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\big\}$, $k \geq 1$. For convenience, we recall here the definition of $\mathbf{HA}^{p,r}_{\varphi}(\mathbb{R})$ via the Beurling algebras $\mathbf{A}^p$.

Definition 1.10. Let f^* be the vertical maximal function, defined by

$$f^*(x) = \sup_{t>0} \big|(f * \psi_t)(x)\big| \,,$$

where $\psi_t(x) = t^{-n}\psi(x/t)$, and ψ is an integrable function on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \psi(x)\, dx = 1$.

Then, we define $\mathbf{HA}^{p,r}_{\varphi}(\mathbb{R})$ by the set of functions f such that $\|f^*\|_{\mathbf{A}^p}$ is finite. Moreover, if we set $\|f\|_{\mathbf{HA}^{p,r}_{\varphi}} = \|f^*\|_{\mathbf{A}^p}$, then $\|\cdot\|_{\mathbf{HA}^{p,r}_{\varphi}}$ is a norm.

The most interesting aspect of the theory constructed by Chen-Lau is the atomic decomposition of $\mathbf{HA}^{p,r}(\mathbb{R})$, for $1 < p \leq 2$. Thanks to this decomposition, they obtained the duality

$$\mathbf{HA}^{p',r'}_{\varphi}(\mathbb{R}^n)^* = \dot{\mathrm{CMO}}^{p,r}(\mathbb{R}^n) \,. \tag{1.17}$$

After that, García-Cuerva [15] extended their results for all $p \in (1,\infty)$ by using the characterizations via the grand maximal functions. Moreover, the associated spaces $\mathbf{HA}^{q,p}$, $0 < q < 1$, $1 < p \leq \infty$ was investigated by the authors in [14].

Remark 1.11. Obviously, for any $1 < p_1 < p_2 \leq \infty$ we have

$$\mathbf{HA}^{p_2,r}_{\varphi}(\mathbb{R}^n) \subset \mathbf{HA}^{p_1,r}_{\varphi}(\mathbb{R}^n) \,. \tag{1.18}$$

It is interesting to emphasize that when $\varphi(t) \equiv 1$ the inclusion in (1.18) is strictly according to (1.16) and (1.17). This observation is different from the point of view of the classical Hardy spaces. That is

$$\mathbf{H}^{1,\infty}(\mathbb{R}^n) = \mathbf{H}^{1,q}(\mathbb{R}^n) \tag{1.19}$$

for $1 < q < \infty$, see Theorem A, [6]. By (1.19), one can define $\mathbf{H}^1(\mathbb{R}^n)$ (the real Hardy space) to be any one of the spaces $\mathbf{H}^{1,q}(\mathbb{R}^n)$ for $1 < q \leq \infty$.

Next, we discuss the commutators of Hardy operators. For any operator T, let us define

$$[b,T](f) := bT(f) - T(bf) \,.$$

Note that b is called the symbol function of $[b,T]$. When T is an operator of Hardy type, the study of $[b,T]$ has been investigated by many authors in [12, 20, 13, 24, 27, 26, 25, 23, 22], and the references therein. In [22], Long–Wang proved Hardy's integral inequalities for commutators $[b,\mathcal{H}]$ and $[b,\mathcal{H}_\beta]$ (the fractional Hardy operator), $\beta \in (0,1)$, with b belongs to the one-sided dyadic functions $\dot{\mathrm{CMO}}^{p,r}(\mathbb{R}^+)$. Moreover, Fu et al., [13] obtained some characterizations of $\dot{\mathrm{CMO}}^{p,r}(\mathbb{R}^n)$ for $1 < p < \infty$ via the L^p-boundedness of $[b,\mathcal{H}]$ and $[b,\mathcal{H}^*]$ in the following theorem.

Theorem 1.12 (Fu et al. [13])**.** *Let $b \in \dot{\mathrm{CMO}}^{\max\{p,p'\},r}(\mathbb{R}^n)$. Then both $[b,\mathcal{H}]$ and $[b,\mathcal{H}^*]$ are bounded on L^p. Conversely,*

- (a) *if $[b,\mathcal{H}]$ is bounded on L^p, then $b \in \dot{\mathrm{CMO}}^{p',r}(\mathbb{R}^n)$;*
- (b) *if $[b,\mathcal{H}^*]$ is bounded on L^p, then $b \in \dot{\mathrm{CMO}}^{p,r}(\mathbb{R}^n)$.*

We also mention that Komori, [20] obtained a characterization of function $b \in \dot{\mathrm{CMO}}^{p,r}(\mathbb{R}^+)$ by means of the L^p-boundedness of $[b,\mathcal{H}]$ and $[b,\mathcal{H}^*]$. Note that his argument can be adapted for the setting of the Euclidean space $\mathbb{R}^n$ instead of $\mathbb{R}^+$. Lu-Zhao, [24] extended Theorem 1.12 to the space $\dot{\mathrm{CMO}}^{\max\{p,q'\},\lambda}(\mathbb{R}^n)$ as follows.

Theorem 1.13 (Lu-Zhao, [24])**.** *Let $1 < q < p < \infty$ be such that $0 < \lambda = \frac{1}{q} - \frac{1}{p} < \frac{1}{n}$. Then $b \in \dot{\mathrm{CMO}}^{\max\{p,q'\},\lambda}(\mathbb{R}^n) \Longleftrightarrow [b,\mathcal{H}],[b,\mathcal{H}^*] : L^q(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.*

1.1. **Main results.** As mentioned at the beginning, our first result is the following duality.

Theorem 1.14. *Let $1 < p < \infty$, and $\varphi(t) : (0,\infty) \to (0,\infty)$. Then, we have*

$$\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)^* = \dot{\mathrm{CMO}}_\varphi^{p,r}(\mathbb{R}^n)\,.$$

Remark 1.15. As a consequence of Theorem 1.14, we observe that $\dot{\mathrm{CMO}}_\varphi^{p,r}(\mathbb{R}^n)$ is a Banach space.

Next, we extend Theorem 1.1 to $\dot{\mathbf{M}}_\varphi^{p}(\mathbb{R}^n)$.

Theorem 1.16. *Let $1 < p < \infty$. Assume that there is a constant $D_0 \in (0,1)$ such that*

$$\varphi(2t) \le D_0\varphi(t), \quad \forall t > 0\,. \tag{1.20}$$

Then, $\mathcal{H}$ (resp. $\mathcal{H}^*$) is a bounded operator from $\dot{\mathbf{M}}_\varphi^{p,r}(\mathbb{R}^n) \to \dot{\mathbf{M}}_\varphi^{p,r}(\mathbb{R}^n)$. In addition, we have

$$\|\mathcal{H}(f)\|_{\dot{\mathbf{M}}_\varphi^{p,r}} \le \Big(\frac{p}{p-1}\Big)\|f\|_{\dot{\mathbf{M}}_\varphi^{p,r}} \tag{1.21}$$

for $f \in \dot{\mathbf{M}}_\varphi^{p,r}(\mathbb{R}^n)$; and there is a constant $C = C(n,p) > 0$ such that

$$\|\mathcal{H}^*(f)\|_{\dot{\mathbf{M}}_\varphi^{p,r}} \le C\|f\|_{\dot{\mathbf{M}}_\varphi^{p,r}} \tag{1.22}$$

for $f \in \dot{\mathbf{M}}_\varphi^{p,r}(\mathbb{R}^n)$.

Remark 1.17. We emphasize that condition (1.20) can be relaxed in the $\dot{\mathbf{M}}_\varphi^{p,r}$-boundedness of $\mathcal{H}$, see the proof of Theorem 1.16. This means that one can take $\varphi(t) \equiv C > 0$ in (1.21).

As a consequence of Theorem 1.16, Remark 1.6, and Theorem1.14, we have the following corollary.

Corollary 1.18. *Theorem 1.16. Then, the following statements hold*

- (a) *$\mathcal{H}$ and $\mathcal{H}^*$ are bounded operators from $\dot{\mathrm{CMO}}_\varphi^{p,r}(\mathbb{R}^n) \to \dot{\mathrm{CMO}}_\varphi^{p,r}(\mathbb{R}^n)$;*
- (b) *$\mathcal{H}$ and $\mathcal{H}^*$ are bounded operators from $\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n) \to \mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)$.*

Concerning the boundedness of commutators of Hardy operator, we have the following theorem.

Theorem 1.19. *Assume hypotheses of Theorem 1.16. If $b \in \dot{\mathrm{CMO}}^{\max\{p,p'\},r}(\mathbb{R}^n)$, then the following statements hold*

- (a) *$[b,\mathcal{H}]$ (resp. $[b,\mathcal{H}^*]$) is a bounded operator from $\dot{\mathbf{M}}_\varphi^{p,r}(\mathbb{R}^n) \to \dot{\mathbf{M}}_\varphi^{p,r}(\mathbb{R}^n)$;*
- (b) *$[b,\mathcal{H}]$ (resp. $[b,\mathcal{H}^*]$) is a bounded operator from $\dot{\mathbf{M}}_\varphi^{p',r}(\mathbb{R}^n) \to \dot{\mathbf{M}}_\varphi^{p',r}(\mathbb{R}^n)$.*

Remark 1.20. Similarly as in Remark 1.17, (1.20) can be relaxed for conclusion (a) of Theorem 1.19.

By duality, we have the following result.

Corollary 1.21. *Assume hypotheses in Corollary 1.18. If* $b \in \dot{\mathrm{CMO}}^{\max\{p,p'\},r}(\mathbb{R}^n)$*, then* $[b,\mathcal{H}]$ *(resp.* $[b,\mathcal{H}^*]$*) is a bounded operator on* $\dot{\mathrm{CMO}}_\varphi^{p,r}(\mathbb{R}^n)$ *and* $\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)$*.*

Typical examples for the Corollaries 1.18, 1.21 are $\varphi(t) = t^{-\lambda}$, and $\varphi(t) = \big(\frac{1}{\log(1+t)}\big)^\lambda$, for $\lambda \in (0, n/p]$. Our last result is a characterization of function b in $\dot{\mathrm{CMO}}^{p,r}(\mathbb{R}^n)$ by means of the boundedness of $[b,\mathcal{H}^*]$ in $\dot{\mathbf{M}}_\varphi^{p,r}(\mathbb{R}^n)$.

Theorem 1.22. *Assume hypotheses in Theorem 1.16. If* $b \in L^p_{\rm loc}(\mathbb{R}^n)$*, and* $[b,\mathcal{H}^*]$ *is a bounded operator on* $\dot{\mathbf{M}}_\varphi^{p,r}(\mathbb{R}^n)$*, then* $b \in \dot{\mathrm{CMO}}^{p,r}(\mathbb{R}^n)$*. Furthermore, there exists a constant* $C > 0$ *depending on* n, p *such that*

$$\|b\|_{\dot{\mathrm{CMO}}^{p,r}} \le C\|[b,\mathcal{H}^*]\|_{\dot{\mathbf{M}}_\varphi^{p,r}\to\dot{\mathbf{M}}_\varphi^{p,r}}\,. \tag{1.23}$$

By duality, we have the following corollary.

Corollary 1.23. *Assume hypotheses in Theorem 1.22. If* $b \in L^p_{\rm loc}(\mathbb{R}^n)$*, and* $[b,\mathcal{H}]$ *is a bounded operator on* $\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)$*, then* $b \in \dot{\mathrm{CMO}}^{p,r}(\mathbb{R}^n)$*. In addition, there exists a constant* $C > 0$ *depending on* n, p *such that*

$$\|b\|_{\dot{\mathrm{CMO}}^{p,r}} \le C\|[b,\mathcal{H}]\|_{\mathbf{HA}_\varphi^{p',r'}\to\mathbf{HA}_\varphi^{p',r'}}\,. \tag{1.24}$$

As a consequence of Theorem 1.22 and Corollary 1.23, we have the following result.

Corollary 1.24. *Assume hypotheses in Theorem 1.22. Suppose that* $t^{\min\{\frac{n}{p},\frac{n}{p'}\}}\varphi(t)$ *is nondecreasing on* $(0,\infty)$*, and* $b \in L^{\max\{p,p'\}}_{\rm loc}(\mathbb{R}^n)$*. Then, the following statements hold:*

(a) *If* $[b,\mathcal{H}^*]$ *is a bounded operator on* $\dot{\mathbf{M}}_\varphi^{p,r}(\mathbb{R}^n)$ *and* $\dot{\mathbf{M}}_\varphi^{p',r}(\mathbb{R}^n)$*, then* $b \in \dot{\mathrm{CMO}}^{\max\{p,p'\},r}(\mathbb{R}^n)$*. In addition, there exists a constant* $C = C(n,p) > 0$ *such that*

$$\|b\|_{\dot{\mathrm{CMO}}^{\max\{p,p'\},r}} \le C\left(\|[b,\mathcal{H}^*]\|_{\dot{\mathbf{M}}_\varphi^{p,r}\to\dot{\mathbf{M}}_\varphi^{p,r}} + \|[b,\mathcal{H}^*]\|_{\dot{\mathbf{M}}_\varphi^{p',r}\to\dot{\mathbf{M}}_\varphi^{p',r}}\right). \tag{1.25}$$

(b) *If* $[b,\mathcal{H}]$ *is a bounded operator on* $\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)$ *and* $\mathbf{HA}_\varphi^{p,r}(\mathbb{R}^n)$*, then* $b \in \dot{\mathrm{CMO}}^{\max\{p,p'\},r}(\mathbb{R}^n)$*. In addition, there exists a constant* $C = C(n,p) > 0$ *such that*

$$\|b\|_{\dot{\mathrm{CMO}}^{\max\{p,p'\},r}} \le C\left(\|[b,\mathcal{H}\|_{\mathbf{HA}_\varphi^{p',r'}\to\mathbf{HA}_\varphi^{p',r'}} + \|[b,\mathcal{H}]\|_{\mathbf{HA}_\varphi^{p,r}\to\mathbf{HA}_\varphi^{p,r}}\right). \tag{1.26}$$

Typical examples of functions satisfying Corollary 1.24 are $\varphi(t) = t^{-\lambda}$, and $\varphi(t) = \big(\frac{1}{\log(1+t)}\big)^\lambda$, for $\lambda \in (0, \min\{n/p, n/p'\}]$.

Our paper is organized as follows. We study the generalized central Hardy space, and prove Theorem 1.14 in the next section. The last section is devoted to the proof of Theorems 1.16-1.22, and of Corollary 1.18-1.24.

2. Space $\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)$ as the predual of $\dot{\mathrm{CMO}}_\varphi^{p,r}(\mathbb{R}^n)$

For any ball B in $\mathbb{R}^n$ we denote $L_0^{p,r}(B)$ by the subspace of $L^p(B)$ of functions having mean value zero. It is not difficult to verify that

$$L_0^{p,r}(B)^* = L^{p',r'}(B)/C(B)\,, \tag{2.1}$$

where $C(B)$ is the set of the functions, which are constant on B. Then, we have the following embedding result.

Proposition 2.1. *For any* $\tau > 0$*, and for* $f \in L_0^p(B_\tau)$*, we have*

$$\|\mathbf{1}_{B_\tau} f\|_{\mathbf{HA}_\varphi^{p,r}} \le |B_\tau|^{1/p'}\varphi(\tau)\|f\|_{L^{p,r}(B_\tau)}\,.$$

Proof. Let us set

$$a(x) = \frac{\mathbf{1}_{B_\tau} f(x)}{|B_\tau|^{1/p'}\varphi(\tau)\|ft\|_{L^{p,r}(B_\tau)}}\,.$$

Since $\int_{B_\tau} f(x)\,dx = 0$, then it is not difficult to verify that a is a central $(1,p,r,\varphi)$-atom. Therefore, the desired result follows from the Definition 1.9. $\square$

Remark 2.2. As a consequence of Proposition 2.1, if $f \in \mathbf{HA}^{p,r}_{\varphi}(\mathbb{R}^n)^*$, then for any $\tau > 0$ we obtain

$$\mathbf{1}_{B_\tau} f \in L^{p,r}_0(B_\tau)^* .$$

Proof of Theorem 1.14. Let a be a central $(1, p', r', \varphi)$-atom with $\operatorname{supp}(a) \subset B_t$ for some $t > 0$. Then, for any $f \in \dot{\mathrm{CMO}}^{p,r}_{\varphi}(\mathbb{R}^n)$ we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} f(x)a(x)\,dx\Big| &= \Big|\int_{B_t} \left(f(x) - f_{B_t}\right) a(x)\,dx\Big| \\ &\le \|f - f_{B_t}\|_{L^{p,r}(B_t)} \|a\|_{L^{p',r'}(B_t)} \\ &\le \frac{\|f - f_{B_t}\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} \le \|f\|_{\dot{\mathrm{CMO}}^{p,r}_{\varphi}} .\end{aligned}$$

For every $g \in \mathbf{HA}^{p',r'}_{\varphi}(\mathbb{R}^n)$, one can decompose $g = \sum_{j=0}^{\infty} \lambda_j a_j$, where $\{a_j\}_{j\ge 0}$ is a sequence of central $(1, p', r', \varphi)$-atoms; and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$. Therefore, we deduce from the last inequality that

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} f(x)g(x)\,dx\Big| &= \Big|\sum_{j=0}^{\infty} \int_{\mathbb{R}^n} \lambda_j f(x)a_j(x)\,dx\Big| \\ &\le \Big(\sum_{j=0}^{\infty} |\lambda_j|\Big) \|f\|_{\dot{\mathrm{CMO}}^{p,r}_{\varphi}} \\ &\le \|g\|_{\mathbf{HA}^{p',r'}_{\varphi}} \|f\|_{\dot{\mathrm{CMO}}^{p,r}_{\varphi}} .\end{aligned} \tag{2.2}$$

This yields

$$\dot{\mathrm{CMO}}^{p,r}_{\varphi}(\mathbb{R}^n) \subset \mathbf{HA}^{p',r'}_{\varphi}(\mathbb{R}^n)^* .$$

It remains to show that

$$\mathbf{HA}^{p',r'}_{\varphi}(\mathbb{R}^n)^* \subset \dot{\mathrm{CMO}}^{p,r}_{\varphi}(\mathbb{R}^n) . \tag{2.3}$$

Let $F \in \mathbf{HA}^{p',r'}_{\varphi}(\mathbb{R}^n)^*$. Thanks to Remark 2.2, we have that $\mathbf{1}_{B_\tau} F \in L^{p',r'}_0(B_\tau)^*$ for $\tau > 0$. By (2.1), there exists $f_\tau \in L^{p,r}(B_\tau)/C(B_\tau)$ such that

$$\langle \mathbf{1}_{B_\tau} F, g\rangle_{L^p, L^{p'}} = \int_{B_\tau} f_\tau(x)g(x)\,dx, \quad \forall g \in L^{p',r'}_0(B_\tau) . \tag{2.4}$$

Therefore, for every $0 < \tau_1 < \tau_2$, we have

$$f_{\tau_1}(x) = f_{\tau_2}(x) \quad \text{for a.e. } x \in B_{\tau_1} ,$$

which makes sense by (2.4). Next, let us define $f(x) = f_\tau(x)$ if $x \in B_\tau$. Obviously, we have $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

Now, we demonstrate that $f \in \dot{\mathrm{CMO}}^{p,r}_{\varphi}(\mathbb{R}^n)$. Indeed, for any ball B_t in $\mathbb{R}^n$, let us fix $\tau_0 > t$. Remind that $f(x) = f_t(x) \in L^{p,r}(B_t)/C(B_t)$ for $x \in B_t$. By duality (2.1), we obtain

$$\begin{aligned}\frac{\|f - f_{B_t}\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} &= \frac{1}{|B_t|^{1/p}\varphi(t)} \sup_{\|h\|_{L^{p',r'}_0(B_t)}=1} \Big|\int_{B_t} \left(f(x) - f_{B_t}\right) h(x)\,dx\Big| \\ &= \frac{1}{|B_t|^{1/p}\varphi(t)} \sup_{\|h\|_{L^{p',r'}_0(B_t)}=1} \Big|\int_{B_t} f(x)\left(h(x) - h_{B_t}\right)\,dx\Big| \\ &= \sup_{\|h\|_{L^{p',r'}_0(B_t)}=1} \Big|\int_{\mathbb{R}^n} f_{\tau_0}(x) \frac{\left(h(x) - h_{B_t}\right)\mathbf{1}_{B_t}}{|B_t|^{1/p}\varphi(t)}\,dx\Big| .\end{aligned} \tag{2.5}$$

Since $h \in L^{p'}_0(B_t)$ and $\|h\|_{L^{p',r'}(B_t)} = 1$, it follows that $\frac{(h(x)-h_{B_t})\mathbf{1}_{B_t}}{|B_t|^{1/p}\varphi(t)}$ is a central $(1, p', r', \varphi)$-atom (see the proof of Proposition 2.1), and

$$\Big\|\frac{\mathbf{1}_{B_t}\left(h(x) - h_{B_t}\right)}{|B_t|^{1/p}\varphi(t)}\Big\|_{\mathbf{HA}^{p',r'}_{\varphi}} \le 1 .$$

With this inequality noted, it follows from (2.5) that

$$\frac{\|f-f_{B_t}\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} \leq \|\mathbf{1}_{B_{\tau_0}}F\|_{(\mathbf{HA}_\varphi^{p',r'})^*}\Big\|\frac{\mathbf{1}_{B_t}(h(x)-h_{B_t})}{|B_t|^{1/p}\varphi(t)}\Big\|_{\mathbf{HA}_\varphi^{p',r'}} \leq \|F\|_{(\mathbf{HA}_\varphi^{p',r'})^*}\,.$$

Since the last inequality holds for every $t>0$, we obtain

$$\|f\|_{\dot{\mathrm{CMO}}_\varphi^{p,r}} \leq \|F\|_{(\mathbf{HA}_\varphi^{p',r'})^*}\,,$$

which yields (2.3). Hence, we have completed the proof of Theorem 1.14. □

Next, we use some properties of $\mathbf{HA}_\varphi^{p,r}(\mathbb{R}^n)$ under certain conditions on φ.

Proposition 2.3. *Suppose that $\varphi(t)$ is non-increasing on $(0,\infty)$, and there exists $\tau_0>0$ such that $t^{\frac{n}{p}}\varphi(t)$ is nondecreasing on (τ_0,∞). Then $\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)$ is the subspace of $L_c^\infty(\mathbb{R}^n)^*$.*

Proof. Let a be a central $(1,p',r',\varphi)$-atom with $\mathrm{supp}(a)\subset B_t$, and let ψ be a test function in $L_c^\infty(\mathbb{R}^n)$ (the space of bounded functions with compact support) with $\mathrm{supp}(\psi)\subset B_{t_0}$. Applying Hölder's inequality yields

$$\Big|\int_{\mathbb{R}^n} a(x)\psi(x)\,dx\Big| \leq \|a\|_{L^{p'}(B_t)}\|\psi\|_{L^p(B_t\cap B_{t_0})} \leq \frac{\|\psi\|_{L^\infty}|B_t\cap B_{t_0}|^{1/p}}{|B_t|^{1/p}\varphi(t)}\,.$$

If $t\leq \max\{t_0,\tau_0\}$, then it follows from the last inequality and the fact $\varphi(t)\geq\min\{\varphi(t_0),\varphi(\tau_0)\}$ that

$$\Big|\int_{\mathbb{R}^n} a(x)\psi(x)\,dx\Big| \leq \frac{\|\psi\|_{L^\infty}}{\min\{\varphi(t_0),\varphi(\tau_0)\}}\,.$$

Otherwise, we have $t_0^{\frac{n}{p}}\varphi(t_0)\leq t^{\frac{n}{p}}\varphi(t)$. Therefore,

$$\Big|\int_{\mathbb{R}^n} a(x)\psi(x)\,dx\Big| \leq \frac{\|\psi\|_{L^\infty}|B(z_0,t_0)|^{1/p}}{|B(z_0,t_0)|^{1/p}\varphi(t_0)} = \frac{\|\psi\|_{L^\infty}}{\varphi(t_0)}\,.$$

By combining the two cases, we obtain

$$\Big|\int_{\mathbb{R}^n} a(x)\psi(x)\,dx\Big| \leq \frac{\|\psi\|_{L^\infty}}{\min\{\varphi(t_0),\varphi(\tau_0)\}}\,. \tag{2.6}$$

Now, for each $h\in\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)$, we can write $h=\sum_{j=0}^\infty\lambda_j a_j$, where a_j, $j\geq 0$ are $(1,p',r',\varphi)$-atoms, and $\sum_{j=0}^\infty|\lambda_j|<\infty$. Then, it follows from (2.6) that

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} h(x)\psi(x)\,dx\Big| &\leq \sum_{j=0}^\infty|\lambda_j|\Big|\int_{\mathbb{R}^n} a_j(x)\psi(x)\,dx\Big|\\ &\leq \Big(\sum_{j=0}^\infty|\lambda_j|\Big)\frac{\|\psi\|_{L^\infty}}{\min\{\varphi(t_0),\varphi(\tau_0)\}}\\ &\leq \|h\|_{\mathbf{HA}_\varphi^{p',r'}}\frac{\|\psi\|_{L^\infty}}{\min\{\varphi(t_0),\varphi(\tau_0)\}}\,.\end{aligned}$$

Thus, we obtain the conclusion. □

Remark 2.4. As a consequence of Proposition 2.3, if $h\in\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)$, $h=\sum_{j=0}^\infty\lambda_j a_j$, then the series converges to h in the norm of $L_c^\infty(\mathbb{R}^n)^*$.

Proposition 2.5. *Under the hypotheses in Proposition 2.3, $\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)$ is a Banach space.*

Proof. Let $\{f_N\}_{N\geq 1}$ be a Cauchy sequence in $\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)$. Then, there exists a subsequence $\{f_{N_k}\}_{k\geq 1}$ such that

$$\|f_{N_k}-f_{N_{k-1}}\|_{\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)} \leq 2^{-k}\,. \tag{2.7}$$

Put

$$f = f_{N_1}+\sum_{k\geq 2}\left(f_{N_k}-f_{N_{k-1}}\right)\,.$$

Note that for each $k \geq 1$, we have

$$f_{N_k} - f_{N_{k-1}} = \sum_{j\geq 0} \lambda_j^k a_j^k\,,$$

where $\{a_j^k\}_{j\geq 0}$ is a sequence of central $(1, p', r', \varphi)$-atoms, and

$$\sum_{j\geq 0} |\lambda_j^k| \leq \big\|f_{N_k} - f_{N_{k-1}}\big\|_{\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)} + 2^{-k}\,.$$

With this inequality noted, and by (2.7), we obtain

$$\sum_{k\geq 1}\sum_{j\geq 0} |\lambda_j^k| \leq \sum_{k\geq 1} 2^{1-k} < \infty\,. \tag{2.8}$$

This implies that f can be decomposed into central $(1, p', r', \varphi)$-atoms.

Next, we claim that $f_{N_k} \to f$ as $k \to \infty$ in the norm of $L_c^\infty(\mathbb{R}^n)^*$. If this is true, then by (2.8) we can conclude that $f_N \to f$ in $\mathbf{HA}_\varphi^{p',r'}(\mathbb{R}^n)$ as $N \to \infty$. Since $f = f_{N_{k_0}} + \sum_{k\geq k_0+1}(f_{N_k} - f_{N_{k-1}})$, it suffices to prove that $\sum_{k\geq k_0+1}(f_{N_k} - f_{N_{k-1}})$ converges to 0 as $k_0 \to \infty$ with respect to the norm of $L_c^\infty(\mathbb{R}^n)^*$. By (2.6), we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} \sum_{k\geq k_0+1} (f_{N_k} - f_{N_{k-1}})(x)\psi(x)\,dx\Big| &\leq \sum_{k\geq k_0+1}\sum_{l\geq 0} |\lambda_l^k| \Big|\int_{\mathbb{R}^n} a_l^k(x)\psi(x)\,dx\Big| \\ &\leq \sum_{k\geq k_0+1}\sum_{l\geq 0} |\lambda_l^k| \frac{\|\psi\|_{L^\infty}}{\min\{\varphi(t_0), \varphi(\tau_0)\}}\,.\end{aligned}$$

With this inequality it follows from (2.8) that

$$\lim_{k_0\to\infty} \big\|\sum_{k\geq k_0+1} (f_{N_k} - f_{N_{k-1}})\big\|_{L_c^\infty(\mathbb{R}^n)^*} = 0\,.$$

Therefore, $f_{N_{k_0}} \to f$ in $L_c^\infty(\mathbb{R}^n)^*$ as $k_0 \to \infty$. This completes the proof. □

3. Boundedness of operators of Hardy type in generalized central function spaces

3.1. Hardy operators in generalized central function spaces.

Proof of Theorem 1.16. We first prove the $\dot{\mathbf{M}}_\varphi^{p,r}$-boundedness of $\mathcal{H}$. For each ball B_t in $\mathbb{R}^n$, let us write

$$\mathcal{H}(f)(x) = \mathcal{H}(f_1)(x) + \mathcal{H}(f_2)(x), \quad \forall x \in \mathbb{R}^n\,,$$

with $f_1 = f\mathbf{1}_{B_t}$, and $f_2 = f\mathbf{1}_{B_t^c}$, $B_t^c = \mathbb{R}^n \setminus B_t$.

For f_1, we apply (1.12) to obtain

$$\begin{aligned}\frac{\|\mathcal{H}(f_1)\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} &\leq \Big(\frac{p}{p-1}\Big)\frac{\|f_1\|_{L^{p,r}}}{|B_t|^{1/p}\varphi(t)} \\ &= \Big(\frac{p}{p-1}\Big)\frac{\|f\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} \\ &\leq \Big(\frac{p}{p-1}\Big)\|f\|_{\dot{\mathbf{M}}_\varphi^{p,r}}\,.\end{aligned} \tag{3.1}$$

Next, since $f_2 = 0$ on B_t, for each $x \in B_t$ we observe that

$$\mathcal{H}(f_2)(x) = \frac{1}{\nu_n|x|^n}\int_{|y|<|x|} f_2(y)\,dy = 0\,. \tag{3.2}$$

A combination of (3.1) and (3.2) yields

$$\frac{\|\mathcal{H}(f)\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} = \frac{\|\mathcal{H}(f_1)\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} \leq \Big(\frac{p}{p-1}\Big)\|f\|_{\dot{\mathbf{M}}_\varphi^{p,r}}\,.$$

Since the last inequality holds for any $t>0$, we obtain

$$\|\mathcal{H}(f)\|_{\dot{\mathbf{M}}^{p,r}_{\varphi}} \le \Big(\frac{p}{p-1}\Big)\|f\|_{\dot{\mathbf{M}}^{p,r}_{\varphi}}\,.$$

It remains to prove the $\dot{\mathbf{M}}^{p,r}_{\varphi}$-boundedness of $\mathcal{H}^*$. We argue similarly as in (3.1) to obtain

$$\frac{\|\mathcal{H}^*(f_1)\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} \lesssim \|f\|_{\dot{\mathbf{M}}^{p,r}_{\varphi}}\,. \tag{3.3}$$

Next, we observe that

$$\begin{aligned}
|\mathcal{H}^*(f_2)(x)| &= \Big|\frac{1}{\nu_n}\int_{|y|\ge 2t}\frac{f(y)}{|y|^n}\,dy\Big| = \frac{1}{\nu_n}\Big|\sum_{k=1}^{\infty}\int_{\{2^kt\le|y|<2^{k+1}t\}}\frac{f(y)}{|y|^n}\,dy\Big|\\
&\lesssim \sum_{k=1}^{\infty}(2^kt)^{-n}\Big|\int_{\{2^kt\le|y|<2^{k+1}t\}} f(y)\,dy\Big|\\
&\le \sum_{k=1}^{\infty}(2^kt)^{-n}\int_{B_{2^{k+1}t}}|f(y)|\,dy\,.
\end{aligned}$$

Thanks to Hölder's inequality, and (1.20), we obtain

$$\begin{aligned}
|\mathcal{H}^*(f_2)(x)| &\lesssim \sum_{k=1}^{\infty}(2^kt)^{-n}\|f\|_{L^p(B_{2^{k+1}t})}|B_{2^{k+1}t}|^{1/p'}\\
&\lesssim \sum_{k=1}^{\infty}\frac{\|f\|_{L^p(B_{2^{k+1}t})}}{|B_{2^{k+1}t}|^{1/p}\varphi(2^{k+1}t)}\varphi(2^{k+1}t)\\
&\le \sum_{k=1}^{\infty}\varphi(2^{k+1}t)\|f\|_{\dot{\mathbf{M}}^{p,r}_{\varphi}}\\
&\le \sum_{k=1}^{\infty}D_0^{k+1}\varphi(t)\|f\|_{\dot{\mathbf{M}}^{p,r}_{\varphi}} \lesssim \varphi(t)\|f\|_{\dot{\mathbf{M}}^{p,r}_{\varphi}}\,.
\end{aligned}$$

Therefore, we deduce that

$$\|\mathcal{H}^*(f_2)\|_{L^{p,r}(B_t)} \lesssim |B_t|^{1/p}\varphi(t)\|f\|_{\dot{\mathbf{M}}^{p,r}_{\varphi}}\,. \tag{3.4}$$

Combing (3.3) and (3.4) yields the desired result. The proof is complete. ☐

Proof of Corollary 1.18. The proof of part (a) follows from Theorem 1.16 and Remark 1.6. It remains to prove (b). Thanks to duality, for every $f\in\mathbf{HA}^{p',r'}_{\varphi}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\|\mathcal{H}(f)\|_{\mathbf{HA}^{p',r'}_{\varphi}} &= \sup_{\|g\|_{\dot{\mathrm{CMO}}^{p,r}_{\varphi}}=1}\Big|\int\mathcal{H}(f)(x)g(x)\,dx\Big|\\
&= \sup_{\|g\|_{\dot{\mathrm{CMO}}^{p,r}_{\varphi}}=1}\Big|\int f(x)\mathcal{H}^*(g)(x)\,dx\Big|\\
&\le \sup_{\|g\|_{\dot{\mathrm{CMO}}^{p,r}_{\varphi}}=1}\|f\|_{\mathbf{HA}^{p',r'}_{\varphi}}\|\mathcal{H}^*(g)\|_{\dot{\mathrm{CMO}}^{p,r}_{\varphi}}\\
&\lesssim \sup_{\|g\|_{\dot{\mathrm{CMO}}^{p,r}_{\varphi}}=1}\|f\|_{\mathbf{HA}^{p',r'}_{\varphi}}\|g\|_{\dot{\mathrm{CMO}}^{p}_{\varphi}} = \|f\|_{\mathbf{HA}^{p',r'}_{\varphi}}\,.
\end{aligned}$$

Hence, we conclude that $\mathcal{H}$ maps $\mathbf{HA}^{p',r'}_{\varphi}(\mathbb{R}^n)\to\mathbf{HA}^{p',r'}_{\varphi}(\mathbb{R}^n)$. Similarly, the conclusion also holds for $\mathcal{H}^*$. Therefore, we complete the proof. ☐

3.2. **Commutators of Hardy operators in generalized central function spaces.** Before proving Theorems 1.19 and 1.22, we recall a fundamental result being useful for our argument later.

Lemma 3.1. *Let $1 \le p < \infty$, and $k \ge 1$. For each ball B_t in $\mathbb{R}^n$, we have*

$$\left\|b - b_{B_{2^k t}}\right\|_{L^{p,r}(B_t)} \le 2^n (k+1) \|b\|_{\dot{\mathrm{CMO}}^{p,r}} |B_t|^{1/p} .$$

Proof. For each $j \ge 1$, we observe that

$$\begin{aligned}
|b_{B_{2^{j+1}t}} - b_{B_{2^j t}}| &\le \frac{1}{|B_{2^j t}|} \int_{B_{2^j t}} |b(y) - b_{B_{2^{j+1}t}}| \, dy \\
&\le \frac{|B_{2^{j+1}t}|}{|B_{2^j t}|} \frac{1}{|B_{2^{j+1}t}|} \int_{B_{2^{j+1}t}} \left|b(y) - b_{B_{2^{j+1}t}}\right| \, dy \\
&\le 2^n \|b\|_{\dot{\mathrm{CMO}}^1} \le 2^n \|b\|_{\dot{\mathrm{CMO}}^{p,r}} .
\end{aligned}$$

From this inequality, we obtain

$$\begin{aligned}
\|b - b_{B_{2^k t}}\|_{L^{p,r}(B_t)} &\le \|b - b_{B_t}\|_{L^{p,r}(B_t)} + \sum_{j=0}^{k-1} \|b_{B_{2^j t}} - b_{B_{2^{j+1}t}}\|_{L^{p,r}(B_t)} \\
&\le |B_t|^{1/p} \frac{\|b - b_{B_t}\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}} + \sum_{j=0}^{k-1} |b_{B_{2^j t}} - b_{B_{2^{j+1}t}}| |B_t|^{1/p} \\
&\le 2^n (k+1) \|b\|_{\dot{\mathrm{CMO}}^{p,r}} |B_t|^{1/p} .
\end{aligned}$$

The proof is complete. □

Next, we estimate $\|\mathbf{1}_{B_r}\|_{\dot{\mathbf{M}}^{p,r}_\varphi}$ for each ball B_r in $\mathbb{R}^n$.

Lemma 3.2. *Suppose that $\varphi(t)$ is non-increasing, and $t^{\frac{n}{p}} \varphi(t)$ is nondecreasing. Then, for any ball B_r in $\mathbb{R}^n$ we have*

$$\|\mathbf{1}_{B_r}\|_{\dot{\mathbf{M}}^{p,r}_\varphi} = \frac{1}{\varphi(r)} .$$

Proof. We consider the term $I(t) := \frac{\|\mathbf{1}_{B_r}\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)}$, $t > 0$. If $t \le r$, then since $\varphi(t)$ is nonincreasing, then we obtain

$$I(t) = \frac{|B_r \cap B_t|^{1/p}}{|B_t|^{1/p}\varphi(t)} = \frac{|B_t|^{1/p}}{|B_t|^{1/p}\varphi(t)} \le \frac{1}{\varphi(r)} .$$

Otherwise, it follows from the monotonicity of $|B_t|^{1/p}\varphi(t)$ that

$$I(t) \le \frac{|B_r|^{1/p}}{|B_r|^{1/p}\varphi(r)} \le \frac{1}{\varphi(r)} .$$

Combining the two inequalities yields

$$\|\mathbf{1}_{B_r}\|_{\dot{\mathbf{M}}^{p,r}_\varphi} \le \frac{1}{\varphi(r)} . \tag{3.5}$$

The reverse of (3.5) is obvious since $I(r) = \frac{1}{\varphi(r)}$. Therefore, the desired result follows. □

Proof of Theorem 1.19. (a) Fix a ball B_t in $\mathbb{R}^n$. We write

$$[b, \mathcal{H}](f) = [b, \mathcal{H}](f_1) + [b, \mathcal{H}](f_2) ,$$

with $f_1 = f \mathbf{1}_{B_t}$ and $f_2 = f \mathbf{1}_{B_t^c}$. Since $[b, \mathcal{H}]$ maps $L^p \to L^p$, we have

$$\|[b, \mathcal{H}](f_1)\|_{L^{p,r}(B_t)} \lesssim \|b\|_{\dot{\mathrm{CMO}}^{\max\{p,p'\},r}} \|f_1\|_{L^p} = \|b\|_{\dot{\mathrm{CMO}}^{\max\{p,p'\},r}} \|f\|_{L^{p,r}(B_{2t})} .$$

It follows from the monotonicity of φ that

$$\begin{aligned}
\frac{\|[b, \mathcal{H}](f_1)\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} &\lesssim \|b\|_{\dot{\mathrm{CMO}}^{\max\{p,p'\},r}} \frac{\|f\|_{L^{p,r}(B_{2t})}}{|B_t|^{1/p}\varphi(t)} \\
&\le \|b\|_{\dot{\mathrm{CMO}}^{\max\{p,p'\},r}} \|f\|_{\dot{\mathbf{M}}^{p,r}_\varphi} .
\end{aligned} \tag{3.6}$$

Next, for any $x \in B_t$ we observe that $[b,\mathcal{H}](f_2)(x) = 0$. A combination of this fact, and (3.6) provides us with

$$\frac{\|[b,\mathcal{H}](f)\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} = \frac{\|[b,\mathcal{H}(f_1)\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} \lesssim \|b\|_{\dot{\mathrm{CMO}}^{\max\{p,p'\},r}}\|f\|_{\dot{\mathbf{M}}_\varphi^{p,r}}\,.$$

Therefore, we obtain the desired result in part (a).

(b) Since $[b,\mathcal{H}^*]$ maps $L^p \to L^p$, then we can mimic the proof of (3.6) to obtain

$$\frac{\|[b,\mathcal{H}^*](f_1)\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} \lesssim \|b\|_{\dot{\mathrm{CMO}}^{\max\{p,p'\},r}}\|f\|_{\dot{\mathbf{M}}_\varphi^{p,r}}\,. \tag{3.7}$$

Concerning f_2, we write

$$\begin{aligned}
&\|[b,\mathcal{H}^*](f_2)\|_{L^{p,r}(B_t)}\\
&= \Big\|\frac{1}{\nu_n}\int_{|y|\ge 2t}(b(x)-b(y))\frac{f(y)}{|y|^n}\,dy\Big\|_{L^{p,r}(B_t)}\\
&\le \Big\|\sum_{k=0}^{\infty}(2^k t)^{-n}\int_{\{2^k t\le |y|<2^{k+1}t\}}|b(x)-b_{B_{2^{k+1}t}}|\,|f(y)|\,dy\Big\|_{L^{p,r}(B_t)}\\
&\quad + \Big\|\sum_{k=0}^{\infty}(2^k t)^{-n}\int_{\{2^k t\le |y|<2^{k+1}t\}}|b(y)-b_{B_{2^{k+1}t}}|\,|f(y)|\,dy\Big\|_{L^{p,r}(B_t)} := \mathbf{I}_1+\mathbf{I}_2\,.
\end{aligned} \tag{3.8}$$

We first treat $\mathbf{I}_1$. Applying the triangle inequality, Minkowski's inequality, and the Hölder inequality yields

$$\begin{aligned}
\mathbf{I}_1 &\le \sum_{k=0}^{\infty}(2^k t)^{-n}\int_{\{2^k t\le |y|<2^{k+1}t\}}\|b-b_{B_{2^{k+1}t}}\|_{L^{p,r}(B_t)}|f(y)|\,dy\\
&\lesssim \sum_{k=0}^{\infty}|B_{2^{k+1}t}|^{-1}\|b-b_{B_{2^{k+1}t}}\|_{L^{p,r}(B_t)}\|f\|_{L^p(B_{2^{k+1}t})}|B_{2^{k+1}t}|^{\frac{1}{p'}}\\
&\le \sum_{k=0}^{\infty}\|b-b_{B_{2^{k+1}t}}\|_{L^{p,r}(B_t)}\varphi(2^{k+1}t)\|f\|_{\dot{\mathbf{M}}_\varphi^{p,r}}\,.
\end{aligned}$$

Thanks to Lemma 3.1 and (1.20), we obtain from the last inequality that

$$\begin{aligned}
\mathbf{I}_1 &\lesssim \sum_{k=0}^{\infty}2^n(k+2)\|b\|_{\dot{\mathrm{CMO}}^{p,r}}|B_t|^{1/p}D_0^{k+1}\varphi(t)\|f\|_{\dot{\mathbf{M}}_\varphi^{p,r}}\\
&\lesssim |B_t|^{1/p}\varphi(t)\|b\|_{\dot{\mathrm{CMO}}^{p,r}}\|f\|_{\dot{\mathbf{M}}_\varphi^{p,r}}\,.
\end{aligned} \tag{3.9}$$

Note that (3.9) was obtained from $\sum_{k=0}^{\infty}(k+2)D_0^{k+1}<\infty$.

For $\mathbf{I}_2$, we use Hölder's inequality, and Lemma 3.1 to obtain

$$\begin{aligned}
\mathbf{I}_2 &\le \Big\|\sum_{k=0}^{\infty}(2^k t)^{-n}\|b-b_{B_{2^{k+1}t}}\|_{L^{p',r}(B_{2^{k+1}t})}\|f\|_{L^{p,r}(B_{2^{k+1}t})}\Big\|_{L^{p,r}(B_t)}\\
&\lesssim \sum_{k=0}^{\infty}\frac{\|b-b_{B_{2^{k+1}t}}\|_{L^{p',r}(B_{2^{k+1}t})}}{|B_{2^{k+1}t}|^{\frac{1}{p'}}}\frac{\|f\|_{L^{p,r}(B_{2^{k+1}t})}}{|B_{2^{k+1}t}|^{1/p}\varphi(2^{k+1}t)}\varphi(2^{k+1}t)|B_t|^{1/p}\\
&\le \sum_{k=0}^{\infty}\|b\|_{\dot{\mathrm{CMO}}^{p',r}}\|f\|_{\dot{\mathbf{M}}_\varphi^{p,r}}D_0^{k+1}\varphi(t)|B_t|^{1/p}\\
&\lesssim |B_t|^{1/p}\varphi(t)\|b\|_{\dot{\mathrm{CMO}}^{p',r}}\|f\|_{\dot{\mathbf{M}}_\varphi^{p,r}}\,.
\end{aligned} \tag{3.10}$$

Combining (3.8), (3.9), and (3.10) yields

$$\frac{\|[b,\mathcal{H}^*](f_2)\|_{L^{p,r}(B_t)}}{|B_t|^{1/p}\varphi(t)} \lesssim \|b\|_{\dot{\mathrm{CMO}}^{\max\{p,p'\},r}}\|f\|_{\dot{\mathbf{M}}_\varphi^{p,r}}\,. \tag{3.11}$$

Therefore, the desired result follows from (3.7) and (3.11). The proof is complete. □

Proof of Corollary 1.21. The proof is similar to the one of Corollary 1.18, then we leave it to the reader. □

Finally we demonstrate Theorem 1.22. The proof follows by way of the following lemma.

Lemma 3.3. *Let a be a central $(1,p')$-atom. Then, there exist two functions $f \in \mathbf{HA}_{\varphi}^{p',r'}(\mathbb{R}^n)$, and $g \in \dot{\mathbf{M}}_{\varphi}^{p,r}(\mathbb{R}^n)$ such that*

$$a(x) = f(x)\mathcal{H}^*(g)(x) - g(x)\mathcal{H}(f)(x)\,, \tag{3.12}$$

$$\|f\|_{\mathbf{HA}_{\varphi}^{p',r'}} \|g\|_{\dot{\mathbf{M}}_{\varphi}^{p,r}} \le \frac{2^{\frac{n}{p}}}{\ln 2}\,. \tag{3.13}$$

Proof. Suppose that $\mathrm{supp}(a) \subset B_\tau$ for some $\tau > 0$. Let us set

$$f(x) = \frac{a(x)}{\varphi(\tau)\ln 2}, \quad \text{and} \quad g(x) = \varphi(\tau)\mathbf{1}_{\{\tau<|x|<2\tau\}}(x)\,.$$

We first claim that the above construction satisfies (3.12). In fact, if $|x| \ge \tau$, then it is clear that

$$f(x) = \mathcal{H}(f)(x) = 0$$

since $\mathrm{supp}(a) \subset B_\tau$, and the cancellation property of a respectively. Therefore, (3.12) is true for all $|x| \ge \tau$. Otherwise, we have $g(x) = 0$, and

$$\begin{aligned}\mathcal{H}^*(g)(x) &= \frac{1}{\nu_n}\int_{|y|\ge|x|} \frac{\varphi(\tau)\mathbf{1}_{\{\tau<|x|<2\tau\}}(y)}{|y|^n}\,dy \\ &= \frac{\varphi(\tau)}{\nu_n}\int_{\tau}^{2\tau} \nu_n s^{-n}s^{n-1}\,ds = \varphi(\tau)\ln 2\,.\end{aligned}$$

This yields the above claim.

Now, we demonstrate (3.13). Since a is a central $(1,p')$-atom, f is a multiple of central $(1,p',\varphi)$-atom, and

$$\|f\|_{\mathbf{HA}_{\varphi}^{p',r'}} \le \frac{1}{\ln 2}\,. \tag{3.14}$$

Moreover, thanks to Lemma 3.2, we obtain

$$\|g\|_{\dot{\mathbf{M}}_{\varphi}^{p,r}} = \varphi(\tau)\|\mathbf{1}_{\{\tau<|x|<2\tau\}}\|_{\dot{\mathbf{M}}_{\varphi}^{p,r}} \le \frac{\varphi(\tau)}{\varphi(2\tau)} = 2^{\frac{n}{p}}\,\frac{\tau^{\frac{n}{p}}\varphi(\tau)}{(2\tau)^{\frac{n}{p}}\varphi(2\tau)} \le 2^{\frac{n}{p}}\,. \tag{3.15}$$

The last inequality follows from the monotonicity of function $t^{\frac{n}{p}}\varphi(t)$.

As a result, (3.13) follows from (3.14) and (3.15). Therefore, we obtain Lemma 3.3. □

Remark 3.4. The above construction demonstrates that $g \in L_c^\infty(\mathbb{R}^n)$, and $f \in L_c^{p'}(\mathbb{R}^n)$. In addition, the result of Lemma 3.2 can be considered as a $\mathbf{HA}^{p',r'}(\mathbb{R}^n)^*$ factorization. Note that the $\mathbf{H}^1(\mathbb{R}^n)$ factorization by means of the Calderón–Zygmund operators has been studied by the authors in [5, 7, 8, 9, 10, 19, 28, 29] and the references therein.

Proof of Theorem 1.22. For thsi purpose, We use a duality argument. Since $\dot{\mathrm{CMO}}^{p,r}(\mathbb{R}^n) = \mathbf{HA}^{p',r'}(\mathbb{R}^n)^*$, it follows that for any $h \in \mathbf{HA}^{p',r'}(\mathbb{R}^n)$, one can decompose

$$h = \sum_{j=0}^{\infty} \lambda_j a_j\,,$$

where $\{a_j\}_{j\ge0}$ is a sequence of central $(1,p')$-atoms; and $\sum_{j=0}^{\infty}|\lambda_j| < \infty$.

For every $j \ge 0$, by applying Lemma 3.3 to a_j we have that there exist two functions $g_j \in \dot{\mathbf{M}}_{\varphi}^{p,r}(\mathbb{R}^n)$, and $f_j \in \mathbf{HA}_{\varphi}^{p',r'}(\mathbb{R}^n)$ such that

$$a_j(x) = f_j(x)\mathcal{H}^*(g_j)(x) - g_j(x)\mathcal{H}(f_j)(x)\,,$$

and

$$\|f_j\|_{\mathbf{HA}_{\varphi}^{p',r'}} \|g_j\|_{\dot{\mathbf{M}}_{\varphi}^{p,r}} \le \frac{2^{\frac{n}{p}}}{\ln 2}\,. \tag{3.16}$$

Since $b \in L^p_{\rm loc}(\mathbb{R}^n)$, and by Remark 3.4, the following integrals are well-defined, and satisfy

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} b(x)a_j(x)\,dx\Big| &= \Big|\int_{\mathbb{R}^n} b(x)[f_j(x)\mathcal{H}^*(g_j)(x) - g_j(x)\mathcal{H}(f_j)(x)]\,dx\Big| \\ &= \Big|\int_{\mathbb{R}^n} f_j(x)[b,\mathcal{H}^*](g_j)(x)\,dx\Big| \\ &\le \|f_j\|_{\mathbf{HA}^{p',r'}_\varphi}\|[b,\mathcal{H}^*](g_j)\|_{\dot{\mathbf{M}}^{p,r}_\varphi}\,. \end{aligned} \tag{3.17}$$

Note that (3.17) was obtained from $\dot{\mathbf{M}}^{p,r}_\varphi(\mathbb{R}^n) = \mathrm{C\dot{M}O}^{p,r}_\varphi(\mathbb{R}^n) = \mathbf{HA}^{p',r'}_\varphi(\mathbb{R}^n)^*$. Since $[b,\mathcal{H}^*]$ is a bounded operator on $\dot{\mathbf{M}}^{p,r}_\varphi(\mathbb{R}^n)$, it follows from (3.17) and (3.16) that

$$\Big|\int_{\mathbb{R}^n} b(x)a_j(x)\,dx\Big| \le \|[b,\mathcal{H}^*]\|_{\dot{\mathbf{M}}^{p,r}_\varphi\to\dot{\mathbf{M}}^{p,r}_\varphi}\|g_j\|_{\dot{\mathbf{M}}^{p,r}_\varphi}\|f_j\|_{\mathbf{HA}^{p',r'}_\varphi} \le \frac{2^{\frac{n}{p}}}{\ln 2}\|[b,\mathcal{H}^*]\|_{\dot{\mathbf{M}}^{p,r}_\varphi\to\dot{\mathbf{M}}^{p,r}_\varphi}\,.$$

With this inequality, for any $h \in \mathbf{HA}^{p',r'}_\varphi(\mathbb{R}^n)$ we obtain

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} b(x)h(x)\,dx\Big| &= \sum_{j=0}^\infty \Big|\lambda_j\int_{\mathbb{R}^n} b(x)a_j(x)\,dx\Big| \\ &\le \Big(\sum_{j=0}^\infty |\lambda_j|\Big)\frac{2^{\frac{n}{p}}}{\ln 2}\|[b,\mathcal{H}^*]\|_{\dot{\mathbf{M}}^{p,r}_\varphi\to\dot{\mathbf{M}}^{p,r}_\varphi} \\ &\le \frac{2^{\frac{n}{p}}}{\ln 2}\|[b,\mathcal{H}^*]\|_{\dot{\mathbf{M}}^{p,r}_\varphi\to\dot{\mathbf{M}}^{p,r}_\varphi}\|h\|_{\mathbf{HA}^{p',r'}}\,. \end{aligned} \tag{3.18}$$

By duality, we obtain

$$\|b\|_{\mathrm{C\dot{M}O}^{p,r}} \le \frac{2^{\frac{n}{p}}}{\ln 2}\|[b,\mathcal{H}^*]\|_{\dot{\mathbf{M}}^{p,r}_\varphi\to\dot{\mathbf{M}}^{p,r}_\varphi}\,. \tag{3.19}$$

The proof is complete. □

Proof of Corollary 1.23. To obtain the result, we can repeat the proof of Theorem 1.22 with a slight modification in (3.17) as follows

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} b(x)a_j(x)\,dx\Big| &= \Big|\int_{\mathbb{R}^n} b(x)\,[f_j(x)\mathcal{H}^*(g_j)(x) - g_j(x)\mathcal{H}(f_j)(x)]\,\,dx\Big| \\ &= \Big|\int_{\mathbb{R}^n} [b,\mathcal{H}](f_j)(x)g_j(x)\,dx\Big| \\ &\le \|[b,\mathcal{H}](f_j)\|_{\mathbf{HA}^{p',r'}_\varphi}\|g_j\|_{\dot{\mathbf{M}}^{p,r}_\varphi}\,. \end{aligned} \tag{3.20}$$

Since $[b,\mathcal{H}]$ maps $\mathbf{HA}^{p',r'}_\varphi \to \mathbf{HA}^{p',r'}_\varphi$, we deduce from (3.20) that

$$\begin{aligned} \Big|\int_{\mathbb{R}^n} b(x)a_j(x)\,dx\Big| &\le \|[b,\mathcal{H}]\|_{\mathbf{HA}^{p',r'}_\varphi\to\mathbf{HA}^{p',r'}_\varphi}\|f_j\|_{\mathbf{HA}^{p',r'}_\varphi}\|g_j\|_{\dot{\mathbf{M}}^{p,r}_\varphi} \\ &\le \frac{2^{\frac{n}{p}}}{\ln 2}\|[b,\mathcal{H}]\|_{\mathbf{HA}^{p',r'}_\varphi\to\mathbf{HA}^{p',r'}_\varphi}\,. \end{aligned}$$

By arguing similarly as in (3.18), for any $h \in \mathbf{HA}^{p',r'}_\varphi(\mathbb{R}^n)$, we also obtain

$$\Big|\int_{\mathbb{R}^n} b(x)h(x)\,dx\Big| \le \frac{2^{\frac{n}{p}}}{\ln 2}\|[b,\mathcal{H}]\|_{\mathbf{HA}^{p',r'}_\varphi\to\mathbf{HA}^{p',r'}_\varphi}\|h\|_{\mathbf{HA}^{p',r'}}\,.$$

This yields (1.24). □

Proof of Corollary 1.24. The proof is just a combination of the results in Theorem 1.22 and Corollary 1.23. Therefore, we leave it to the reader. □

4. Applications of Hardy's inequality

We present here several applications of Hardy's inequality. In [2], Brezis-Vázquez studied the problem

$$\begin{aligned} -\Delta u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, and f is a continuous, positive, increasing and convex function defined for $u \geq 0$ with $f(0) > 0$ and

$$\lim_{s\to\infty} \frac{f(s)}{s} = \infty.$$

The authors established a characterization of the singular H^1 extremal solutions and the extremal value λ^* by a criterion consisting of two conditions:

(i) They must be energy solutions, not in L^∞.
(ii) They must satisfy

$$\lambda \int_\Omega f'(u)\phi^2 dx \leq \int_\Omega |\nabla \phi|^2 dx, \quad \forall \phi \in \mathcal{C}_0^\infty(\Omega). \tag{4.2}$$

Roughly speaking, this formula means that the first eigenvalue of $-\Delta - \lambda f'(u)$ is nonnegative, is a version of Hardy's inequality. To obtain the desired result, they improved a version of the classical Hardy's inequality.

Another application of Hardy's inequality is to study negative eigenvalues of the self-adjoint operator $-\Delta - V$ in $L^2(\mathbb{R}^n)$, where potential V satisfies $V \geq 0$, $V \in L^{n/2}(\mathbb{R}^n)$, $n \geq 3$. This has important implications in semi-classical spectral analysis, in which the transition between classical and quantum mechanics is studied.

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Le Trung Nghia
Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.
Ho Chi Minh City University of Education, Ho Chi Minh City, Vietnam
Email address: `letrungnghia@tdtu.edu.vn, letrungnghia85@yahoo.com`