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LOCAL AND GLOBAL SOLVABILITY OF FRACTIONAL POROUS MEDIUM EQUATIONS IN CRITICAL BESOV-MORREY SPACES

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Abstract. In this article we study fractional porous medium equations in Besov-Morrey spaces. Using the Littlewood-Paley theory and the smoothing effect of the heat semi-group, we obtain local well-posedness of this model. Also, we obtain global well-posedness for small initial data in the critical Besov-Morrey spaces $\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}(\mathbb{R}^n)$ with $1/2 < m < 1$, $\max\{1, \frac{n}{2m}\} < p < \infty$ and $1 \le h \le p$.

1. Introduction

We study the fractional porous medium equation given by the nonlinear diffusion model with fractional Laplacian operators,

$$\begin{gathered} u_t - \mu\Delta u = \kappa\nabla\cdot(u\nabla\mathcal{K}u) \quad \text{in } \mathbb{R}^n\times(0,\infty), \\ \mathcal{K}u = (-\Delta)^{-m}u \quad \text{in } \mathbb{R}^n\times(0,\infty), \\ u(x,0) = u_0(x) \quad \text{for } x\in\mathbb{R}^n, \end{gathered} \tag{1.1}$$

where $n \ge 1$, $u = u(x,t)$ denotes the density or concentration, and therefore non-negative, u_0 is the initial data, ∇ is the gradient operator, $\mu > 0$ is the dissipative coefficient which corresponds the viscous case, while $\mu = 0$ represents the inviscid case, $\kappa = \pm 1$, and here for simplify the notation, we take $\mu = \kappa = 1$, $(-\Delta)^{-m}$ is the inverse fractional Laplacian operator, and $0 < m < 1$, that is, the abnormal (normal) diffusion is modeled by a fractional power of the Laplacian. We mention here that, The interest in using fractional Laplacians in modeling diffusive processes has a wide literature, especially when one wants to model long-range diffusive interaction, and this interest has been activated by the recent progress in the mathematical theory, see [28, 9, 7, 33, 13, 15] and related references cited therein.

When $\mu = 0$, $\kappa = 1$ and $m = 1$, the model (1.1) corresponds to the mean field equation

$$\begin{gathered} u_t = \nabla\cdot(u\nabla\mathcal{K}u) \quad \text{in } \mathbb{R}^n\times(0,\infty), \\ \mathcal{K}u = (-\Delta)^{-1}u \quad \text{in } \mathbb{R}^n\times(0,\infty), \\ u(x,0) = u_0(x) \quad \text{for } x\in\mathbb{R}^n, \end{gathered} \tag{1.2}$$

which was introduced for the first time by Lin and Zhang [24]. They demonstrated the existence and uniqueness of positive L^∞ solution in two dimensions. There are many studies on well-posedness results of this equation. For instance, refer to [31, 37] and related references cited therein.

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When $\mu = 1$, $\kappa = -1$ and $m = 1$, system (1.1) reduces to the classical Keller-Segel system

$$\begin{gathered} u_t - \Delta u + \nabla \cdot (u \nabla \mathcal{K} u) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ -\Delta \mathcal{K} u = u \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}^n, \end{gathered} \tag{1.3}$$

which describes a model of chemotaxis. System (1.3) has been introduce by Keller and Segel [19]. The well-posedness of Keller-Segel models has been studied by several researchers in various spaces. Biler and Karch [4] have established, in the critical Lebesgue space $L^{\frac{n}{2}}(\mathbb{R}^n)$, the existence of both local and global solutions of this equation with small initial data. Additionally, they have demonstrated the finite-time blowup of non-negative solutions with specific initial data that satisfy high-concentration or large-mass conditions. Making use of the Chemin mono-norm methods, Zhao [36] obtained well-posedness results in the Besov spaces $\dot{B}^{-2+\frac{n}{p}}_{p,r}(\mathbb{R}^n)$ with $1 \le p, r \le \infty$. Making use of the smoothing effect of the heat semigroup, Iwabuchi [17] proved the global well-posedness of the system (1.3) in $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ where $n \ge 1$ and $\max\{1, n/2\} < p < \infty$, under the condition of smallness of the initial data. He also demonstrated, with a sufficient condition, the local well-posedness in $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ [18]. Later, by the same method, Nogayama and Sawano [25] extended these well-posedness results, where they established global well-posedness in the Besov-Morrey spaces $\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}(\mathbb{R}^n)$ with $\max\{1, \frac{n}{2}\} < p < \infty$ and $1 \le h \le p$, and local well-posedness in closed subspaces of these spaces. And here we mention that certain aspects of these results were also extended to the fractional power bipolartype drift-diffusion system. Further information on this topic can be found in [23, 11] and the relevant references cited therein.

When $\kappa = 1$ and $0 < m < 1$, system (1.1) has been derived starting from the same origin as the fractional porous medium equation initially proposed by Caffarelli and Vázquez [6]. Indeed, the model is conceived through the incorporation of the dissipative term $\mu(-\Delta)v$ into the continuity equation

$$\partial_t u + \nabla \cdot (uV) = 0, \tag{1.4}$$

where $V = -\nabla p$ is the velocity, and p represents the gas pressure which is related to v through a linear integral operator $p = \mathcal{K}u$, with kernel $K(x,y) = c|x-y|^{-(d-2m)}$. For $\mu = 0$, that is, the fractional porous medium equation in the inviscid case, we have many studies on this equation. In [6] the authors stated that the notable feature of this equation is the finite speed of propagation, and they established, the existence of a weak solution with bounded initial data that exponentially decays at infinity, the property of compact support, and also the relevant integral estimates. See [31, 28, 37] for more information on this equation. For the viscous case ($\mu > 0$), El Idrissi et al.[10] considered the system (1.1) for initial data in the critical Besov spaces $\dot{B}^{-2m+\frac{d}{p}}_{p,\infty}(\mathbb{R}^n)$ with $\frac{1}{2} < m < 1$ and $\max\{1, \frac{n}{2m}\} < p < \infty$. They established sufficient conditions for the existence and uniqueness of local solutions, and also proved the existence of global solutions for small initial data in the same spaces. Furthermore, the well-posedness of this model has been demonstrated in various functional settings: in the Besov spaces $\dot{B}^{-2m+\frac{n}{p}}_{p,r}(\mathbb{R}^n)$ with $1 \le p, r \le \infty$, in the Fourier-Besov spaces $F\dot{B}^{-2m+\frac{n}{p'}}_{p,r}(\mathbb{R}^n)$ with $1 \le p, r \le \infty$, in the critical Fourier-Besov-Morrey spaces $F\dot{N}^{-2m+\frac{\lambda}{p}+3(1-\frac{1}{p})}_{p,\lambda,r}(\mathbb{R}^3)$ with $1 \le p < \infty$ and $1 \le r \le \infty$, and in the critical variable exponent Besov-Morrey spaces $\dot{\mathcal{N}}^{-2m+\frac{n}{q(\cdot)}}_{r(\cdot),q(\cdot),h}(\mathbb{R}^n)$ with $\frac{1-\varepsilon}{2} < m < 1 + \frac{n}{2q(\cdot)}$, $0 < \varepsilon < 1$, $1 \le r(\cdot) \le q(\cdot) < \infty$ and $1 \le h \le \infty$. These results were obtained by Xiao and Zhang [35], El Idrissi et al. [8], Toumlilin [34], and El Idrissi et al. [12], respectively, by applying the Chemin mono-norm methods.

Using a different method from the latter, which is the smoothing effect of the heat semigroup, we seek to establish the existence of local solutions for initial data in a closed subspace of Besov-Morrey spaces, which will be define later. Also we show the existence of global solutions of system (1.1) in the critical Besov-Morrey spaces $\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}(\mathbb{R}^n)$ with $\frac{1}{2} < m < 1$, $\max\{1, \frac{n}{2m}\} < p < \infty$, $1 \le h \le p$, under the smallness condition of initial data. The idea of our work is motivated by the papers [17, 18, 25], which dealt with the classical Keller-Segel systems.

The Besov-Morrey space $\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}$ is critical for the system (1.1). In fact, if $u(x,t)$ is the solution of system (1.1), then

$$u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t)$$

is also a solution of the same equation and

$$\|u(\cdot,0)\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} \sim \|u_\lambda(\cdot,0)\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}.$$

The Besov-Morrey space, introduced by Kozono and Yamazaki [22], extends the concept of classical Besov spaces by incorporating Morrey spaces as a fundamental element, thus providing a larger functional framework. In particular, Besov-Morrey spaces are strictly broader than classical Besov spaces (also refer to Remark 2.11). It is important to note that replacing the L^p-norm by the $\mathcal{M}^p_h$-norm is not sufficient to ensure a direct transition from Besov spaces to Besov-Morrey spaces. One of the main difficulties comes from the collapse of certain essential embedding features when working in the Besov-Morrey framework (see, for example, [14, 25] for a more detailed discussion). Within this extended functional framework, numerous recent works have been carried out on a variety of fluid dynamics systems. The reader is referred to [14, 20, 22, 25, 29] and related references cited therein for further information. However, Besov-Morrey spaces are generally not separable. Therefore, the compatibility of initial data must be taken into account. With this in mind, in order to consider the local existence result for the the system (1.1), we impose a vanishing condition on the high frequency components of the initial data u_0. i.e., we take $u_0 \in \widetilde{\dot{\mathcal{N}}^s_{p,h,r}} = \widetilde{\dot{\mathcal{N}}^s_{p,h,r}}(\mathbb{R}^n)$, where

$$\widetilde{\dot{\mathcal{N}}^s_{p,h,r}} := \Big\{ f \in \dot{\mathcal{N}}^s_{p,h,r} : \lim_{N\to\infty} \| \sum_{j>N} \dot{\Delta}_j f \|_{\dot{\mathcal{N}}^s_{p,h,r}} = 0 \Big\},$$

which is a subspace of Besov-Morrey spaces (the idea is inspired by [18]). See Section 2 for definitions of other notation.

To address the system (1.1), we consider the integral equation

$$u = e^{t\Delta}u_0 + \int_0^t e^{(t-t')\Delta}\nabla\cdot\big(u\nabla(-\Delta)^{-m}u\big)\,\mathrm{d}t', \tag{1.5}$$

where $e^{t\Delta} := \mathcal{F}^{-1}(e^{-t|\xi|^2}\mathcal{F})$ and $(-\Delta)^{-m} := \mathcal{F}^{-1}(|\xi|^{-2m}\mathcal{F})$ are the heat semigroup and the inverse fractional Laplacian operators, respectively. By using the contraction mapping approach to the map below, we can solve (1.5),

$$\Psi(u) := e^{t\Delta}u_0 + \int_0^t e^{(t-t')\Delta}\nabla\cdot\big(u\nabla(-\Delta)^{-m}u\big)\,\mathrm{d}t'. \tag{1.6}$$

Throughout this paper, C will represent constants that may differ at different places, $E \lesssim F$ denotes the existence of a constant $C > 0$ such that $E \le CF$ and $E \sim F$ denotes the existence of constants $C_1, C_2 > 0$ such that $C_1F \le E \le C_2F$. $BC(0,T;X)$ is the set of all functions bounded on $[0,T)$ and continuous on $(0,T)$ with values in the space X. We define for $v \in \mathcal{S}(\mathbb{R}^n)$, the Fourier transform as

$$\mathcal{F}v(\xi) = \widehat{v}(\xi) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}v(x)dx,$$

and its inverse Fourier transform as

$$\mathcal{F}^{-1}v(x) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^d} e^{ix\cdot\xi}v(\xi)d\xi.$$

Organization of the paper: In Section 2, some basic facts about Littlewood-Paley theory and some product laws in Besov-Morrey spaces are presented, and then presenting the statements of our two main results. To prove these results, we give their key estimates and prove them in Section 3. Lastly, Theorem 2.9 and Theorem 2.10 are established in Section 4.

2. Preliminaries and main results

We introduce some basic knowledge of Littlewood-Paley theory and Besov-Morreyspaces and reviews some lemmas that are pertinent to our purposes.

2.1. **Littlewood-Paley theory and Besov-Morrey spaces.** We start by recalling the Littlewood-Paley decomposition (see [3] for more details). Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a smooth radial function such that

$$\begin{gathered} 0 \le \varphi \le 1, \\ \operatorname{supp} \varphi \subset \big\{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\big\}, \\ \sum_{j \in Z} \varphi\left(2^{-j}\xi\right) = 1, \quad \text{for all } \xi \neq 0, \end{gathered}$$

and we denote $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. So, for all $u \in \mathcal{S}'(\mathbb{R}^n)$, let's set the frequency localization operators for all $j \in \mathbb{Z}$, to be as below

$$\dot{\Delta}_j u = \mathcal{F}^{-1}\varphi_j * u \quad \text{and} \quad \dot{S}_j u = \sum_{k \le j-1} \dot{\Delta}_k u. \tag{2.1}$$

Then, we have the homogeneous Littlewood-Paley decomposition

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \quad \text{for all } u \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n),$$

with $\mathcal{P}(\mathbb{R}^n)$ denoting the collection of all polynomials over $\mathbb{R}^n$. One observes here that $\dot{\Delta}_j$ has frequency $[|\xi| \sim 2^j\}$ and that $\dot{S}_j$ has frequency $[|\xi| \lesssim 2^j\}$, and one also notes that the quasi-orthogonality property holds for the Littlewood-Paley decomposition, that is, for every $u, v \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$,

$$\begin{gathered} \dot{\Delta}_i \dot{\Delta}_j u = 0 \quad \text{if } |i-j| \ge 2, \\ \dot{\Delta}_i\big(\dot{S}_{j-1} u \dot{\Delta}_j v\big) = 0 \quad \text{if } |i-j| \ge 5. \end{gathered} \tag{2.2}$$

Next, before we give the definition of Besov-Morrey spaces introduced by Kozono and Yamazaki [22], we first present that of Morrey spaces, which serve as foundation for these spaces. Refer to [22, 29, 30, 16, 2, 32], for more information on these spaces.

Definition 2.1. Let $1 \le h \le p < \infty$. The Morrey space $\mathcal{M}_h^p = \mathcal{M}_h^p(\mathbb{R}^n)$ is defined to be the set of all $u \in L^h_{\rm loc}(\mathbb{R}^n)$ such that

$$\|u\|_{\mathcal{M}_h^p} := \sup_{x_0 \in \mathbb{R}^d,\, R>0} R^{\frac{d}{p} - \frac{d}{h}} \|u\|_{L^h(B(x_0,R))} < \infty,$$

where $B(x_0, R)$ represents an open ball in $\mathbb{R}^d$ with center x_0 and radius R.

The space $\mathcal{M}_h^p$ equipped with the norm $\|\cdot\|_{\mathcal{M}_h^p}$ is a Banach space. Furthermore, we have $\mathcal{M}_{h_1}^p \hookrightarrow \mathcal{M}_{h_2}^p$ for $1 \le h_2 \le h_1 \le p < \infty$, $\mathcal{M}_p^p = L^p(\mathbb{R}^n)$ for $1 < p \le \infty$, and that $\mathcal{M}_1^1 = \mathcal{M}(\mathbb{R}^n)$ where $\mathcal{M}(\mathbb{R}^n)$ stands for the space of finite Radon measures on $\mathbb{R}^d$.

Definition 2.2. For $s \in \mathbb{R}$, $1 \le h \le p \le \infty$, $1 \le r \le \infty$ and $u \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$, we set

$$\|u\|_{\dot{\mathcal{N}}^s_{p,h,r}} := \begin{cases} \big(\sum_{j \in Z} 2^{jsr} \|\dot{\Delta}_j u\|^r_{\mathcal{M}_h^p}\big)^{1/r} & \text{if } r < \infty, \\ \sup_{j \in Z} 2^{js} \|\dot{\Delta}_j u\|_{\mathcal{M}_h^p} & \text{if } r = \infty. \end{cases}$$

Then the homogeneous Besov-Morrey space $\dot{\mathcal{N}}^s_{p,h,r} = \dot{\mathcal{N}}^s_{p,h,r}(\mathbb{R}^n)$ is defined by

$$\dot{\mathcal{N}}^s_{p,h,r} = \big\{u \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P} : \|u\|_{\dot{\mathcal{N}}^s_{p,h,r}} < \infty\big\}.$$

Note that the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^n)$ is the particular case of $p = h$;

Definition 2.3. For $s \in \mathbb{R}$, $1 \le p,\, r \le \infty$ and $u \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$, we set

$$\|u\|_{\dot{B}^s_{p,r}} := \begin{cases} \big(\sum_{j \in Z} 2^{jsr} \|\dot{\Delta}_j u\|^r_{L^p}\big)^{1/r} & \text{if } r < \infty, \\ \sup_{j \in Z} 2^{js} \|\dot{\Delta}_j u\|_{L^p} & \text{if } r = \infty. \end{cases}$$

Then the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^n)$ is defined by

$$\dot{B}^s_{p,r}(\mathbb{R}^n) = \big\{u \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P} : \|u\|_{\dot{B}^s_{p,r}} < \infty\big\}.$$

For $s \in \mathbb{R}$, $1 \le h_0 \le h \le p < \infty$ and $1 \le r_0 \le r \le \infty$, we have the following embeddings (refer to [22, 29]):

$$\dot{\mathcal{N}}^0_{p,h,1} \hookrightarrow \mathcal{M}^p_h \hookrightarrow \dot{\mathcal{N}}^0_{p,h,\infty}, \tag{2.3}$$

$$\dot{\mathcal{N}}^s_{p,h,r_0} \hookrightarrow \dot{\mathcal{N}}^s_{p,h,r}, \tag{2.4}$$

$$\dot{\mathcal{N}}^s_{p,h,r} \hookrightarrow \dot{\mathcal{N}}^s_{p,h_0,r}. \tag{2.5}$$

Throughout this paper, the following Bony paraproduct decomposition will be used,

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v), \tag{2.6}$$

with

$$\dot{T}_u v = \sum_j \dot{S}_{j-1} u \dot{\Delta}_j v, \quad \dot{R}(u,v) = \sum_j \sum_{|j-l|\le 1} \dot{\Delta}_j u \dot{\Delta}_l v.$$

For more details, see [3, 5].

2.2. **Essential lemmas.** We invoke the following lemmas.

Lemma 2.4 ([29]). (1) *(Hölder's inequality) Let $1 \le p, p_1, p_2 < \infty$ and $1 \le h, h_1, h_2 \le \infty$ satisfying $h \le p$, $h_i \le p_i (i = 1,2)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{h} = \frac{1}{h_1} + \frac{1}{h_2}$. Then for all $u \in \mathcal{M}^{p_1}_{h_1}$ and $v \in \mathcal{M}^{p_2}_{h_2}$, there exists a constant C such that*

$$\|uv\|_{\mathcal{M}^p_h} \le C\|u\|_{\mathcal{M}^{p_1}_{h_1}} \|v\|_{\mathcal{M}^{p_2}_{h_2}}. \tag{2.7}$$

(2) *(Young's inequality) Let $1 \le h \le p < \infty$. Then for all $u \in L^1$ and $v \in \mathcal{M}^p_h$, one has*

$$\|u * v\|_{\mathcal{M}^p_h} \le \|u\|_{L^1} \|v\|_{\mathcal{M}^p_h}. \tag{2.8}$$

(3) *(Sobolev embedding) Let $1 \le h \le p < \infty$, $1 \le h_0 \le p_0 < \infty$, $1 \le r \le \infty$, and let $s, s_0 \in \mathbb{R}$ with $s < s_0$. If*

$$s_0 - \frac{n}{p_0} = s(\cdot) - \frac{n}{p} \quad \text{and} \quad \frac{h_0}{p_0} = \frac{h}{p},$$

then

$$\dot{\mathcal{N}}^{s_0}_{p_0,h_0,r} \hookrightarrow \dot{\mathcal{N}}^s_{p,h,r}. \tag{2.9}$$

Lemma 2.5. *∂^α_ξ is a bounded operator from $\dot{\mathcal{N}}^{s+|\alpha|}_{p,h,r}$ to $\dot{\mathcal{N}}^s_{p,h,r}$.*

Proof. Using that $[|\xi| \sim 2^j\}$ for all $j \in \mathbb{Z}$, one has

$$\begin{aligned}
\|\partial^\alpha_\xi u\|_{\dot{\mathcal{N}}^s_{p,h,r}} &= \Big(\sum_{j\in Z} 2^{srj} \|\dot{\Delta}_j \partial^\alpha_\xi u\|^r_{\mathcal{M}^p_h}\Big)^{1/r} \\
&= \Big(\sum_{j\in Z} 2^{srj} \||\xi|^\alpha \dot{\Delta}_j u\|^r_{\mathcal{M}^p_h}\Big)^{1/r} \\
&\lesssim \Big(\sum_{j\in Z} 2^{srj} 2^{|\alpha|jr} \|\dot{\Delta}_j u\|^r_{\mathcal{M}^p_h}\Big)^{1/r} \\
&\lesssim \|u\|_{\dot{\mathcal{N}}^{s+|\alpha|}_{p,h,r}}.
\end{aligned}$$

□

Remark 2.6. Lemma 2.5 gives the following inequalities:

$$\begin{gathered}
\|\nabla^m u\|_{\dot{\mathcal{N}}^s_{p,h,r}} \lesssim \|u\|_{\dot{\mathcal{N}}^{s+m}_{p,h,r}}, \\
\|\nabla^m \cdot u\|_{\dot{\mathcal{N}}^s_{p,h,r}} \lesssim \|u\|_{\dot{\mathcal{N}}^{s+m}_{p,h,r}}, \\
\|\Delta^m u\|_{\dot{\mathcal{N}}^s_{p,h,r}} \lesssim \|u\|_{\dot{\mathcal{N}}^{s+2m}_{p,h,r}}.
\end{gathered}$$

Lemma 2.7 ([25, 22, 21])**.** *Let $1 \le h \le p < \infty$, $1 \le h_0 \le p_0 < \infty$, $1 \le r \le \infty$ and $s \in \mathbb{R}$. If $p \le p_0$ and $\frac{h_0}{p_0} = \frac{h}{p}$. Then for all $u \in \dot{\mathcal{N}}^s_{p,h,\infty}$, one has*

$$\|e^{t\Delta}u\|_{\dot{\mathcal{N}}^s_{p_0,h_0,r}} \lesssim t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{p_0}\right)}\|u\|_{\dot{\mathcal{N}}^s_{p,h,r}}. \tag{2.10}$$

Lemma 2.8 ([25, 22, 21])**.** *Let $1 \le h \le p < \infty$, $s \in \mathbb{R}$ and $\varepsilon > 0$. Then for all $u \in \dot{\mathcal{N}}^s_{p,h,\infty}$, one has*

$$\|e^{t\Delta}u\|_{\dot{\mathcal{N}}^{s+\varepsilon}_{p,h,1}} \lesssim t^{-\frac{\varepsilon}{2}}\|u\|_{\dot{\mathcal{N}}^s_{p,h,\infty}}. \tag{2.11}$$

2.3. **Main results.** In this subsection, we state the two main theorems of this work. Our local well-posedness theorem for the system (1.1) with $u_0 \in \widetilde{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}$ is the following.

Theorem 2.9. *Let $n \ge 1$, $1/2 < m < 1$ and $1 \le h \le p < \infty$. Assume that $u_0 \in \widetilde{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}$. Then we have the following results:*

(1) *Let $\max\{1, \frac{n}{2m}\} < p \le n$ and let p_1, p_2, h_1, h_2 be arbitrary real numbers satisfying*

$$\begin{gathered} p \le p_1, p_2 < \infty, \quad h_1 \le p_1, \quad p_2 \in (d,\infty) \cap [h_2,\infty), \\ \frac{1}{p} < \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{h} = \frac{1}{h_1} + \frac{1}{h_2}, \quad \frac{h}{p} = \frac{h_1}{p_1} = \frac{h_2}{p_2}. \end{gathered} \tag{2.12}$$

Then, there exist $\delta > 0$ and $T > 0$ such that the system (1.1) admits a unique time-local solution $u \in BC\big(0,T;\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}\big)$, and satisfy

$$\sup_{t\in(0,T)} \|u(t)\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} < \infty,$$

$$\sup_{t\in(0,T)} t^{m-\frac{n}{2p_1}}\|u(t)\|_{\mathcal{M}^{p_1}_{h_1}} + \sup_{t\in(0,T)} t^{\frac{1}{2}-\frac{n}{2p_2}}\||\nabla|^{1-2m}u(t)\|_{\mathcal{M}^{p_2}_{h_2}} \le \delta.$$

(2) *Let $n < p < \infty$ and let $\rho, \hbar$ be arbitrary real numbers satisfying*

$$\max\{p,\hbar\} \le \rho \le 2p, \quad \frac{\hbar}{\rho} = \frac{h}{p}. \tag{2.13}$$

Then, there exist $\delta > 0$ and $T > 0$ such that the system (1.1) admits a unique time-local solution $u \in BC\big(0,T;\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}\big)$, and satisfy

$$\sup_{t\in(0,T)} \|u(t)\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} < \infty, \quad \sup_{t\in(0,T)} t^{\frac{1}{2}-\frac{n}{2\rho}}\|u(t)\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,1}} \le \delta.$$

Next, we present the global well-posedness theorem.

Theorem 2.10. *Let $n \ge 1$, $1/2 < m < 1$ and $1 \le h \le p < \infty$.*

(1) *Let $\max\{1, \frac{n}{2m}\} < p \le n$ and let p_1, p_2, h_1, h_2 be arbitrary real numbers satisfying (2.12). Then there is a constant $\delta > 0$ such that for any $u_0 \in \dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}$ satisfying $\|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} \le \delta$, the system (1.1) has a unique global solution $u \in BC\big(0,\infty;\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}\big)$, and satisfy*

$$\begin{aligned} &\sup_{t\in(0,\infty)} \|u(t)\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} + \sup_{t\in(0,\infty)} t^{m-\frac{n}{2p_1}}\|u(t)\|_{\mathcal{M}^{p_1}_{h_1}} \\ &+ \sup_{t\in(0,\infty)} t^{\frac{1}{2}-\frac{n}{2p_2}}\||\nabla|^{1-2m}u(t)\|_{\mathcal{M}^{p_2}_{h_2}} \le C_0, \end{aligned}$$

where C_0 is a constant depending on δ.

(2) *Let $n < p < \infty$ and let $\rho, \hbar$ satisfy (2.13). Then there exists a constant $\delta > 0$ such that for any $u_0 \in \dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}$ satisfying $\|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} \le \delta$, the system (1.1) has a unique global solution $u \in BC\big(0,\infty;\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}\big)$, and satisfies*

$$\sup_{t\in(0,\infty)} \|u(t)\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} + \sup_{t\in(0,\infty)} t^{\frac{1}{2}-\frac{n}{2\rho}}\|u(t)\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,1}} \le C_0,$$

where C_0 is a constant depending on δ.

Remark 2.11. *The method used in this paper, for proving local and global well-posedness results, is different from the one in [35] and [36]. There, the authors used the Chemin mono-norm method, to study systems (1.1) and (1.1) with $m = 1$, respectively. In Besov spaces $\dot{B}^s_{p,r}$, their functional framework is distinct from our own. Indeed, one has $\dot{\mathcal{N}}^s_{p,h,r} \not\subset \dot{B}^{s'}_{p',r'}$ for any $1 \le p' < \infty$, $1 \le r' < \infty$ and $s' \in \mathbb{R}$. However, the results of this work remain valid if we take Besov spaces $\dot{B}^s_{p,r}$ instead of Besov-Morrey spaces $\dot{\mathcal{N}}^s_{p,h,r}$. In fact, if we have $p = h$, then $\dot{\mathcal{N}}^s_{p,p,r} = \dot{B}^s_{p,r}$.*

3. Key estimates

We get our critical estimates out of the way in this section. First, the following linear estimate is used.

Lemma 3.1. *Let $0 < T \le \infty$, $\frac{1}{2} < m < 1$ and $1 \le h \le p < \infty$.*

(1) *Let $\frac{n}{2m} < p \le n$ and p_1, p_2, h_1, h_2 satisfy (2.12), and let $\|\cdot\|_{X_T}$ be given as*

$$\begin{aligned} \|u\|_{X_T} := &\sup_{t\in(0,T)} \|u(t)\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} + \sup_{t\in(0,T)} t^{m-\frac{n}{2p_1}} \|u(t)\|_{\mathcal{M}^{p_1}_{h_1}} \\ &+ \sup_{t\in(0,T)} t^{\frac{1}{2}-\frac{n}{2p_2}} \||\nabla|^{1-2m} u(t)\|_{\mathcal{M}^{p_2}_{h_2}}. \end{aligned} \tag{3.1}$$

Then

$$\|e^{t\Delta}u_0\|_{X_T} \lesssim \|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}. \tag{3.2}$$

(2) *Let $p > n$ and $\rho, \hbar$ satisfy (2.13), and let $\|\cdot\|_{Y_T}$ be defined by*

$$\|u\|_{Y_T} := \sup_{t\in(0,T)} \|u(t)\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} + \sup_{t\in(0,T)} t^{\frac{1}{2}-\frac{n}{2\rho}} \|u(t)\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,1}}. \tag{3.3}$$

Then

$$\|e^{t\Delta}u_0\|_{Y_T} \lesssim \|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}. \tag{3.4}$$

Proof. (1) On the one hand, we have

$$\|e^{t\Delta}u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} \lesssim \|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}. \tag{3.5}$$

On the other hand, since $p_1 \ge p > \frac{n}{2m}$ and $\frac{h_1}{p_1} = \frac{h}{p}$, using the embedding (2.3) and the embedding (2.9), and Lemma 2.8 with $\varepsilon = 2m - \frac{n}{p_1} > 0$, we obtain

$$\begin{aligned} t^{m-\frac{n}{2p_1}} \|e^{t\Delta}u_0\|_{\mathcal{M}^{p_1}_{h_1}} &\lesssim t^{m-\frac{n}{2p_1}} \|e^{t\Delta}u_0\|_{\dot{\mathcal{N}}^{\frac{n}{p}-\frac{n}{p_1}}_{p,h,1}} \\ &\lesssim t^{m-\frac{n}{2p_1}} t^{-\frac{2m-\frac{n}{p_1}}{2}} \|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} \\ &\lesssim \|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}, \end{aligned} \tag{3.6}$$

and since $p_2 > n$, $p \le p_2$, and $\frac{h_2}{p_2} = \frac{h}{p}$, using the embeddings (2.3) and (2.9), Remark 2.6 and Lemma 2.8 with $\varepsilon = 1 - \frac{n}{p_2} > 0$, one has

$$\begin{aligned} t^{\frac{1}{2}-\frac{n}{2p_2}} \||\nabla|^{1-2m} e^{t\Delta}u_0\|_{\mathcal{M}^{p_2}_{h_2}} &\lesssim t^{\frac{1}{2}-\frac{n}{2p_2}} \|e^{t\Delta}u_0\|_{\dot{\mathcal{N}}^{\frac{n}{p}-\frac{n}{p_2}+1-2m}_{p,h,1}} \\ &\lesssim t^{\frac{1}{2}-\frac{n}{2p_2}} t^{-\frac{1-\frac{n}{p_2}}{2}} \|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} \\ &\lesssim \|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}. \end{aligned} \tag{3.7}$$

Finally, estimates (3.5)-(3.7) yield (3.2).

(2) Since $n < p \le \rho$ and $\frac{\hbar}{\rho} = \frac{h}{p}$, by using Lemmas 2.7 and 2.8 with $\varepsilon = 1 - \frac{n}{p} > 0$, we obtain
$$t^{\frac12 - \frac{n}{2\rho}} \|e^{t\Delta} u_0\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,1}} \lesssim t^{\frac12 - \frac{n}{2\rho}} t^{-\frac{1-\frac{n}{p}}{2} - \frac{n}{2}\left(\frac1p - \frac1\rho\right)} \|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} \lesssim \|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}. \tag{3.8}$$
Thus, estimates (3.5) and (3.8) yield (3.4). □

The second lemma is a bilinear estimation.

Lemma 3.2. *Let $1/2 < m < 1$ and let $1 \le \hbar \le \rho < \infty$, $1 \le \hbar_1 \le \rho_2 < \infty$ and $1 \le \hbar_2 \le \rho_2 < \infty$ satisfying $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$ and $\frac{1}{\hbar} = \frac{1}{\hbar_1} + \frac{1}{\hbar_2}$. Then for all $f \in \dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,1}$ and $g \in \dot{\mathcal{N}}^{1-2m}_{\rho_2,\hbar_2,1}$, one has*
$$\|f\nabla(-\Delta)^{-m} g + g\nabla(-\Delta)^{-m} f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,\infty}} \lesssim \|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,1}} \|g\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_2,\hbar_2,1}}. \tag{3.9}$$

Proof. Using the paraproduct decomposition due to Bony [5], we have
$$f\nabla(-\Delta)^{-m} g + g\nabla(-\Delta)^{-m} f := J_1 + J_2 + J_3, \tag{3.10}$$
where
$$\begin{aligned} J_1 &:= \sum_{l\in\mathbb{Z}} \dot{\Delta}_l f \nabla(-\Delta)^{-m} \dot{S}_{l-1} g + \dot{\Delta}_l g \nabla(-\Delta)^{-m} \dot{S}_{l-1} f, \\ J_2 &:= \sum_{l\in\mathbb{Z}} \dot{S}_{l-1} f \nabla(-\Delta)^{-m} \dot{\Delta}_l g + \dot{S}_{l-1} g \nabla(-\Delta)^{-m} \dot{\Delta}_l f, \\ J_3 &:= \sum_{l\in\mathbb{Z}} \sum_{|l-k'|\le 1} \dot{\Delta}_l f \nabla(-\Delta)^{-m} \dot{\Delta}_{k'} g + \dot{\Delta}_l g \nabla(-\Delta)^{-m} \dot{\Delta}_{k'} f. \end{aligned}$$
Below, we estimate J_1, J_2 and J_3 separately. For J_1, we consider the estimate of its first term only, while the second one can be treated similarly. So, by (2.1) and (2.2), the embedding (2.3), Hölder's inequality (2.7), and Remake 2.6, we obtain
$$\begin{aligned} &\|\dot{\Delta}_j \sum_{l\in\mathbb{Z}} \dot{\Delta}_l f \nabla(-\Delta)^{-m} \dot{S}_{l-1} g\|_{\mathcal{M}^{\rho}_{\hbar}} \\ &\lesssim \sum_{|l-j|\le 4} \|\dot{\Delta}_l f \nabla(-\Delta)^{-m} \dot{S}_{l-1} g\|_{\mathcal{M}^{\rho}_{\hbar}} \\ &\lesssim \sum_{|l-j|\le 4} \|\dot{\Delta}_l f\|_{\mathcal{M}^{\rho_1}_{\hbar_1}} \|\nabla(-\Delta)^{-m} \dot{S}_{l-1} g\|_{\mathcal{M}^{\rho_2}_{\hbar_2}} \\ &\lesssim 2^{-(1-2m)j} \sum_{|l-j|\le 4} 2^{-(1-2m)(l-j)} 2^{(1-2m)l} \|\dot{\Delta}_l f\|_{\mathcal{M}^{\rho_1}_{\hbar_1}} \|\nabla(-\Delta)^{-m} g\|_{\mathcal{M}^{\rho_2}_{\hbar_2}} \\ &\lesssim 2^{-(1-2m)j} \Big(\sup_{|l-j|\le 4} 2^{-(1-2m)(l-j)} \Big) \|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,1}} \|\nabla(-\Delta)^{-m} g\|_{\dot{\mathcal{N}}^{0}_{\rho_2,\hbar_2,1}} \\ &\lesssim 2^{-(1-2m)j} \|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,1}} \|g\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_2,\hbar_2,1}}. \end{aligned}$$
Similarly,
$$\|\dot{\Delta}_j \sum_{l\in\mathbb{Z}} \dot{\Delta}_l g \nabla(-\Delta)^{-m} \dot{S}_{l-1} f\|_{\mathcal{M}^{\rho}_{\hbar}} \lesssim 2^{-(1-2m)j} \|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,1}} \|g\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_2,\hbar_2,1}}.$$
Which implies that
$$\|\dot{\Delta}_j J_1\|_{\mathcal{M}^{\rho}_{\hbar}} \lesssim 2^{-(1-2m)j} \|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,1}} \|g\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_2,\hbar_2,1}}. \tag{3.11}$$
Analogously for J_2, applying the embedding (2.3), Hölder's inequality (2.7), and Remark 2.6 again, since $m > 1/2$, one has
$$\begin{aligned} &\|\dot{\Delta}_j \sum_{l\in\mathbb{Z}} \dot{S}_{l-1} f \nabla(-\Delta)^{-m} \dot{\Delta}_l g\|_{\mathcal{M}^{\rho}_{\hbar}} \\ &\lesssim \sum_{|l-j|\le 4} \|\dot{S}_{l-1} f \nabla(-\Delta)^{-m} \dot{\Delta}_l g\|_{\mathcal{M}^{\rho}_{\hbar}} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{|l-j|\le 4}\sum_{k\le l-2}\|\dot\Delta_k f\|_{\mathcal{M}^{\rho_1}_{\hbar_1}}\|\nabla(-\Delta)^{-m}\dot\Delta_l g\|_{\mathcal{M}^{\rho_2}_{\hbar_2}}\\
&\lesssim \sum_{|j-l|\le 4}2^{-(1-2m)l}\sum_{k\le l-2}2^{-(1-2m)(k-l)}2^{(1-2m)k}\|\dot\Delta_k f\|_{\mathcal{M}^{\rho_1}_{\hbar_1}}\|\nabla(-\Delta)^{-m}g\|_{\mathcal{M}^{\rho_2}_{\hbar_2}}\\
&\lesssim \sum_{|j-l|\le 4}2^{-(1-2m)l}\Big(\sup_{k\le l-2}2^{-(1-2m)(k-l)}\Big)\|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,1}}\|\nabla(-\Delta)^{-m}g\|_{\dot{\mathcal{N}}^{0}_{\rho_2,\hbar_2,1}}\\
&\lesssim 2^{-(1-2m)j}\Big(\sum_{|j-l|\le 4}2^{-(1-2m)(l-j)}\Big)\|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,1}}\|g\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_2,\hbar_2,1}}\\
&\lesssim 2^{-(1-2m)j}\|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,1}}\|g\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_2,\hbar_2,1}}.
\end{aligned}
$$

Thus, when $m>1/2$, we have

$$
\|\dot\Delta_j J_2\|_{\mathcal{M}^{\rho}_{\hbar}}\lesssim 2^{-(1-2m)j}\|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,1}}\|g\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_2,\hbar_2,1}}. \tag{3.12}
$$

We are now moving on to the last term J_3. We use the following formula, based on an analysis of the algebraic structure of the system (1.1):

$$
(J_3)_i=\sum_{l\in\mathbb{Z}}\sum_{|l-k'|\le 1}\dot\Delta_l f\partial_i(-\Delta)^{-m}\dot\Delta_{k'}g+\dot\Delta_l g\partial_i(-\Delta)^{-m}\dot\Delta_{k'}f=K_i^1+K_i^2+K_i^3,
$$

for $i=1,2,\dots,n$. Where $(J_3)_i$ is the ith exponent of (J_3) and

$$
\begin{aligned}
K_i^1&:=\sum_{l\in\mathbb{Z}}\sum_{|l-k'|\le 1}(-\Delta)^m\Big[\Big((-\Delta)^{-m}\dot\Delta_l f\Big)\Big(\partial_i(-\Delta)^{-m}\dot\Delta_{k'}g\Big)\Big],\\
K_i^2&:=\sum_{l\in\mathbb{Z}}\sum_{|l-k'|\le 1}2\nabla^m\cdot\Big[\Big((-\Delta)^{-m}\dot\Delta_l f\Big)\Big(\partial_i\nabla^m(-\Delta)^{-m}\dot\Delta_{k'}g\Big)\Big],\\
K_i^3&:=\sum_{l\in\mathbb{Z}}\sum_{|l-k'|\le 1}\partial_i\Big[\Big((-\Delta)^{-m}\dot\Delta_l f\Big)\dot\Delta_{k'}g\Big].
\end{aligned}
$$

To estimate the above three terms, we use Hölder's inequality (2.7), Remark 2.6 and Cauchy-Schwartz's inequality in the following way: From (2.2), there is $d_0\in\mathbb{N}$ such that

$$
\begin{aligned}
\|\dot\Delta_j K_i^1\|_{\mathcal{M}^{\rho}_{\hbar}}&\lesssim 2^{2mj}\sum_{\substack{|l-k'|\le 1\\ l,k'\ge j-d_0}}\|\big((-\Delta)^{-m}\dot\Delta_l f\big)\big(\partial_i(-\Delta)^{-m}\dot\Delta_{k'}g\big)\|_{\mathcal{M}^{\rho}_{\hbar}}\\
&\lesssim 2^{2mj}\sum_{\substack{|l-k'|\le 1\\ l,k'\ge j-d_0}}2^{-2ml}\|\dot\Delta_l f\|_{\mathcal{M}^{\rho_1}_{\hbar_1}}2^{(1-2m)k'}\|\dot\Delta_{k'}g\|_{\mathcal{M}^{\rho_2}_{\hbar_2}}\\
&\lesssim 2^{2mj}\sum_{\substack{|l-k'|\le 1\\ l,k'\ge j-d_0}}2^{-j}\Big(\sup_{l\ge j-d_0}2^{-(l-j)}\Big)2^{(1-2m)l}\|\dot\Delta_l f\|_{\mathcal{M}^{\rho_1}_{\hbar_1}}2^{(1-2m)k'}\|\dot\Delta_{k'}g\|_{\mathcal{M}^{\rho_2}_{\hbar_2}}\\
&\lesssim 2^{-(1-2m)j}\|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,2}}\|g\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_2,\hbar_2,2}}.
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
\|\dot\Delta_j K_i^2\|_{\mathcal{M}^{\rho}_{\hbar}}&\lesssim 2^{mj}\sum_{\substack{|l-k'|\le 1\\ l,k'\ge j-d_0}}\|\Big((-\Delta)^{-m}\dot\Delta_l f\Big)\Big(\partial_i\nabla^m(-\Delta)^{-m}\dot\Delta_{k'}g\Big)\|_{\mathcal{M}^{\rho}_{\hbar}}\\
&\lesssim 2^{mj}\sum_{\substack{|l-k'|\le 1\\ l,k'\ge j-d_0}}2^{-2ml}\|\dot\Delta_l f\|_{\mathcal{M}^{\rho_1}_{\hbar_1}}2^{(1-m)k'}\|\dot\Delta_{k'}g\|_{\mathcal{M}^{\rho_2}_{\hbar_2}}\\
&\lesssim 2^{mj}\sum_{\substack{|l-k'|\le 1\\ l,k'\ge j-d_0}}2^{-2ml}\|\dot\Delta_l f\|_{\mathcal{M}^{\rho_1}_{\hbar_1}}2^{ml}\Big(\sup_{|l-k'|\le 1}2^{m(k'-l)}\Big)2^{(1-2m)k'}\|\dot\Delta_{k'}g\|_{\mathcal{M}^{\rho_2}_{\hbar_2}}
\end{aligned}
$$

$$\lesssim 2^{-(1-2m)j} \sum_{\substack{|l-k'|\le 1\\ l,k'\ge j-d_0}} \Big(\sup_{l\ge j-d_0} 2^{-(1-m)(l-j)}\Big) 2^{(1-2m)l}\|\dot\Delta_l f\|_{\mathcal{M}^{\rho_1}_{\hbar_1}} 2^{(1-2m)k'}\|\dot\Delta_{k'} g\|_{\mathcal{M}^{\rho_2}_{\hbar_2}}.$$

Since $m<1$, then

$$\|\dot\Delta_j K_i^2\|_{\mathcal{M}^{\rho}_{\hbar}} \lesssim 2^{-(1-2m)j}\|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,2}}\|g\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_2,\hbar_2,2}}. \tag{3.14}$$

$$\begin{aligned}
&\|\dot\Delta_j K_i^3\|_{\mathcal{M}^{\rho}_{\hbar}}\\
&\lesssim 2^j \sum_{\substack{|l-k'|\le 1\\ l,k'\ge j-d_0}} \|\Big((-\Delta)^{-m}\dot\Delta_l f\Big)\dot\Delta_{k'} g\|_{\mathcal{M}^{\rho}_{\hbar}}\\
&\lesssim 2^j \sum_{\substack{|l-k'|\le 1\\ l,k'\ge j-d_0}} 2^{-2ml}\|\dot\Delta_l f\|_{\mathcal{M}^{\rho_1}_{\hbar_1}}\|\dot\Delta_{k'} g\|_{\mathcal{M}^{\rho_2}_{\hbar_2}}\\
&\lesssim 2^j \sum_{\substack{|l-k'|\le 1\\ l,k'\ge j-d_0}} 2^{-2ml}\|\dot\Delta_l f\|_{\mathcal{M}^{\rho_1}_{\hbar_1}} 2^{-(1-2m)l}\Big(\sup_{|l-k'|\le 1} 2^{-(1-2m)(k'-l)}\Big) 2^{(1-2m)k'}\|\dot\Delta_{k'} g\|_{\mathcal{M}^{\rho_2}_{\hbar_2}}\\
&\lesssim 2^{-(1-2m)j} \sum_{\substack{|l-k'|\le 1\\ l,k'\ge j-d_0}} \Big(\sup_{l\ge j-d_0} 2^{-2(1-m)(l-j)}\Big) 2^{(1-2m)l}\|\dot\Delta_l f\|_{\mathcal{M}^{\rho_1}_{\hbar_1}} 2^{(1-2m)k'}\|\dot\Delta_{k'} g\|_{\mathcal{M}^{\rho_2}_{\hbar_2}}.
\end{aligned}$$

Since $m<1$, we have

$$\|\dot\Delta_j K_i^3\|_{\mathcal{M}^{\rho}_{\hbar}} \lesssim 2^{-(1-2m)j}\|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,2}}\|g\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_2,\hbar_2,2}}. \tag{3.15}$$

Thus, (3.13), (3.14), (3.15) and the embedding (2.4), give us

$$\|\dot\Delta_j J_3\|_{\mathcal{M}^{\rho}_{\hbar}} \le \sum_{i=1}^{n}\sum_{k=1}^{3}\|\dot\Delta_j K_i^k\|_{\mathcal{M}^{\rho}_{\hbar}} \lesssim 2^{-(1-2m)j}\|f\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_1,\hbar_1,1}}\|g\|_{\dot{\mathcal{N}}^{1-2m}_{\rho_2,\hbar_2,1}}. \tag{3.16}$$

Eventually, we combine (3.11), (3.12) and (3.16), then multiply by $2^{(1-2m)j}$ and take $l^\infty-$norm on both sides of the resultant estimate to obtain (3.9). The proof of Lemma 3.2 is complete. □

The last estimate is a corollary of the following lemma.

Lemma 3.3 ([25])**.** *Let $s\in\mathbb{R}$, $\beta>0$, $1\le h\le p<\infty$, $1\le h_0\le p_0<\infty$ and $1\le r\le\infty$. If $p_0\ge p$ and $\frac{h_0}{p_0}=\frac{h}{p}$. Then for all $u\in\widetilde{\dot{\mathcal{N}}^{s}_{p,h,r}}$, we have*

$$t^{\frac{\beta}{2}+\frac{n}{2}\left(\frac{1}{p}-\frac{1}{p_0}\right)}\|e^{t\Delta}u\|_{\dot{\mathcal{N}}^{s+\beta}_{p_0,h_0,r}} \xrightarrow{t\to 0} 0.$$

Lemma 3.4. *Let $0<T<\infty$, $\frac12<m<1$, $1\le h\le p<\infty$ and $u_0\in\widetilde{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}$.*

(1) *Let $\frac{n}{2m}<p\le n$ and let p_1,p_2,h_1,h_2 satisfy (2.12). Then, one has*

$$t^{m-\frac{n}{2p_1}}\|e^{t\Delta}u_0\|_{\mathcal{M}^{p_1}_{h_1}} + t^{\frac12-\frac{n}{2p_2}}\||\nabla|^{1-2m}e^{t\Delta}u_0\|_{\mathcal{M}^{p_2}_{h_2}} \xrightarrow{t\to 0} 0.$$

(2) *Let $p>n$, the parameters $\rho,\hbar$ satisfy (2.13) and $1\le r\le\infty$. Then, one has*

$$t^{\frac12-\frac{n}{2\rho}}\|e^{t\Delta}u_0\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,r}} \xrightarrow{t\to 0} 0.$$

Proof. (1) By applying the embedding (2.3) and Lemma 3.3 for $s=-2m+\frac{n}{p}$, $\beta=2m-\frac{n}{p}>0$, $p_0=p_1>p$, $h_0=h_1$ and $r=1$, we obtain

$$t^{m-\frac{n}{2p_1}}\|e^{t\Delta}u_0\|_{\mathcal{M}^{p_1}_{h_1}} \lesssim t^{m-\frac{n}{2p_1}}\|e^{t\Delta}u_0\|_{\dot{\mathcal{N}}^{0}_{p_1,h_1,1}} \xrightarrow{t\to 0} 0. \tag{3.17}$$

For the second term, by the Adams theorem [1], we have

$$t^{\frac12-\frac{n}{2p_2}}\||\nabla|^{1-2m}e^{t\Delta}u_0\|_{\mathcal{M}^{p_2}_{h_2}} \lesssim t^{\frac{1+\zeta}{2}-\frac{n}{2p_3}}\||\nabla|^{\zeta+1-2m}e^{t\Delta}u_0\|_{\mathcal{M}^{p_3}_{h_3}},$$

where $\zeta > 0$ and p_3, h_3 satisfy

$$1 \le h_3 \le p_3 < \infty, \quad p_3 > p, \quad \frac{h_3}{p_3} = \frac{h_2}{p_2}, \quad \frac{1}{p_3} = \frac{1}{p_2} + \frac{\zeta}{d}.$$

Using again the embedding (2.3) and Lemma 3.3 for $s = -2m + \frac{n}{p}$, $\beta = \zeta + 1 - \frac{n}{p} > 0$, $p_0 = p_3 > p$, $h_0 = h_3$ and $r = 1$, we obtain

$$\begin{aligned} t^{\frac{1+\zeta}{2} - \frac{n}{2p_3}} \| |\nabla|^{\zeta+1-2m} e^{t\Delta} u_0 \|_{\mathcal{M}^{p_3}_{h_3}} &\lesssim t^{\frac{1+\zeta}{2} - \frac{n}{2p_3}} \| |\nabla|^{\zeta+1-2m} e^{t\Delta} u_0 \|_{\dot{\mathcal{N}}^{0}_{p_3,h_3,1}} \\ &\lesssim t^{\frac{1+\zeta}{2} - \frac{n}{2p_3}} \| |e^{t\Delta} u_0 \|_{\dot{\mathcal{N}}^{\zeta+1-2m}_{p_3,h_3,1}} \xrightarrow{t\to 0} 0. \end{aligned}$$

Thus,

$$t^{\frac{1}{2} - \frac{n}{2p_2}} \| |\nabla|^{1-2m} e^{t\Delta} u_0 \|_{\mathcal{M}^{p_2}_{h_2}} \xrightarrow{t\to 0} 0. \tag{3.18}$$

Then (3.17) and (3.18) yield the desired result.

(2) By choosing $s = -2m + \frac{n}{p}$, $\beta = 1 - \frac{n}{p} > 0$, $p_0 = \rho \ge p$ and $h_0 = \hbar$, Lemma 3.3 gives us the desired limit. □

4. Proof of main theorems

We find local solutions with any initial data u_0 in the subspaces $\widetilde{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}$ and global solutions with small initial data u_0 in the critical Besov-Morrey spaces $\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}$, for the gathered $n < p < \infty$ and $\frac{n}{2m} < p \le n$ in Subsection 4.1 and Subsection 4.2, respectively.

4.1. Case $n < p < \infty$: Proof Theorems 2.9 (2) and 2.10 (2).

. We first prove the following bilinear estimate.

Proposition 4.1. *Let $0 < T \le \infty$, $\frac{1}{2} < m < 1$, $n < p < \infty$, $1 \le h \le p$, the parameters $\rho, \hbar$ satisfy* (2.13)*, and let*

$$\|u\|_{Y^1_T} := \sup_{t\in(0,T)} t^{\frac{1}{2} - \frac{n}{2\rho}} \|u(t)\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,1}}. \tag{4.1}$$

Then

$$\| \int_0^t e^{(t-t')\Delta} \nabla \cdot \big(u\nabla(-\Delta)^{-m} w + w\nabla(-\Delta)^{-m} u \big)\, \mathrm{d}t' \|_{Y_T} \lesssim \|u\|_{Y^1_T} \|w\|_{Y^1_T}.$$

Proof. Recall that for $0 < T \le \infty$, $\|\cdot\|_{Y_T}$ is given by

$$\|u\|_{Y_T} := \sup_{t\in(0,T)} \|u(t)\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} + \sup_{t\in(0,T)} t^{\frac{1}{2} - \frac{n}{2\rho}} \|u(t)\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,1}}.$$

We denote $H := v\nabla(-\Delta)^{-m} w + w\nabla(-\Delta)^{-m} u$. Applying Remark 2.6, the embedding (2.4) and Lemma 2.8, we obtain

$$\begin{aligned} \| \int_0^t e^{(t-t')\Delta} \nabla \cdot H \mathrm{d}t' \|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} &\lesssim \int_0^t \| e^{(t-t')\Delta} H \mathrm{d}t' \|_{\dot{\mathcal{N}}^{1-2m+\frac{n}{p}}_{p,\hbar,1}} \mathrm{d}t' \\ &\lesssim \int_0^t (t-t')^{-\frac{n}{2p} - \frac{n}{2}\left(\frac{2}{\rho} - \frac{1}{p}\right)} \|H\|_{\dot{\mathcal{N}}^{1-2m}_{\frac{\rho}{2},\frac{\hbar}{2},\infty}} \mathrm{d}t'. \end{aligned}$$

From Lemma 3.2, we have

$$\begin{aligned} \| \int_0^t e^{(t-t')\Delta} \nabla \cdot H \mathrm{d}t' \|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} &\lesssim \int_0^t (t-t')^{-\frac{n}{\rho}} \|u(t')\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,1}} \|w(t')\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,1}} \mathrm{d}t' \\ &\lesssim \int_0^t (t-t')^{-\frac{n}{\rho}} (t')^{2(-\frac{1}{2} + \frac{n}{2\rho})} \|u\|_{Y^1_T} \|w\|_{Y^1_T} \mathrm{d}t' \\ &\lesssim \|u\|_{Y^1_T} \|w\|_{Y^1_T} \int_0^t (t-t')^{-\frac{n}{\rho}} (t')^{-1+\frac{n}{\rho}} \mathrm{d}t' \\ &\lesssim \|u\|_{Y^1_T} \|w\|_{Y^1_T}. \end{aligned} \tag{4.2}$$

For the second term of $\|\cdot\|_{Y_T}$, we apply Remark 2.6 and Lemmas 2.8 and 3.2 again, to obtain

$$\begin{aligned}
\|\int_0^t e^{(t-t')\Delta}\nabla\cdot H\mathrm{d}t'\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,1}} &\lesssim \int_0^t \|e^{(t-t')\Delta}H\mathrm{d}t'\|_{\dot{\mathcal{N}}^{2-2m}_{\rho,\hbar,1}} \\
&\lesssim \int_0^t (t-t')^{-\frac{1}{2}-\frac{n}{2}\left(\frac{2}{\rho}-\frac{1}{\rho}\right)}\|H\|_{\dot{\mathcal{N}}^{1-2m}_{\frac{\rho}{2},\frac{\hbar}{2},\infty}}\mathrm{d}t' \\
&\lesssim \int_0^t (t-t')^{-\frac{1}{2}-\frac{n}{2\rho}}\|u(t')\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,1}}\|w(t')\|_{\dot{\mathcal{N}}^{1-2m}_{\rho,\hbar,1}}\mathrm{d}t' \\
&\lesssim \|u\|_{Y_T^1}\|w\|_{Y_T^1}\int_0^t (t-t')^{-\frac{1}{2}-\frac{n}{2\rho}}(t')^{-\frac{1}{2}+\frac{n}{2\rho}}(t')^{-\frac{1}{2}+\frac{n}{2\rho}}\mathrm{d}t' \\
&\lesssim t^{-\frac{1}{2}+\frac{n}{2\rho}}\|u\|_{Y_T^1}\|w\|_{Y_T^1}.
\end{aligned}$$

Finally, we obtain the estimate of $\|\cdot\|_{Y_T}$ by multiplying the two sides of the above inequality by $t^{\frac{1}{2}-\frac{n}{2\rho}}$, and combine the resultant estimate with (4.2). The proof of Proposition 4.1 is complete. □

Now we can prove the existence of local solutions of system (1.1) for any initial data u_0 in $\widetilde{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}$ with $1\le h\le p<\infty$, in the case $n<p<\infty$.

Proof of Theorem 2.9 (2) For $T>0$ and $t\in[0,T]$, we define the map

$$\Psi : u(t)\to e^{t\Delta}u_0+\int_0^t e^{(t-t')\Delta}\nabla\cdot\big(u\nabla(-\Delta)^{-m}u\big)\,\mathrm{d}t', \tag{4.3}$$

in the following metric space, for a small $\delta>0$,

$$Y_T^1 := \{u\in BC\big(0,T;\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}\big) : \|u\|_{Y_T^1}\le\delta\},$$

equipped with the distance

$$d(u,w) := \|u-w\|_{Y_T^1}.$$

According to Lemma 3.4 (2), we have the existence of $T>0$ such that

$$\|e^{t\Delta}u_0\|_{Y_T^1}\le\frac{\delta}{2},$$

and we choose T satisfying this. Then by Proposition 4.1, there exists C_1 such that for every $u\in Y_T^1$, we have

$$\|\Psi(u)\|_{Y_T^1}\le\|e^{t\Delta}u_0\|_{Y_T^1}+C_1\big(\|u\|_{Y_T^1}\big)^2\le\frac{\delta}{2}+C_1\delta^2,$$

and, by setting $A(u,w):=\int_0^t e^{(t-t')\Delta}\nabla\cdot\big(u\nabla(-\Delta)^{-m}w\big)\mathrm{d}t'$, there exists C_2 such that for every $u,w\in Y_T^1$, we obtain

$$\begin{aligned}
d\left(\Psi(u),\Psi(w)\right) &= \|A(u,u)-A(w,w)\|_{Y_T^1} \\
&\le \|A(u,v-w)\|_{Y_T^1}+\|A(u-w,w)\|_{Y_T^1} \\
&\le C_2\left(\|u\|_{Y_T^1}+\|w\|_{Y_T^1}\right)\|u-w\|_{Y_T^1} \\
&\le 2C_2\delta\|u-w\|_{Y_T^1}.
\end{aligned}$$

Then, by choosing δ small enough such that

$$\delta\le\min\big\{\frac{1}{2C_1},\frac{1}{4C_2}\big\},$$

we have

$$\|\Psi(u)\|_{Y_T^1}\le\delta,\quad d\left(\Psi(u),\Psi(w)\right)\le\frac{1}{2}d(u,w).$$

Thus, according to Banach's fixed point theorem, we obtain a unique fixed point $u\in Y_\infty$ of Ψ, that is, the local solution of system (1.1). Furthermore, by setting $\|u\|_{Y_T^0}:=\sup_{t\in(0,T)}\|u(t)\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}$, and since $\Psi(u) = v$, then from Lemma 3.1 (2) and Proposition 4.1, we obtain

$$\|u\|_{Y_T^0} = \|\Psi(u)\|_{Y_T^0} \lesssim \|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} + \big(\|u\|_{Y_T^1}\big)^2 \lesssim \|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} + \delta^2 < \infty.$$

The proof of the second statement of the Theorem 2.9 is thus complete.

Next, we demonstrate the existence of global solutions of Equation 1.1 for small initial data u_0 in $\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}$ with $1 \le h \le p < \infty$, in the case $n < p < \infty$.

Proof of Theorem 2.10 (2). For small $C_0 > 0$, we define the metric space

$$Y_\infty := \{u \in BC\Big(0,\infty;\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}\Big) : \|u\|_{Y_\infty} \le C_0\},$$

equipped with the distance $d(u,w) := \|u-w\|_{Y_\infty}$. Returning to the map (4.3), and according to Lemma 3.1 (2) and Proposition 4.1, there exist $C_3, C_4 > 0$ such that for every $u \in Y_\infty$, we have

$$\|\Psi(u)\|_{Y_\infty} \le C_3\|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} + C_4\left(\|u\|_{Y_\infty}\right)^2 \le C_3\|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} + C_4C_0^2,$$

and, there exists C_5 such that for every $u, w \in Y_\infty$, we obtain

$$d\left(\Psi(u),\Psi(w)\right) \le C_5\left(\|u\|_{Y_\infty} + \|w\|_{Y_\infty}\right)\|u-w\|_{Y_\infty} \le 2C_5C_0\|u-w\|_{Y_\infty}.$$

Assume that $\|u_0\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} \le \delta$, for small $\delta > 0$ satisfy

$$\delta \le \frac{1}{2C_3}C_0,$$

and choosing C_0 small enough such that

$$C_0 \le \min\big\{\frac{1}{2C_4}, \frac{1}{4C_5}\big\}.$$

Then, for every $u, w \in Y_\infty$, one has

$$\|\Psi(u)\|_{Y_\infty} \le C_0, \quad d\left(\Psi(u),\Psi(w)\right) \le \frac{1}{2}d(u,w).$$

Eventually, using the Banach fixed point theorem, we obtain a unique fixed point $u \in Y_\infty$ of Ψ, that is, the global solution of the system (1.1). This completes the proof.

4.2. **The case $\frac{n}{2m} < p \le n$: Proof of Theorems 2.9 (1) and 2.10 (1).** To finish the proofs of Theorem 2.9 and Theorem 2.10, it suffices to prove the following crucial bilinear estimate. With this estimate, by using Lemma 3.4 (1), Lemma 3.1 (1) and the same argument as in Subsection 4.1, we can obtain the local and global solutions for the case $\frac{n}{2m} < p \le n$.

Proposition 4.2. *Let $0 < T \le \infty$, $\frac{1}{2} < m < 1$, $\max\{1, \frac{n}{2m}\} < p \le n$, $1 \le h \le p$ the parameters p_1, p_2, h_1, h_2 satisfy (2.12), and let $\|\cdot\|_{X_T^1}$ be given as*

$$\|u\|_{X_T^1} := \sup_{t\in(0,T)} t^{m-\frac{n}{2p_1}}\|u(t)\|_{\mathcal{M}^{p_1}_{h_1}} + \sup_{t\in(0,T)} t^{\frac{1}{2}-\frac{n}{2p_2}}\||\nabla|^{1-2m}u(t)\|_{\mathcal{M}^{p_2}_{h_2}}. \tag{4.4}$$

Then, we have

$$\|\int_0^t e^{(t-t')\Delta}\nabla\cdot\big(u\nabla(-\Delta)^{-m}w\big)\,dt'\|_{X_T} \lesssim \|u\|_{X_T^1}\|w\|_{X_T^1}.$$

Proof. We have

$$\begin{aligned}\|u\|_{X_T} := &\sup_{t\in(0,T)}\|u(t)\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}} + \sup_{t\in(0,T)} t^{m-\frac{n}{2p_1}}\|u(t)\|_{\mathcal{M}^{p_1}_{h_1}}\\ &+ \sup_{t\in(0,T)} t^{\frac{1}{2}-\frac{n}{2p_2}}\||\nabla|^{1-2m}u(t)\|_{\mathcal{M}^{p_2}_{h_2}}.\end{aligned}$$

Let ϑ be defined by $\frac{1}{\vartheta} := \frac{1}{p_1} + \frac{1}{p_2}$, so that $\vartheta < p$ and $\frac{n}{1+2m} < \vartheta < d$. Applying Remark 2.6, the embeddings (2.9), (2.5), (2.4) and (2.3), and Lemma 2.8, we obtain

$$\|\int_0^t e^{(t-t')\Delta}\nabla\cdot\big(u\nabla(-\Delta)^{-m}w\big)\,dt'\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}$$

$$
\begin{aligned}
&\lesssim \int_0^t \|e^{(t-t')\Delta}\left(u\nabla(-\Delta)^{-m}w\right)\mathrm{d}t'\|_{\dot{\mathcal{N}}^{1-2m+\frac{n}{\vartheta}}_{\vartheta,h,\infty}}\\
&\lesssim \int_0^t (t-t')^{-\frac{1-2m+\frac{n}{\vartheta}}{2}}\|v\nabla(-\Delta)^{-m}w\|_{\mathcal{M}^{\vartheta}_h}\mathrm{d}t'.
\end{aligned}
$$

Using Hölder's inequality (2.7) and since $m<1$, $\frac{n}{1+2m}<\vartheta<d$, then we have

$$
\begin{aligned}
&\|\int_0^t e^{(t-t')\Delta}\nabla\cdot\left(u\nabla(-\Delta)^{-m}w\right)\mathrm{d}t'\|_{\dot{\mathcal{N}}^{-2m+\frac{n}{p}}_{p,h,\infty}}\\
&\lesssim \int_0^t (t-t')^{-\frac12+m-\frac{n}{2\vartheta}}\|u(t')\|_{\mathcal{M}^{p_1}_{h_1}}\|\nabla(-\Delta)^{-m}w(t')\|_{\mathcal{M}^{p_2}_{h_2}}\mathrm{d}t'\\
&\lesssim \int_0^t (t-t')^{-\frac12+m-\frac{n}{2\vartheta}}(t')^{-m+\frac{n}{2p_1}-\frac12+\frac{n}{2p_2}}\|u\|_{X^1_T}\|w\|_{X^1_T}\mathrm{d}t'\\
&\lesssim \|u\|_{X^1_T}\|w\|_{X^1_T}\int_0^t (t-t')^{-\frac12+m-\frac{n}{2\vartheta}}(t')^{-\frac12-m+\frac{n}{2\vartheta}}\mathrm{d}t'\\
&\lesssim \|u\|_{X^1_T}\|w\|_{X^1_T}.
\end{aligned}
\tag{4.5}
$$

For the other terms of $\|\cdot\|_{X_T}$, we have

$$
\begin{aligned}
&\|\int_0^t e^{(t-t')\Delta}\nabla\cdot\left(u\nabla(-\Delta)^{-m}w\right)\mathrm{d}t'\|_{\mathcal{M}^{p_1}_{h_1}}\\
&\lesssim \int_0^t \|e^{(t-t')\Delta}\left(u\nabla(-\Delta)^{-m}w\right)\mathrm{d}t'\|_{\dot{\mathcal{N}}^{1}_{p_1,h_1,1}}\\
&\lesssim \int_0^t (t-t')^{-\frac12-\frac{n}{2}\left(\frac{1}{\vartheta}-\frac{1}{p_1}\right)}\|v\nabla(-\Delta)^{-m}w\|_{\dot{\mathcal{N}}^{0}_{\vartheta,\frac{h_1}{p_1}\vartheta,\infty}}\mathrm{d}t'\\
&\lesssim \int_0^t (t-t')^{-\frac12-\frac{n}{2p_2}}\|v\nabla(-\Delta)^{-m}w\|_{\mathcal{M}^{\vartheta}_{\frac{h_1}{p_1}\vartheta}}\mathrm{d}t'\\
&\lesssim \int_0^t (t-t')^{-\frac12-\frac{n}{2p_2}}\|u(t')\|_{\mathcal{M}^{p_1}_{h_1}}\|\nabla(-\Delta)^{-m}w(t')\|_{\mathcal{M}^{p_2}_{h_2}}\mathrm{d}t'\\
&\lesssim \int_0^t (t-t')^{-\frac12-\frac{n}{2p_2}}(t')^{-\frac12+\frac{n}{2p_2}-m+\frac{n}{2p_1}}\|u\|_{X^1_T}\|w\|_{X^1_T}\mathrm{d}t'\\
&\lesssim t^{-m+\frac{n}{2p_1}}\|u\|_{X^1_T}\|w\|_{X^1_T},
\end{aligned}
$$

and

$$
\begin{aligned}
&\||\nabla|^{1-2m}\int_0^t e^{(t-t')\Delta}\nabla\cdot\left(u\nabla(-\Delta)^{-m}w\right)\mathrm{d}t'\|_{\mathcal{M}^{p_2}_{h_2}}\\
&\lesssim \int_0^t \|e^{(t-t')\Delta}\left(u\nabla(-\Delta)^{-m}w\right)\mathrm{d}t'\|_{\dot{\mathcal{N}}^{2-2m}_{p_2,h_2,1}}\\
&\lesssim \int_0^t (t-t')^{-\frac12(2-2m)-\frac{n}{2}\left(\frac{1}{\vartheta}-\frac{1}{p_2}\right)}\|v\nabla(-\Delta)^{-m}w\|_{\mathcal{M}^{\vartheta}_{\frac{h_2}{p_2}\vartheta}}\mathrm{d}t'\\
&\lesssim \int_0^t (t-t')^{-1+m-\frac{n}{2p_1}}\|u(t')\|_{\mathcal{M}^{p_1}_{h_1}}\|\nabla(-\Delta)^{-m}w(t')\|_{\mathcal{M}^{p_2}_{h_2}}\mathrm{d}t'\\
&\lesssim \int_0^t (t-t')^{-1+m-\frac{n}{2p_1}}(t')^{-m+\frac{n}{2p_1}-\frac12+\frac{n}{2p_2}}\|u\|_{X^1_T}\|w\|_{X^1_T}\mathrm{d}t'\\
&\lesssim t^{-\frac12+\frac{n}{2p_2}}\|u\|_{X^1_T}\|w\|_{X^1_T}.
\end{aligned}
$$

Thus, we arrive at

$$
t^{m-\frac{n}{2p_1}}\|\int_0^t e^{(t-t')\Delta}\nabla\cdot\left(u\nabla(-\Delta)^{-m}w\right)\mathrm{d}t'\|_{\mathcal{M}^{p_1}_{h_1}}\lesssim \|u\|_{X^1_T}\|w\|_{X^1_T},
\tag{4.6}
$$

$$t^{\frac{1}{2}-\frac{n}{2p_2}}\||\nabla|^{1-2m}\int_0^t e^{(t-t')\Delta}\nabla\cdot\big(u\nabla(-\Delta)^{-m}w\big)\,dt'\|_{\mathcal{M}^{p_2}_{h_2}}\lesssim\|u\|_{X^1_T}\|w\|_{X^1_T}. \tag{4.7}$$

Therefore, we obtain Proposition 4.2. □

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