Electronic Journal of Differential Equations, Vol. 2025 (2025), No. 66, pp. 1–15.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2025.66

SOLUTIONS TO NONLINEAR ELLIPTIC PROBLEMS WITH NONHOMOGENEOUS OPERATORS AND MIXED NONLOCAL BOUNDARY CONDITIONS

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Abstract. We investigate the existence, multiplicity and nonexistence of positive solutions to nonlinear (singular) elliptic problems involving nonhomogeneous operators and mixed nonlocal boundary conditions based on the behaviors of the nonlinear term near 0 and ∞. In particular, we discuss the existence of at least three positive solutions to the mixed nonlocal boundary problems, which is new finding even for the problems involving homogeneous operators. The novelty of this study lies in constructing completely continuous operators related to nonlinear elliptic problems involving complicated boundary conditions. We emphasize that only one fixed point theorem is used to obtain the existence and multiplicity results, despite generalizing and extending most of the problems in previous literature.

1. Introduction and main results

We consider the nonlinear (singular) elliptic problems with nonhomogeneous operators and mixed nonlocal boundary conditions:

$$\begin{gathered}
-(w(t)\phi(u'))' = \lambda h(t) f(u), \quad t \in (0,1), \\
u(0) - au'(0) = \int_0^1 g_0(s) k_0^0(s,u)ds + \sum_{i=1}^{m} \alpha_i k_i^0(\zeta_i, u(\zeta_i)), \\
u(1) + bu'(1) = \int_0^1 g_1(s) k_0^1(s,u)ds + \sum_{j=1}^{n} \beta_j k_j^1(\xi_j, u(\xi_j)),
\end{gathered} \tag{1.1}$$

where $\lambda > 0$, $a \geq 0$, $b \geq 0$, $0 \leq \alpha_i < 1$, $0 \leq \beta_j < 1$, $m \in \mathbb{N}$, $n \in \mathbb{N}$, and $\{\zeta_i\}_{i=1}^m$ and $\{\xi_j\}_{j=1}^n$ are increasing sequences in $(0,1)$. Here ϕ, f, w, h, g_0, g_1, k_i^0 and k_j^1 satisfy the following conditions:

- (H1) $\phi \in C(\mathbb{R},\mathbb{R})$ is an odd increasing homeomorphism such that there exists $\psi \in C((0,\infty),(0,\infty))$ such that $\psi(0) = 0$ and $\phi(rs) \leq \psi(r)\phi(s)$ for $r > 0$ and $s > 0$,
- (H2) $f \in C((0,\infty),(0,\infty))$ with $\liminf_{s\to\infty} f(s) > 0$ and there exist $c > 0$ and $\gamma \geq 0$ such that $f(s) < \frac{c}{s^\gamma}$ for $0 < s < 1$,
- (H3) $w \in C([0,1],(0,\infty))$ and $h \in C((0,1),(0,\infty))$ with $\int_0^1 \frac{h(r)}{d(r)^\gamma} dr < \infty$, where $d(r) := \min\{r, 1-r\}$,
- (H4) $g_0, g_1 \in C((0,1),[0,\infty))$ with $G := \max\{\|g_0\|_1 + \sum_{i=1}^m \alpha_i, \|g_1\|_1 + \sum_{j=1}^n \beta_j\} < 1$,
- (H5) $k_i^0, k_j^1 \in C((0,1)\times[0,\infty),[0,\infty))$ are such that $k_i^0(s,r) \leq r$ and $k_j^1(s,r) \leq r$, where $i \in \{0,1,2,\dots,m\}$ and $j \in \{0,1,2,\dots,n\}$.

Nonlocal boundary value problems of ordinary differential equations arise in various areas of applied mathematics and physics. In particular, multipoint boundary value problems arise in a fluid flow problem [11] and the theory of elastic stability [20], and integral boundary value problems

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2020 *Mathematics Subject Classification.* 34B10, 34B16, 34B18.
Key words and phrases. Singular elliptic problem; nonhomogeneous operator; integral boundary condition; multipoint boundary condition; positive solution.
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Submitted February 20, 2025. Published June 30, 2025.

arise in blood flow problems [17, 24] and thermal conduction problems [5, 13]. Recently, many studies have been conducted on these two nonlocal boundary value problems. One can find several works on multipoint and integral boundary value problems in a series of papers [6, 7, 15, 16, 21] and [1, 3, 4, 12, 14, 22, 23], respectively. In [16], Ma studied the existence of positive solutions of (1.1) with the homogeneous operator $u \mapsto (w(t)u')'$ (i.e., $\phi(s) = s$), a nonsingular nonlinear term f and the following m-point boundary condition

$$au(0) - bw(0)u'(0) = \sum_{i=1}^{m-2} \alpha_i u(\zeta_i) \quad \text{and} \quad cu(1) + dw(1)u'(1) = \sum_{i=1}^{m-2} \beta_i u(\zeta_i)$$

with $ac + ad + bc > 0$. They used the Guo-Krasnoselskii fixed point theorem in a cone to get the result. Webb and Infante [22, 23] provided a unified method of establishing the existence and multiplicity of positive solutions of (1.1) with the homogeneous operator $u \mapsto u''$ (i.e., $w(t) \equiv 1$ and $\phi(s) = s$), a nonsingular nonlinear term f and various nonlocal boundary conditions involving Stieltjes integrals (thus allowing for m-point and integral boundary conditions). But one can see that our mixed nonlocal boundary conditions contain their boundary conditions. Recently, Hai and Wang [10] addressed the problem (1.1) with the homogeneous operator $u \mapsto (|u'|^{p-2}u')'$ (i.e., $w(t) \equiv 1$ and $\phi(s) = |s|^{p-2}s$), a singular nonlinear term f and the following boundary conditions:

$$au(0) - bu'(0) = \int_0^1 g(s)u(s)ds \quad \text{and} \quad u'(1) = 0,$$

or

$$au(0) - bu'(0) = \int_0^1 g(s)u(s)ds \quad \text{and} \quad u(1) = 0,$$

with $a > 0$ and $b \geq 0$. Assuming that the nonlinear term $f(s)$ could have negative values near 0, they showed the existence of positive solutions for the cases $\lim_{s\to\infty} \frac{f(s)}{\phi(s)} = 0$ and $\lim_{s\to\infty} \frac{f(s)}{\phi(s)} = \infty$ by applying the Krasnoselskii fixed point theorem in a Banach space. We note that they did not discuss multiplicity results. The problem involving both multipoint and integral boundary conditions simultaneously was initially introduced in [2] as follows:

$$\begin{gathered}
-u''(t) = f(t,u), \quad t \in (T_1, T_2), \\
\alpha_1 u(T_1) + \alpha_2 u(T_2) = \alpha_3 \int_{T_1}^{\zeta} u(s)ds + \sum_{i=1}^{m} \gamma_i u(\nu_i), \\
\beta_1 u'(T_1) + \beta_2 u'(T_2) = \beta_3 \int_{T_1}^{\zeta} u'(s)ds + \sum_{i=1}^{m} \rho_i u'(\nu_i),
\end{gathered}$$

where $0 < T_1 < \zeta \leq \nu_i \leq T_2$, $\alpha_i, \beta_i \in \mathbb{R}$, and $\gamma_i, \rho_i \in (0, \infty)$. The existence of solutions (which may not be positive solutions) was discussed via three different fixed point theorems: Schaefer, Krasnoselskii and Leray-Schauder.

Motivated by the aforementioned studies, we extend these results to the more general case (1.1), which has a nonhomogeneous operator, a (non)singular nonlinear term, and mixed multipoint and integral boundary conditions. Our objective is to study the existence, multiplicity and nonexistence of positive solutions of (1.1) in $C^1[0,1]$ according to the behaviors of f near 0 and ∞, that is, the values of $f_0 := \lim_{s\to 0} \frac{f(s)}{\phi(s)}$ and $f_\infty := \lim_{s\to\infty} \frac{f(s)}{\phi(s)}$. In particular, to discuss the existence of three positive solutions, we assume

- (H6) $f(s) := \frac{f_\gamma(s)}{s^\gamma}$, where f_γ is continuous and nondecreasing,
- (H7) there exist $\eta > 0$ and $\theta > 0$ such that

$$\frac{f(\theta)}{\phi(\theta)} \Big/ \frac{f(\eta)}{\phi(\eta)} > \frac{4^\gamma w^* \|h_\gamma\|_1 \psi(\frac{16}{\delta})\psi(\frac{1+a}{1-G})}{w_* h_* \delta^{2\gamma}},$$

and one or two of the following:

- (H8a) $\eta < \frac{4\theta}{\delta}$,
- (H8b) $\eta > \frac{4\theta}{\delta}$,

(H9a) $\frac{f(\theta)}{\phi(\theta)} > \frac{2^{1+2\gamma} w^* \|h\|_1 \psi(\frac{16}{\delta})\psi(\frac{1+a}{1-G}) \min\{f_0, f_\infty\}}{w_* h_* \delta^\gamma}$,

(H9b) $\frac{f(\eta)}{\phi(\eta)} < \frac{w_* h_* \delta^\gamma \max\{f_0, f_\infty\}}{2w^* \|h_\gamma\|_1 \psi(\frac{16}{\delta})\psi(\frac{1+a}{1-G})}$,

(H10a) $\frac{\max\{f_0, f_\infty\}}{\min\{f_0, f_\infty\}} > \frac{4w^* \|h\|_1 \psi(\frac{16}{\delta})\psi(\frac{1+a}{1-G})}{w_* h_*}$ (> 1),

(H10b) $f_0 > f_\infty$, (H8a), (H9a) and (H9b),

(H10c) $f_0 < f_\infty$, (H8b), (H9a) and (H9b),

where $w_* := \min_{t\in[0,1]} w(t)$, $w^* := \max_{t\in[0,1]} w(t)$, $\|h_\gamma\|_1 := \int_0^1 \frac{h(r)}{d(r)^\gamma} dr$,

$$h_* := \min\Big\{ \int_{\frac{1}{4}}^{\frac{1}{2}} h(r)dr, \int_{\frac{1}{2}}^{\frac{3}{4}} h(r)dr \Big\},$$

and δ is the largest constant such that $\max_{s\in[0,\delta]} \psi(s) \le \frac{w_*}{2w^*}$. We note that $\psi(1) \ge 1$ since $\phi(1) \le \psi(1)\phi(1)$ by (H1). This implies $\delta < 1$.

Noting that $f_0 = \infty$ implies $f(s)$ could be singular at 0, we state theorems according to the following value of f_0: $f_0 = \infty$, $f_0 = 0$ and $f_0 \in (0, \infty)$.

1. Case $f_0 = \infty$. Let $\lambda_* := \frac{4^\gamma w^* \psi(\frac{16}{\delta})\phi(\theta)}{h_* \delta^\gamma f(\theta)}$, $\lambda^* := \frac{w_* \delta^\gamma \phi(\eta)}{\|h_\gamma\|_1 \psi(\frac{1+a}{1-G}) f(\eta)}$ and $\lambda^\infty := \frac{w_*}{2\|h\|_1 \psi(\frac{1+a}{1-G}) f_\infty}$. We establish the following results.

Theorem 1.1. *Assume* (H1)–(H5), $f_0 = \infty$ *and* $f_\infty = \infty$. *Then* (1.1) *has no positive solution for* $\lambda \gg 1$ *and has two positive solutions* u_1 *and* u_2 *for* $\lambda \approx 0$ *such that* $\|u_1\|_\infty \to 0$ *and* $\|u_2\|_\infty \to \infty$ *as* $\lambda \to 0$.

Theorem 1.2. *Assume* (H1)–(H5), $f_0 = \infty$ *and* $f_\infty = 0$. *Then* (1.1) *has a positive solution* u *for* $\lambda > 0$ *such that* $\|u\|_\infty \to 0$ *as* $\lambda \to 0$ *and* $\|u\|_\infty \to \infty$ *as* $\lambda \to \infty$. *In addition, if* (H6), (H7), (H8a) *are satisfied, then* (1.1) *has three positive solutions* u_1, u_2 *and* u_3 *for* $\lambda \in (\lambda_*, \lambda^*)$ *such that* $\|u_1\|_\infty < \eta < \|u_2\|_\infty < \frac{4\theta}{\delta} < \|u_3\|_\infty$.

Theorem 1.3. *Assume* (H1)–(H5), $f_0 = \infty$ *and* $f_\infty \in (0, \infty)$. *Then* (1.1) *has no positive solution for* $\lambda \gg 1$ *and has a positive solution* u *for* $\lambda < \lambda^\infty$ *such that* $\|u\|_\infty \to 0$ *as* $\lambda \to 0$. *In addition, if* (H6), (H7), (H8a), (H9a) *are satisfied, then* (1.1) *has three positive solutions* u_1, u_2 *and* u_3 *for* $\lambda \in (\lambda_*, \min\{\lambda^*, \lambda^\infty\})$ *such that* $\|u_1\|_\infty < \eta < \|u_2\|_\infty < \frac{4\theta}{\delta} < \|u_3\|_\infty$.

2. Case $f_0 = 0$. Let $\lambda_\infty := \frac{2w^* \psi(\frac{16}{\delta})}{h_* f_\infty}$. We establish the following results.

Theorem 1.4. *Assume* (H1)–(H5), $f_0 = 0$ *and* $f_\infty = \infty$. *Then* (1.1) *has a positive solution* u *for* $\lambda > 0$ *such that* $\|u\|_\infty \to \infty$ *as* $\lambda \to 0$ *and* $\|u\|_\infty \to 0$ *as* $\lambda \to \infty$. *In addition, if* (H6), (H7), (H8b) *are satisfied, then* (1.1) *has three positive solutions* u_1, u_2 *and* u_3 *for* $\lambda \in (\lambda_*, \lambda^*)$ *such that* $\|u_1\|_\infty < \frac{4\theta}{\delta} < \|u_2\|_\infty < \eta < \|u_3\|_\infty$.

Theorem 1.5. *Assume* (H1)–(H5), $f_0 = 0$ *and* $f_\infty = 0$. *Then* (1.1) *has no positive solution for* $\lambda \approx 0$ *and has two positive solutions* u_1 *and* u_2 *for* $\lambda \gg 1$ *such that* $\|u_1\|_\infty \to 0$ *and* $\|u_2\|_\infty \to \infty$ *as* $\lambda \to \infty$.

Theorem 1.6. *Assume* (H1)–(H5), $f_0 = 0$ *and* $f_\infty \in (0, \infty)$. *Then* (1.1) *has no positive solution for* $\lambda \approx 0$ *and has a positive solution* u *for* $\lambda > \lambda_\infty$ *such that* $\|u\|_\infty \to 0$ *as* $\lambda \to \infty$. *In addition, if* (H6), (H7), (H8b), (H9b) *are satisfied, then* (1.1) *has three positive solutions* u_1, u_2 *and* u_3 *for* $\lambda \in (\max\{\lambda_*, \lambda_\infty\}, \lambda^*)$ *such that* $\|u_1\|_\infty < \frac{4\theta}{\delta} < \|u_2\|_\infty < \eta < \|u_3\|_\infty$.

3. Case $f_0 \in (0, \infty)$. Let $\lambda^0 := \frac{w_*}{2\|h\|_1 \psi(\frac{1+a}{1-G}) f_0}$ and $\lambda_0 := \frac{2w^* \psi(\frac{16}{\delta})}{h_* f_0}$. We establish the following results.

Theorem 1.7. *Assume* (H1)–(H5), $f_0 \in (0, \infty)$ *and* $f_\infty = \infty$. *Then* (1.1) *has no positive solution for* $\lambda \gg 1$ *a nd has a positive solution* u *for* $\lambda < \lambda^0$ *such that* $\|u\|_\infty \to \infty$ *as* $\lambda \to 0$. *In addition, if* (H6), (H7), (H8b), (H9a) *are satisfied, then* (1.1) *has three positive solutions* u_1, u_2 *and* u_3 *for* $\lambda \in (\lambda_*, \min\{\lambda^*, \lambda^0\})$ *such that* $\|u_1\|_\infty < \frac{4\theta}{\delta} < \|u_2\|_\infty < \eta < \|u_3\|_\infty$.

Theorem 1.8. *Assume* (H1)–(H5), $f_0 \in (0,\infty)$ *and* $f_\infty = 0$. *Then* (1.1) *has no positive solution for* $\lambda \approx 0$ *and has a positive solution* u *for* $\lambda > \lambda_0$ *such that* $\|u\|_\infty \to \infty$ *as* $\lambda \to \infty$. *In addition, if* (H6), (H7), (H8a), (H9b) *are satisfied, then* (1.1) *has three positive solutions* u_1, u_2 *and* u_3 *for* $\lambda \in (\max\{\lambda_*, \lambda_0\}, \lambda^*)$ *such that* $\|u_1\|_\infty < \eta < \|u_2\|_\infty < \frac{4\theta}{\delta} < \|u_3\|_\infty$.

Theorem 1.9. *Assume* (H1)–(H5), $f_0 \in (0,\infty)$ *and* $f_\infty \in (0,\infty)$. *Then* (1.1) *has no positive solution for* $\lambda \approx 0$ *and* $\lambda \gg 1$. *If* (H10a) *is satisfied, then* (1.1) *has a positive solution for* $\lambda \in (\min\{\lambda_0, \lambda_\infty\}, \max\{\lambda^0, \lambda^\infty\})$. *If* (H6), (H7), (H10a), (H10b) *are satisfied, then* (1.1) *has three positive solutions* u_1, u_2 *and* u_3 *for* $\lambda \in (\max\{\lambda_*, \lambda_0\}, \min\{\lambda^*, \lambda^\infty\})$. *If* (H6), (H7), (H10a), (H10c) *are satisfied, then* (1.1) *has three positive solutions* u_1, u_2 *and* u_3 *for* $\lambda \in (\max\{\lambda_*, \lambda_\infty\}, \min\{\lambda^*, \lambda^0\})$.

We use the following Krasnoselskii-type fixed point theorem to get the existence and multiplicity results (see [9, Lemma A]).

Proposition 1.10. *Let* X *be a Banach space and* $I : X \to X$ *be a completely continuous operator. Suppose that there exist a nonzero element* $z \in X$ *and positive constants* r *and* R *with* $r \neq R$ *such that*

(a) *if* $y \in X$ *satisfies* $y = \sigma I y$ *for* $\sigma \in (0,1]$, *then* $\|y\|_X \neq r$,
(b) *if* $y \in X$ *satisfies* $y = Iy + \tau z$ *for* $\tau \geq 0$, *then* $\|y\|_X \neq R$.

Then I *has a fixed point* $y \in X$ *with* $\min\{r, R\} < \|y\|_X < \max\{r, R\}$.

The main challenges of this study are constructing the completely continuous operator (T_λ in Section 2) for the modified problem of (1.1) that reflects the mixed nonlocal boundary conditions and finding lower estimates of functions obtained through the operator ($T_\lambda y$, where $y \in C[0,1]$). In general, due to the boundary conditions, completely continuous operators related to nonlocal boundary value problems are more complicated than those related to local boundary value problems, and functions obtained through the operators related to nonlocal boundary value problems could have maximums either in $(0,1)$ or at a boundary of $(0,1)$. To overcome the difficulty of constructing the operator T_λ, we modify the completely continuous operators for the local (Dirichlet and nonlinear) boundary value problems in [18, 19] by adding some functions representing the boundary conditions ($A(y)$, $B(y)$ and $C(s)$ in Section 2). To find necessary lower estimates of $T_\lambda y$, we use different representations of $T_\lambda y$ depending on where it has a maximum.

It is also noteworthy that this study complements the existing results by dealing with the nonlocal boundary value problem in which it has a nonhomogeneous operator and a (non)singular nonlinear term. In particular, we discuss the existence of at least three positive solutions that has not been treated much in previous studies.

In Section 2, we construct the completely continuous operator T_λ to find fixed points of the modified problem of (1.1) and show that the fixed points are eventually positive solutions of (1.1). We prove Theorems 1.1 - 1.3, 1.4 - 1.6, and 1.7 - 1.9 in Sections 3, 4, and 5, respectively. Section 6 provides an example of (1.1) with a nonhomogeneous operator and a singular nonlinear term.

2. Preliminaries

In this section, we construct the completely continuous operator T_λ for the modified problem of (1.1) and show that fixed points of T_λ are positive solutions of (1.1).

We first construct the completely continuous operator T_λ. To do this, we extend the ideas in [18, 19], which are the local boundary value problems. For $y \in C[0,1]$, we define the following functions related to the boundary conditions:

$$A(y) := \int_0^1 g_0(s) k_0^0(s, |y(s)|) ds + \sum_{i=1}^m \alpha_i k_i^0(\zeta_i, |y(\zeta_i)|),$$

$$B(y) := \int_0^1 g_1(s) k_0^1(s, |y(s)|) ds + \sum_{j=1}^n \beta_j k_j^1(\xi_j, |y(\xi_j)|),$$

$$C(s) := \phi^{-1}\Big(\frac{w(1)\phi(s)}{w(0)} + \frac{\lambda}{w(0)} \int_0^1 h(r) f^*(y) dr\Big) \quad \text{for } s \in \mathbb{R},$$

where $f^*(y(t)) := f(\max\{y(t), \delta\rho_\lambda d(t)\})$ and $\rho_\lambda > 0$ is to be determined in each section. Then we define the operator $T_\lambda : C[0,1] \to C[0,1]$ by

$$T_\lambda y(t) := A(y) + aC(m_y) + \int_0^t \phi^{-1}\Big(\frac{w(1)\phi(m_y)}{w(s)} + \frac{\lambda}{w(s)}\int_s^1 h(r)f^*(y)dr\Big)ds,$$

where $y \in C[0,1]$ and $m_y \in \mathbb{R}$ is the constant such that

$$B(y) - bm_y = A(y) + aC(m_y) + \int_0^1 \phi^{-1}\Big(\frac{w(1)\phi(m_y)}{w(s)} + \frac{\lambda}{w(s)}\int_s^1 h(r)f^*(y)dr\Big)ds.$$

It can be shown that $T_\lambda y \in C^1[0,1]$, $m_y = (T_\lambda y)'(1)$, $C(m_y) = (T_\lambda y)'(0)$, and m_y is continuous for y. Further, $T_\lambda : C[0,1] \to C[0,1]$ is completely continuous, $T_\lambda y(t)$ is the solution to the boundary value problem

$$\begin{gathered}
-(w(t)\phi(x'))' = \lambda h(t)f^*(y),\ t \in (0,1),\\
x(0) - ax'(0) = A(y),\\
x(1) + bx'(1) = B(y),
\end{gathered}$$

and $T_\lambda y$ satisfies the following property.

Lemma 2.1. *Assume* (H1)–(H5). *Then* $T_\lambda y(t) \ge \delta\|T_\lambda y\|_\infty d(t)$.

Proof. Let $x(t) := T_\lambda y(t)$. We first show $x(0) \ge 0$. If $a = 0$, then it is clear because $x(0) = A(y) \ge 0$. Let $a > 0$. Assume to the contrary that $x(0) < 0$. Then $x'(0) < 0$ by the boundary condition at 0. If there exists $t_x \in (0,1]$ such that $x'(t_x) = 0$ and $x'(t) < 0$ for $t \in (0,t_x)$, then we have

$$x'(t) = \phi^{-1}\Big(\frac{\lambda}{w(t)}\int_t^{t_x} h(s)f^*(y)ds\Big) \ge 0,$$

which is a contradiction. Thus $x'(t) < 0$ for $t \in (0,1]$. This implies $x'(1) < 0$ and $x(1) < 0$. However, this is a contradiction since $x(1) = B(y) - bx'(1) \ge 0$. Hence $x(0) \ge 0$.

We can show $x(1) \ge 0$ by similar arguments. Then we obtain $x(t) \ge \delta\|x\|_\infty d(t)$ by Lemma 2.1 in [8]. □

Next we find a condition for fixed points of T_λ to be positive solutions of (1.1).

Lemma 2.2. *Assume* (H1)–(H5). *If* $T_\lambda y = y$ *for some* $y \in C[0,1]$ *with* $\|y\|_\infty \ge \rho_\lambda$*, then* y *is a positive solution of* (1.1).

Proof. It is clear that $y \in C^1[0,1]$ since $T_\lambda y \in C^1[0,1]$. Further, we have $y(t) \ge \delta\|y\|_\infty d(t) \ge \delta\rho_\lambda d(t) \ge 0$ by Lemma 2.1. Thus y satisfies

$$\begin{gathered}
-(w(t)\phi(y'))' = \lambda h(t)f^*(y) = \lambda h(t)f(y), \quad t \in (0,1),\\
y(0) - ay'(0) = A(y) = \int_0^1 g_0(s)k_0^0(s,y(s))ds + \sum_{i=1}^m \alpha_i k_i^0(\zeta_i, y(\zeta_i)),\\
y(1) + by'(1) = B(y) = \int_0^1 g_1(s)k_0^1(s,y(s))ds + \sum_{j=1}^n \beta_j k_j^1(\xi_j, y(\xi_j)).
\end{gathered}$$

Hence y is a positive solution of (1.1). □

Now we introduce different representations of $T_\lambda y$ to be used to find lower estimates. If $\|T_\lambda y\|_\infty = T_\lambda y(t_m)$ for some $t_m \in (0,1)$, then $(T_\lambda y)'(t) \ge 0$ for $t \in (0,t_m)$, $(T_\lambda y)'(t) \le 0$ for $t \in (t_m,1)$ and $T_\lambda y$ can be written as

$$T_\lambda y(t) = \begin{cases} A(y) + a(T_\lambda y)'(0) + \int_0^t \phi^{-1}\Big(\frac{\lambda}{w(s)}\int_s^{t_m} h(r)f^*(y)dr\Big)ds, & 0 \le t \le t_m,\\ B(y) - b(T_\lambda y)'(1) + \int_t^1 \phi^{-1}\Big(\frac{\lambda}{w(s)}\int_{t_m}^s h(r)f^*(y)dr\Big)ds, & t_m \le t \le 1. \end{cases}$$

If $\|T_\lambda y\|_\infty = T_\lambda y(0)$, then $(T_\lambda y)'(t) \le 0$ for $t \in (0,1)$ and $T_\lambda y$ can be written as

$$T_\lambda y(t) = B(y) - b(T_\lambda y)'(1) + \int_t^1 \phi^{-1}\Big(-\frac{w(0)\phi((T_\lambda y)'(0))}{w(s)} + \frac{\lambda}{w(s)}\int_0^s h(r)f^*(y)dr\Big)ds.$$

If $\|T_\lambda y\|_\infty = T_\lambda y(1)$, then $(T_\lambda y)'(t) \geq 0$ for $t \in (0,1)$ and $T_\lambda y$ can be written as

$$T_\lambda y(t) = A(y) + a(T_\lambda y)'(0) + \int_0^t \phi^{-1}\Big(\frac{w(1)\phi((T_\lambda y)'(1))}{w(s)} + \frac{\lambda}{w(s)}\int_s^1 h(r)f^*(y)dr\Big)ds.$$

3. Proofs of Theorems 1.1–1.3

We use Proposition 1.10 with $I = T_\lambda$, $X = C[0,1]$ and $z \equiv 1$ to show the existence and multiplicity results in Theorems 1.1–1.3. Let $f_m(s) := \inf_{r\in(s,\infty)} f(r)$. Noting that $\lim_{s\to 0} \frac{f_m(s)}{\phi(s)} = \infty$ since $f_0 = \infty$, we can choose $r_\lambda \in (0,1)$ such that $\frac{f_m(s)}{\phi(s)} > \frac{2w^*\psi(\frac{16}{\delta})}{\lambda h_*}$ for $s \leq r_\lambda$. Then we define $\rho_\lambda = r_\lambda$ in T_λ. Additionally, we assume $r_\lambda < \min\{\eta, \frac{4\theta}{\delta}\}$ to prove the existence of three positive solutions in Theorems 1.2 - 1.3.

Proof of Theorem 1.1. We first show the multiplicity result for $\lambda \approx 0$. Let $\sigma \in (0,1]$ and $u \in C[0,1]$ be a solution of $u = \sigma T_\lambda u$ and $\|u\|_\infty \geq r_\lambda$. Then $u(t) = \sigma T_\lambda u(t) \geq 0$ by Lemma 2.1. If $\|u\|_\infty = u(0)$, then $u'(0) = \sigma(T_\lambda u)'(0) \leq 0$ and u satisfies

$$\begin{aligned}
\|u\|_\infty &= u(0) \leq T_\lambda u(0)\\
&\leq A(u) = \int_0^1 g_0(s)k_0^0(s,|u(s)|)ds + \sum_{i=1}^m \alpha_i k_i^0(\zeta_i,|u(\zeta_i)|)\\
&\leq G\|u\|_\infty.
\end{aligned}$$

However, this is a contradiction since $G < 1$. If $\|u\|_\infty = u(1)$, then $u'(1) = \sigma(T_\lambda u)'(1) \geq 0$ and u satisfies

$$\begin{aligned}
\|u\|_\infty &= u(1) \leq T_\lambda u(1)\\
&\leq B(u) = \int_0^1 g_1(s)k_0^1(s,|u(s)|)ds + \sum_{j=1}^n \beta_j k_j^1(\xi_j,|u(\xi_j)|)\\
&\leq G\|u\|_\infty.
\end{aligned}$$

However, this is a contradiction. Hence there exists $t_m \in (0,1)$ such that $\|u\|_\infty = u(t_m)$. Then $u'(1) = \sigma(T_\lambda u)'(1) \leq 0$. We note that $\delta < 1$, $A(u) \leq G\|u\|_\infty$, $(T_\lambda u)'(0) = C((T_\lambda u)'(1))$, and $u(t) = \sigma T_\lambda u(t) \geq \sigma\delta\|T_\lambda u\|_\infty d(t) = \delta\|u\|_\infty d(t)$ by Lemma 2.1. Thus u satisfies

$$\begin{aligned}
&\|u\|_\infty\\
&\leq A(u) + a(T_\lambda u)'(0) + \int_0^{t_m} \phi^{-1}\Big(\frac{\lambda}{w(s)}\int_s^{t_m} h(r)f^*(u)dr\Big)ds\\
&= A(u) + a\phi^{-1}\Big(\frac{w(1)\phi((T_\lambda u)'(1))}{w(0)} + \frac{\lambda}{w(0)}\int_0^1 h(r)f^*(u)dr\Big) + \int_0^{t_m} \phi^{-1}\Big(\frac{\lambda}{w(s)}\int_s^{t_m} h(r)f^*(u)dr\Big)ds\\
&\leq A(u) + (1+a)\phi^{-1}\Big(\frac{\lambda}{w_*}\int_0^1 h(r)f^*(u)dr\Big)\\
&\leq A(u) + (1+a)\phi^{-1}\Big(\frac{\lambda}{w_*}\int_0^1 h(r)\Big(\frac{c}{\max\{u,\delta r_\lambda d(r)\}^\gamma} + f_M(\max\{u,\delta r_\lambda d(r)\})\Big)dr\Big)\\
&\leq A(u) + (1+a)\phi^{-1}\Big(\frac{\lambda}{w_*}\int_0^1 h(r)\Big(\frac{c}{(\delta\|u\|_\infty d(r))^\gamma} + f_M(\|u\|_\infty)\Big)dr\Big)\\
&\leq G\|u\|_\infty + (1+a)\phi^{-1}\Big(\frac{\lambda}{w_*}\Big(\frac{c\|h_\gamma\|_1}{\delta^\gamma\|u\|_\infty^\gamma} + \|h\|_1 f_M(\|u\|_\infty)\Big)\Big),
\end{aligned}$$

where $f_M \in C([0,\infty),[0,\infty))$ is such that $f_M(0) = 0$, $f_M(s)$ is nondecreasing for $s \leq 1$ and $f_M(s) := \max_{r\in[1,s]} f(r)$ for $s > 1$. Then we obtain

$$\phi(\|u\|_\infty) \leq \psi\Big(\frac{1+a}{1-G}\Big)\phi\Big(\frac{(1-G)\|u\|_\infty}{1+a}\Big) \leq \frac{\lambda}{w_*}\psi\Big(\frac{1+a}{1-G}\Big)\Big(\frac{c\|h_\gamma\|_1}{\delta^\gamma\|u\|_\infty^\gamma} + \|h\|_1 f_M(\|u\|_\infty)\Big).$$

This implies

$$1 \le \frac{\lambda}{w_*}\psi\Big(\frac{1+a}{1-G}\Big)\Big(\frac{c\|h_\gamma\|_1}{\delta^\gamma\|u\|_\infty^\gamma\phi(\|u\|_\infty)} + \frac{\|h\|_1 f_M(\|u\|_\infty)}{\phi(\|u\|_\infty)}\Big). \tag{3.1}$$

If $\|u\|_\infty = 1$, then we have $1 \le \frac{\lambda}{w_*}\psi(\frac{1+a}{1-G})(\frac{c\|h_\gamma\|_1}{\delta^\gamma\phi(1)} + \frac{\|h\|_1 f_M(1)}{\phi(1)})$. However, this is a contradiction for $\lambda \approx 0$. Hence $\|u\|_\infty \ne 1$ for $\lambda \approx 0$.

Now we show that there exist two constants (one is greater than 1 and one is ess than 1) satisfying (b) in Proposition 1.10. Let $\tau \ge 0$ and $u \in C[0,1]$ be a solution of $u = T_\lambda u + \tau$. Then there are three cases: (i) $\|u\|_\infty = u(t_m)$ for some $t_m \in (0,1)$, (ii) $\|u\|_\infty = u(0)$, and (iii) $\|u\|_\infty = u(1)$. We first consider the case (i). Then $(T_\lambda u)'(t_m) = 0$. We note that

$$\begin{aligned} u(t) &= T_\lambda u(t) + \tau \ge \delta\|T_\lambda u\|_\infty d(t) + \tau \\ &\ge \delta(\|T_\lambda u\|_\infty + \tau)d(t) = \delta\|u\|_\infty d(t) \\ &\ge \frac{\delta\|u\|_\infty}{4} \end{aligned} \tag{3.2}$$

for $t \in [\frac14, \frac34]$ by $\delta < 1$ and Lemma 2.1. If $t_m \ge \frac12$, then $(T_\lambda u)'(0) \ge 0$ and u satisfies

$$\begin{aligned} \|u\|_\infty &\ge A(u) + a(T_\lambda u)'(0) + \int_0^{t_m} \phi^{-1}\Big(\frac{\lambda}{w(s)}\int_s^{t_m} h(r)f^*(u)dr\Big)ds \\ &\ge \int_0^{1/4} \phi^{-1}\Big(\frac{\lambda}{w(s)}\int_s^{t_m} h(r)f^*(u)dr\Big)ds \\ &\ge \int_0^{1/4} \phi^{-1}\Big(\frac{\lambda}{w^*}\int_{\frac14}^{\frac12} h(r)f_m(u)dr\Big)ds \\ &\ge \frac14\phi^{-1}\Big(\frac{\lambda h_*}{w^*}f_m(\frac{\delta\|u\|_\infty}{4})\Big). \end{aligned}$$

By similar arguments, we can show that if $t_m < \frac12$ then $\|u\|_\infty \ge \frac14\phi^{-1}(\frac{\lambda h_*}{w^*}f_m(\frac{\delta\|u\|_\infty}{4}))$. For case (ii), $(T_\lambda u)'(0) \le 0$, $(T_\lambda u)'(1) \le 0$ and u satisfies

$$\begin{aligned} \|u\|_\infty &\ge B(u) - b(T_\lambda u)'(1) + \int_{3/4}^1 \phi^{-1}\Big(-\frac{w(0)\phi((T_\lambda u)'(0))}{w(s)} + \frac{\lambda}{w(s)}\int_0^s h(r)f^*(u)dr\Big)ds \\ &\ge \int_{3/4}^1 \phi^{-1}\Big(\frac{\lambda}{w(s)}\int_0^s h(r)f^*(u)dr\Big)ds \\ &\ge \int_{3/4}^1 \phi^{-1}\Big(\frac{\lambda}{w^*}\int_{\frac12}^{\frac34} h(r)f_m(u)dr\Big)ds \\ &\ge \frac14\phi^{-1}\Big(\frac{\lambda h_*}{w^*}f_m(\frac{\delta\|u\|_\infty}{4})\Big). \end{aligned}$$

For case (iii), $(T_\lambda u)'(0) \ge 0$, $(T_\lambda u)'(1) \ge 0$ and u satisfies

$$\begin{aligned} \|u\|_\infty &\ge A(u) + a(T_\lambda u)'(0) + \int_0^{1/4} \phi^{-1}\Big(\frac{w(1)\phi((T_\lambda u)'(1))}{w(s)} + \frac{\lambda}{w(s)}\int_s^1 h(r)f^*(u)dr\Big)ds \\ &\ge \int_0^{1/4} \phi^{-1}\Big(\frac{\lambda}{w(s)}\int_s^1 h(r)f^*(u)dr\Big)ds \\ &\ge \int_0^{1/4} \phi^{-1}\Big(\frac{\lambda}{w^*}\int_{\frac14}^{\frac12} h(r)f_m(u)dr\Big)ds \\ &\ge \frac14\phi^{-1}\Big(\frac{\lambda h_*}{w^*}f_m(\frac{\delta\|u\|_\infty}{4})\Big). \end{aligned}$$

Hence we obtain $\|u\|_\infty \ge \frac14\phi^{-1}(\frac{\lambda h_*}{w^*}f_m(\frac{\delta\|u\|_\infty}{4}))$ for all cases. Then we have

$$\frac{\lambda h_*}{w^*}f_m\Big(\frac{\delta\|u\|_\infty}{4}\Big) \le \phi(4\|u\|_\infty) \le \psi\Big(\frac{16}{\delta}\Big)\phi\Big(\frac{\delta\|u\|_\infty}{4}\Big).$$

This implies

$$\frac{f_m(\frac{\delta\|u\|_\infty}{4})}{\phi(\frac{\delta\|u\|_\infty}{4})} \le \frac{w^*\psi(\frac{16}{\delta})}{\lambda h_*}. \tag{3.3}$$

By the definition of r_λ, we obtain $r_\lambda < \frac{\delta\|u\|_\infty}{4}$. Thus $\|u\|_\infty \neq r_\lambda$ for $\lambda > 0$. Noting that $\lim_{s\to\infty}\frac{f_m(s)}{\phi(s)} = \infty$ since $f_\infty = \infty$, we can also find $R_\lambda \gg 1$ such that $R_\lambda > 1$ and

$$\frac{f_m(\frac{\delta s}{4})}{\phi(\frac{\delta s}{4})} > \frac{w^*\psi(\frac{16}{\delta})}{\lambda h_*}$$

for $s \ge R_\lambda$. Thus $\|u\|_\infty \neq R_\lambda$ for $\lambda > 0$.

By Proposition 1.10, T_λ has two fixed points $v_1, v_2 \in C[0,1]$ for $\lambda \approx 0$ such that $r_\lambda < \|v_1\|_\infty < 1 < \|v_2\|_\infty < R_\lambda$. Hence v_1 and v_2 are positive solutions of (1.1) by Lemma 2.2. Further, we obtain $\|v_1\|_\infty \to 0$ and $\|v_2\|_\infty \to \infty$ as $\lambda \to 0$ from (3.1).

Next we show the nonexistence result for $\lambda \gg 1$. Assume that (1.1) has a positive solution u. Then u satisfies (3.3). Since $\lim_{s\to 0}\frac{f_m(s)}{\phi(s)} = \infty = \lim_{s\to\infty}\frac{f_m(s)}{\phi(s)}$, we have

$$0 < \inf_{s\in(0,\infty)}\frac{f_m(s)}{\phi(s)} \le \frac{f_m(\frac{\delta\|u\|_\infty}{4})}{\phi(\frac{\delta\|u\|_\infty}{4})} \le \frac{w^*\psi(\frac{16}{\delta})}{\lambda h_*}. \tag{3.4}$$

However, this is a contradiction for $\lambda \gg 1$. Hence (1.1) has no positive solution for $\lambda \gg 1$. □

Proof of Theorem 1.2. We first show the existence result for $\lambda > 0$. If $u \in C[0,1]$ is a solution of $u = T_\lambda u + \tau$ with $\tau \ge 0$, then u satisfies (3.3). This implies $\|u\|_\infty \neq r_\lambda$ for $\lambda > 0$.

If $u \in C[0,1]$ is a solution of $u = \sigma T_\lambda u$ with $\sigma \in (0,1]$ and $\|u\|_\infty \ge r_\lambda$, then u satisfies (3.1). Noting that $\lim_{s\to\infty}\frac{f_M(s)}{\phi(s)} = 0$ since $f_\infty = 0$, we can find $\widehat{R}_\lambda \gg 1$ such that $\widehat{R}_\lambda > r_\lambda$ and

$$1 > \frac{\lambda}{w_*}\psi\Big(\frac{1+a}{1-G}\Big)\Big(\frac{c\|h_\gamma\|_1}{\delta^\gamma s^\gamma\phi(s)} + \frac{\|h\|_1 f_M(s)}{\phi(s)}\Big) \quad \text{for } s \ge \widehat{R}_\lambda.$$

Thus we obtain $\|u\|_\infty \neq \widehat{R}_\lambda$ for $\lambda > 0$ from (3.1). By Proposition 1.10, T_λ has a fixed point v for $\lambda > 0$ such that $r_\lambda < \|v\|_\infty < \widehat{R}_\lambda$. By Lemma 2.2, v is a positive solution of (1.1). Further, we obtain $\|v\|_\infty \to 0$ as $\lambda \to 0$ from (3.1) and $\|v\|_\infty \to \infty$ as $\lambda \to \infty$ from (3.3).

Next we show the multiplicity result for $\lambda \in (\lambda_*, \lambda^*)$. Let $\sigma \in (0,1]$ and $u \in C[0,1]$ be a solution of $u = \sigma T_\lambda u$. We note that

$$u(t) = \sigma T_\lambda u(t) \ge \sigma\delta\|T_\lambda u\|_\infty d(t) = \delta\|u\|_\infty d(t)$$

by Lemma 2.1. By (H6), if $\|u\|_\infty = \eta$ $(\ge r_\lambda)$ then u satisfies

$$\begin{aligned}
\|u\|_\infty &\le A(u) + a(T_\lambda u)'(0) + \int_0^{t_m}\phi^{-1}\Big(\frac{\lambda}{w(s)}\int_s^{t_m} h(r)f^*(u)dr\Big)ds \\
&\le A(u) + (1+a)\phi^{-1}\Big(\frac{\lambda}{w_*}\int_0^1 h(r)f^*(u)dr\Big) \\
&= A(u) + (1+a)\phi^{-1}\Big(\frac{\lambda}{w_*}\int_0^1 h(r)\frac{f_\gamma(\max\{u, \delta r_\lambda d(r)\})}{\max\{u, \delta r_\lambda d(r)\}^\gamma}dr\Big)ds \\
&\le A(u) + (1+a)\phi^{-1}\Big(\frac{\lambda}{w_*}\int_0^1 h(r)\frac{f_\gamma(\|u\|_\infty)}{(\delta\|u\|_\infty d(r))^\gamma}dr\Big) \\
&\le G\|u\|_\infty + (1+a)\phi^{-1}\Big(\frac{\lambda\|h_\gamma\|_1 f(\|u\|_\infty)}{w_*\delta^\gamma}\Big).
\end{aligned}$$

Therefore,

$$\phi(\|u\|_\infty) \le \psi\Big(\frac{1+a}{1-G}\Big)\phi\Big(\frac{(1-G)\|u\|_\infty}{1+a}\Big) \le \frac{\lambda\|h_\gamma\|_1 f(\|u\|_\infty)}{w_*\delta^\gamma}\psi\Big(\frac{1+a}{1-G}\Big).$$

This implies

$$\frac{w_*\delta^\gamma}{\lambda\|h_\gamma\|_1\psi(\frac{1+a}{1-G})}\le\frac{f(\|u\|_\infty)}{\phi(\|u\|_\infty)}=\frac{f(\eta)}{\phi(\eta)}.$$

However, this is a contradiction for $\lambda<\lambda^*$. Hence $\|u\|_\infty\ne\eta$ for $\lambda<\lambda^*$.

Let $\tau\ge0$ and $u\in C[0,1]$ be a solution of $u=T_\lambda u+\tau$. Assume $\|u\|_\infty=\frac{4\theta}{\delta}$. Then three cases can occur: (i) $\|u\|_\infty=u(t_m)$ for some $t_m\in(0,1)$, (ii) $\|u\|_\infty=u(0)$ and (iii) $\|u\|_\infty=u(1)$. We first consider case (i). Noting that $u(t)\ge\frac{\delta\|u\|_\infty}{4}$ for $t\in[\frac14,\frac34]$ from (3.2), if $t_m\ge\frac12$, then

$$\begin{aligned}
\|u\|_\infty&\ge A(u)+a(T_\lambda u)'(0)+\int_0^{t_m}\phi^{-1}\Big(\frac{\lambda}{w(s)}\int_s^{t_m}h(r)f^*(u)dr\Big)ds\\
&\ge\int_0^{1/4}\phi^{-1}\Big(\frac{\lambda}{w(s)}\int_s^{t_m}h(r)\frac{f_\gamma(\max\{u,\delta r_\lambda d(r)\})}{\max\{u,\delta r_\lambda d(r)\}^\gamma}dr\Big)ds\\
&\ge\int_0^{1/4}\phi^{-1}\Big(\frac{\lambda}{w^*}\int_{\frac14}^{\frac12}h(r)\frac{f_\gamma(\frac{\delta\|u\|_\infty}{4})}{\|u\|_\infty^\gamma}dr\Big)ds\\
&\ge\frac14\phi^{-1}\Big(\frac{\lambda h_*\delta^\gamma}{4^\gamma w^*}f\Big(\frac{\delta\|u\|_\infty}{4}\Big)\Big).
\end{aligned}$$

By similar arguments, we can show that if $t_m<\frac12$ then $\|u\|_\infty\ge\frac14\phi^{-1}(\frac{\lambda h_*\delta^\gamma}{4^\gamma w^*}f(\frac{\delta\|u\|_\infty}{4}))$. For case (ii), we have

$$\begin{aligned}
\|u\|_\infty&\ge B(u)-b(T_\lambda u)'(1)+\int_{3/4}^1\phi^{-1}\Big(-\frac{w(0)\phi((T_\lambda u)'(0))}{w(s)}+\frac{\lambda}{w(s)}\int_0^s h(r)f^*(u)dr\Big)ds\\
&\ge\int_{3/4}^1\phi^{-1}\Big(\frac{\lambda}{w(s)}\int_0^s h(r)\frac{f_\gamma(\max\{u,\delta r_\lambda d(r)\})}{\max\{u,\delta r_\lambda d(r)\}^\gamma}dr\Big)ds\\
&\ge\int_{3/4}^1\phi^{-1}\Big(\frac{\lambda}{w^*}\int_{\frac12}^{\frac34}h(r)\frac{f_\gamma(\frac{\delta\|u\|_\infty}{4})}{\|u\|_\infty^\gamma}dr\Big)ds\\
&\ge\frac14\phi^{-1}\Big(\frac{\lambda h_*\delta^\gamma}{4^\gamma w^*}f\Big(\frac{\delta\|u\|_\infty}{4}\Big)\Big).
\end{aligned}$$

For case (iii), we have

$$\begin{aligned}
\|u\|_\infty&\ge A(u)+a(T_\lambda u)'(0)+\int_0^{1/4}\phi^{-1}\Big(\frac{w(1)\phi((T_\lambda u)'(1))}{w(s)}+\frac{\lambda}{w(s)}\int_s^1 h(r)f^*(u)dr\Big)ds\\
&\ge\int_0^{1/4}\phi^{-1}\Big(\frac{\lambda}{w(s)}\int_s^1 h(r)\frac{f_\gamma(\max\{u,\delta r_\lambda d(r)\})}{\max\{u,\delta r_\lambda d(r)\}^\gamma}dr\Big)ds\\
&\ge\int_0^{1/4}\phi^{-1}\Big(\frac{\lambda}{w^*}\int_{\frac14}^{\frac12}h(r)\frac{f_\gamma(\frac{\delta\|u\|_\infty}{4})}{\|u\|_\infty^\gamma}dr\Big)ds\\
&\ge\frac14\phi^{-1}\Big(\frac{\lambda h_*\delta^\gamma}{4^\gamma w^*}f\Big(\frac{\delta\|u\|_\infty}{4}\Big)\Big).
\end{aligned}$$

Hence we obtain $\|u\|_\infty\ge\frac14\phi^{-1}(\frac{\lambda h_*\delta^\gamma}{4^\gamma w^*}f(\frac{\delta\|u\|_\infty}{4}))$ for all cases. Since $\|u\|_\infty=\frac{4\theta}{\delta}$, we have

$$\frac{\lambda h_*\delta^\gamma f(\theta)}{4^\gamma w^*}=\frac{\lambda h_*\delta^\gamma}{4^\gamma w^*}f\Big(\frac{\delta\|u\|_\infty}{4}\Big)\le\phi\Big(4\|u\|_\infty\Big)\le\psi\Big(\frac{16}{\delta}\Big)\phi\Big(\frac{\delta\|u\|_\infty}{4}\Big)=\psi\Big(\frac{16}{\delta}\Big)\phi(\theta).$$

However, this is a contradiction for $\lambda>\lambda_*$. Hence $\|u\|_\infty\ne\frac{4\theta}{\delta}$ for $\lambda>\lambda_*$.

We can choose $\widehat{R}_\lambda\gg1$ such that $\widehat{R}_\lambda>\frac{4\theta}{\delta}$. Further, $r_\lambda<\eta<\frac{4\theta}{\delta}$ and (λ_*,λ^*) is nonempty by (H8a) and (H7), respectively. Thus (1.1) has three positive solutions v_1, v_2 and v_3 for $\lambda\in(\lambda_*,\lambda^*)$ such that $r_\lambda<\|v_1\|_\infty<\eta<\|v_2\|_\infty<\frac{4\theta}{\delta}<\|v_3\|_\infty<\widehat{R}_\lambda$ by Proposition 1.10 and Lemma 2.2. □

Proof of Theorem 1.3. We first show the existence result for $\lambda<\lambda^\infty$. If $u\in C[0,1]$ is a solution of $u=T_\lambda u+\tau$ with $\tau\ge0$, then u satisfies (3.3). Thus $\|u\|_\infty\ne r_\lambda$ for $\lambda>0$.

Let $\sigma \in (0,1]$ and $u \in C[0,1]$ be a solution of $u = \sigma T_\lambda u$. Assume $\|u\|_\infty \geq r_\lambda$. Then u satisfies (3.1). Since $\lim_{s\to\infty} \frac{f_M(s)}{\phi(s)} = f_\infty \in (0,\infty)$, there exists $\overline{R}_\lambda \gg 1$ such that $\overline{R}_\lambda > r_\lambda$ and

$$\frac{\lambda}{w_*}\psi\Big(\frac{1+a}{1-G}\Big)\Big(\frac{c\|h_\gamma\|_1}{\delta^\gamma s^\gamma \phi(s)} + \frac{\|h\|_1 f_M(s)}{\phi(s)}\Big) < \frac{2\lambda\|h\|_1 f_\infty}{w_*}\psi\Big(\frac{1+a}{1-G}\Big) \quad \text{for } s \geq \overline{R}_\lambda.$$

If $\|u\|_\infty = \overline{R}_\lambda$, then $1 < \frac{2\lambda\|h\|_1 f_\infty}{w_*}\psi(\frac{1+a}{1-G})$ from (3.1). However, this is a contradiction for $\lambda < \lambda^\infty$. Hence $\|u\|_\infty \neq \overline{R}_\lambda$ for $\lambda < \lambda^\infty$.

By Proposition 1.10 and Lemma 2.2, (1.1) has a positive solution v for $\lambda < \lambda^\infty$ such that $r_\lambda < \|v\|_\infty < \overline{R}_\lambda$. Further, we obtain $\|v\|_\infty \to 0$ as $\lambda \to 0$ from (3.1).

Next we show the multiplicity result for $\lambda \in (\lambda_*, \min\{\lambda^*, \lambda^\infty\})$. The following were proven in the proof of Theorem 1.2: (i) if $u \in C[0,1]$ is a solution of $u = \sigma T_\lambda u$ with $\sigma \in (0,1]$, then $\|u\|_\infty \neq \eta$ for $\lambda < \lambda^*$, and (ii) if $u \in C[0,1]$ is a solution of $u = T_\lambda u + \tau$ with $\tau \geq 0$, then $\|u\|_\infty \neq \frac{4\theta}{\delta}$ for $\lambda > \lambda_*$. Further, we can choose $\overline{R}_\lambda \gg 1$ such that $\overline{R}_\lambda > \frac{4\theta}{\delta}$. Since $r_\lambda < \eta < \frac{4\theta}{\delta}$ and $(\lambda_*, \min\{\lambda^*, \lambda^\infty\})$ is nonempty by (H7), (H8a), and (H9a), (1.1) has three positive solutions v_1, v_2 and v_3 for $\lambda \in (\lambda_*, \min\{\lambda^*, \lambda^\infty\})$ such that $r_\lambda < \|v_1\|_\infty < \eta < \|v_2\|_\infty < \frac{4\theta}{\delta} < \|v_3\|_\infty < \overline{R}_\lambda$.

Now we show the nonexistence result for $\lambda \gg 1$. Assume to the contrary that (1.1) has a positive solution u for $\lambda \gg 1$. Then u satisfies (3.4) since $\lim_{s\to 0} \frac{f_m(s)}{\phi(s)} = \infty$ and $\lim_{s\to\infty} \frac{f_m(s)}{\phi(s)} = f_\infty > 0$. However, this is a contradiction for $\lambda \gg 1$. Hence (1.1) has no positive solution for $\lambda \gg 1$. □

4. Proofs of Theorems 1.4–1.6

In this section, we consider the case $f_0 = 0$. Since $f_0 = 0$ implies $\lim_{s\to 0} f(s) = 0$, we can define $f_M^*(s) := \max_{r\in[0,s]} f(r)$. Noting that $\lim_{s\to 0} \frac{f_M^*(s)}{\phi(s)} = 0$ since $f_0 = 0$, there exists $r_\lambda^* \in (0,1)$ such that $\frac{f_M^*(s)}{\phi(s)} < \frac{w_*}{\lambda\|h\|_1\psi(\frac{1+a}{1-G})}$ for $s \leq r_\lambda^*$. Then we define $\rho_\lambda = r_\lambda^*$ in T_λ. Additionally, $r_\lambda^* < \min\{\eta, \frac{4\theta}{\delta}\}$ is assumed to show the multiplicity results in Theorem 1.4 and Theorem 1.6.

Proof of Theorem 1.4. We first show the existence result for $\lambda > 0$. Let $\sigma \in (0,1]$ and $u \in C[0,1]$ be a solution of $u = \sigma T_\lambda u$. Assume $\|u\|_\infty \geq r_\lambda^*$. Following the arguments in the proof of Theorem 1.1, we obtain

$$\begin{aligned}
\|u\|_\infty &\leq A(u) + (1+a)\phi^{-1}\Big(\frac{\lambda}{w_*}\int_0^1 h(r)f^*(u)dr\Big) \\
&\leq A(u) + (1+a)\phi^{-1}\Big(\frac{\lambda}{w_*}\int_0^1 h(r)f_M^*(\max\{u, \delta r_\lambda^* d(r)\})dr\Big) \\
&\leq G\|u\|_\infty + (1+a)\phi^{-1}\Big(\frac{\lambda\|h\|_1 f_M^*(\|u\|_\infty)}{w_*}\Big).
\end{aligned}$$

Then u satisfies

$$\phi(\|u\|_\infty) \leq \psi\Big(\frac{1+a}{1-G}\Big)\phi\Big(\frac{(1-G)\|u\|_\infty}{1+a}\Big) \leq \frac{\lambda\|h\|_1 f_M^*(\|u\|_\infty)}{w_*}\psi\Big(\frac{1+a}{1-G}\Big).$$

This implies

$$\frac{w_*}{\lambda\|h\|_1\psi(\frac{1+a}{1-G})} \leq \frac{f_M^*(\|u\|_\infty)}{\phi(\|u\|_\infty)}. \tag{4.1}$$

Thus $\|u\|_\infty \neq r_\lambda^*$ for $\lambda > 0$ by the definition of r_λ^*.

If $u \in C[0,1]$ is a solution of $u = T_\lambda u + \tau$ with $\tau \geq 0$, then u satisfies (3.3). Then $\|u\|_\infty \neq R_\lambda$ (> 1) for $\lambda > 0$, where R_λ is the constant found in the proof of Theorem 1.1. By Proposition 1.10 and Lemma 2.2, (1.1) has a positive solution v for $\lambda > 0$ such that $r_\lambda^* < \|v\|_\infty < R_\lambda$. Further, we obtain $\|v\|_\infty \to 0$ as $\lambda \to \infty$ from (3.3) and $\|v\|_\infty \to \infty$ as $\lambda \to 0$ from (4.1).

Next we show the multiplicity result for $\lambda \in (\lambda_*, \lambda^*)$. Following the arguments in the proof of Theorem 1.2, we can show that (i) if $u \in C[0,1]$ is a solution of $u = \sigma T_\lambda u$ with $\sigma \in (0,1]$, then $\|u\|_\infty \neq \eta$ for $\lambda < \lambda^*$, and (ii) if $u \in C[0,1]$ is a solution of $u = T_\lambda u + \tau$ with $\tau \geq 0$, then $\|u\|_\infty \neq \frac{4\theta}{\delta}$ for $\lambda > \lambda_*$. Further, we can choose $R_\lambda \gg 1$ such that $R_\lambda > \eta$. We note that $r_\lambda^* < \frac{4\theta}{\delta} < \eta$ and

(λ_*, λ^*) is nonempty by (H8b) and (H7), respectively. Hence (1.1) has three positive solutions v_1, v_2 and v_3 for $\lambda \in (\lambda_*, \lambda^*)$ such that $r_\lambda^* < \|v_1\|_\infty < \frac{4\theta}{\delta} < \|v_2\|_\infty < \eta < \|v_3\|_\infty < R_\lambda$. ☐

Proof of Theorem 1.5. We first show the multiplicity result for $\lambda \gg 1$. Let $\tau \geq 0$ and $u \in C[0,1]$ be a solution of $u = T_\lambda u + \tau$. Then u satisfies (3.3). If $\|u\|_\infty = 1$, we have $\frac{f_m(\frac{\delta}{4})}{\phi(\frac{\delta}{4})} \leq \frac{w^*\psi(\frac{16}{\delta})}{\lambda h_*}$. However, this is a contradiction for $\lambda \gg 1$. Hence $\|u\|_\infty \neq 1$ for $\lambda \gg 1$.

Let $\sigma \in (0,1]$ and $u \in C[0,1]$ be a solution of $u = \sigma T_\lambda u$ and $\|u\|_\infty \geq r_\lambda^*$. Then u satisfies (4.1). This implies $\|u\|_\infty \neq r_\lambda^*$ for $\lambda > 0$. Further, since $\lim_{s\to\infty} \frac{f_M^*(s)}{\phi(s)} = 0$, we can find $\widetilde{R}_\lambda \gg 1$ such that $\widetilde{R}_\lambda > 1$ and

$$\frac{f_M^*(s)}{\phi(s)} < \frac{w_*}{\lambda \|h\|_1 \psi(\frac{1+a}{1-G})} \quad \text{for } s \geq \widetilde{R}_\lambda.$$

Thus we obtain $\|u\|_\infty \neq \widetilde{R}_\lambda$ for $\lambda > 0$ from (4.1). By Proposition 1.10 and Lemma 2.2, there exist positive solutions v_1 and v_2 for $\lambda \gg 1$ such that $r_\lambda^* < \|v_1\|_\infty < 1 < \|v_2\|_\infty < \widetilde{R}_\lambda$. Further, we obtain $\|v_1\|_\infty \to 0$ and $\|v_2\|_\infty \to \infty$ as $\lambda \to \infty$ from (3.3).

Next we show the nonexistence result for $\lambda \approx 0$. If u is a positive solution of (1.1), then u satisfies (4.1). Thus we have

$$\frac{w_*}{\lambda \|h\|_1 \psi(\frac{1+a}{1-G})} \leq \sup_{s\in(0,\infty)} \frac{f_M^*(s)}{\phi(s)} < \infty. \tag{4.2}$$

However, this is a contradiction for $\lambda \approx 0$. Hence (1.1) has no positive solution for $\lambda \approx 0$. ☐

Proof of Theorem 1.6. We first show the existence result for $\lambda > \lambda_\infty$. If $u \in C[0,1]$ is a solution of $u = \sigma T_\lambda u$ with $\sigma \in (0,1]$ and $\|u\|_\infty \geq r_\lambda^*$, then u satisfies (4.1). This implies $\|u\|_\infty \neq r_\lambda^*$ for $\lambda > 0$.

Let $\tau \geq 0$ and $u \in C[0,1]$ be a solution of $u = T_\lambda u + \tau$. Then u satisfies (3.3). We note that there exists $R_\lambda^* \gg 1$ such that $R_\lambda^* > 1$ and $\frac{f_m(s)}{\phi(s)} > \frac{f_\infty}{2}$ for $s \geq R_\lambda^*$ since $\lim_{s\to\infty} \frac{f_m(s)}{\phi(s)} = f_\infty \in (0,\infty)$. If $\|u\|_\infty = \frac{4R_\lambda^*}{\delta}$, then we have

$$\frac{f_\infty}{2} \leq \frac{f_m(R_\lambda^*)}{\phi(R_\lambda^*)} \leq \frac{w^*\psi(\frac{16}{\delta})}{\lambda h_*} \tag{4.3}$$

from (3.3). However, this is a contradiction for $\lambda > \lambda_\infty$. Hence $\|u\|_\infty \neq \frac{4R_\lambda^*}{\delta}$ for $\lambda > \lambda_\infty$. By Proposition 1.10 and Lemma 2.2, (1.1) has a positive solution v for $\lambda > \lambda_\infty$ such that $r_\lambda^* < \|v\|_\infty < \frac{4R_\lambda^*}{\delta}$. Further, we obtain $\|v\|_\infty \to 0$ as $\lambda \to \infty$ from (3.3).

Next we show the multiplicity result for $\lambda \in (\max\{\lambda_*, \lambda_\infty\}, \lambda^*)$. The following were proven in the proof of Theorem 1.2: (i) if $u \in C[0,1]$ is a solution of $u = \sigma T_\lambda u$ with $\sigma \in (0,1]$, then $\|u\|_\infty \neq \eta$ for $\lambda < \lambda^*$ and (ii) if $u \in C[0,1]$ is a solution of $u = T_\lambda u + \tau$ with $\tau \geq 0$, then $\|u\|_\infty \neq \frac{4\theta}{\delta}$ for $\lambda > \lambda_*$. Further, $r_\lambda^* < \frac{4\theta}{\delta} < \eta$ and $(\max\{\lambda_*, \lambda_\infty\}, \lambda^*)$ is nonempty by (H7), (H8b), and (H9b), and we can choose $R_\lambda^* \gg 1$ such that $\frac{4R_\lambda^*}{\delta} > \eta$. Hence (1.1) has three positive solutions v_1, v_2 and v_3 for $\lambda \in (\max\{\lambda_*, \lambda_\infty\}, \lambda^*)$ such that $r_\lambda^* < \|v_1\|_\infty < \frac{4\theta}{\delta} < \|v_2\|_\infty < \eta < \|v_3\|_\infty < \frac{4R_\lambda^*}{\delta}$.

Now we show the nonexistence result for $\lambda \approx 0$. If u is a positive solution of (1.1), then we can show that u satisfies (4.2) following the arguments in Theorem 1.5. However, this is a contradiction for $\lambda \approx 0$. Hence (1.1) has no positive solution for $\lambda \approx 0$. ☐

5. Proofs of Theorems 1.7–1.9

Proof of Theorem 1.7. We choose $\rho_\lambda = \overline{r}_\lambda$ in T_λ, where $\overline{r}_\lambda \in (0,1)$ is such that $\frac{f_M^*(s)}{\phi(s)} < 2f_0$ for $s \leq \overline{r}_\lambda$. To show the multiplicity result, we also assume $\overline{r}_\lambda < \min\{\eta, \frac{4\theta}{\delta}\}$.

We first show the existence result for $\lambda < \lambda^0$. Let $\sigma \in (0,1]$ and $u \in C[0,1]$ be a solution of $u = \sigma T_\lambda u$. If $\|u\|_\infty = \overline{r}_\lambda$, then u satisfies (4.1). Thus we have

$$\frac{w_*}{\lambda \|h\|_1 \psi(\frac{1+a}{1-G})} \leq \frac{f_M^*(\overline{r}_\lambda)}{\phi(\overline{r}_\lambda)} < 2f_0. \tag{5.1}$$

However, this is a contradiction for $\lambda < \lambda^0$. Hence $\|u\|_\infty \neq \overline{r}_\lambda$ for $\lambda < \lambda^0$.

If $u \in C[0,1]$ is a solution of $u = T_\lambda u + \tau$ with $\tau \geq 0$, then u satisfies (3.3). Then $\|u\|_\infty \neq R_\lambda$ (> 1) for $\lambda > 0$, where R_λ is the constant found in the proof of Theorem 1.1. By Proposition 1.10 and Lemma 2.2, (1.1) has a positive solution v for $\lambda < \lambda^0$ such that $\bar{r}_\lambda < \|v\|_\infty < R_\lambda$. Further, we obtain $\|v\|_\infty \to \infty$ as $\lambda \to 0$ from (4.1).

Next we show the multiplicity result for $\lambda \in (\lambda_*, \min\{\lambda^*, \lambda^0\})$. Following the arguments in the proof of Theorem 1.2, we can show that (i) if $u \in C[0,1]$ is a solution of $u = \sigma T_\lambda u$ with $\sigma \in (0,1]$, then $\|u\|_\infty \neq \eta$ for $\lambda < \lambda^*$, and (ii) if $u \in C[0,1]$ is a solution of $u = T_\lambda u + \tau$ with $\tau \geq 0$, then $\|u\|_\infty \neq \frac{4\theta}{\delta}$ for $\lambda > \lambda_*$. Further, $\bar{r}_\lambda < \frac{4\theta}{\delta} < \eta$ and $(\lambda_*, \min\{\lambda^*, \lambda^0\})$ is nonempty by (H7), (H8b) and (H9a), and we can choose $R_\lambda \gg 1$ such that $R_\lambda > \eta$. Hence (1.1) has three positive solutions v_1, v_2 and v_3 for $\lambda \in (\lambda_*, \min\{\lambda^*, \lambda^0\})$ such that $\bar{r}_\lambda < \|v_1\|_\infty < \frac{4\theta}{\delta} < \|v_2\|_\infty < \eta < \|v_3\|_\infty < R_\lambda$.

Now we show the nonexistence result for $\lambda \gg 1$. If u is a positive solution of (1.1), then u satisfies (3.4). However, this is a contradiction for $\lambda \gg 1$. Hence (1.1) has no positive solution for $\lambda \gg 1$. □

Proof of Theorem 1.8. We choose $\rho_\lambda = \tilde{r}_\lambda$ in T_λ, where $\tilde{r}_\lambda \in (0,1)$ is such that $\frac{f_m(s)}{\phi(s)} > \frac{f_0}{2}$ for $s \leq \tilde{r}_\lambda$. To show the multiplicity result, we also assume $\tilde{r}_\lambda < \min\{\eta, \frac{4\theta}{\delta}\}$.

We first show the existence result for $\lambda > \lambda_0$. Let $\tau \geq 0$ and $u \in C[0,1]$ be a solution of $u = T_\lambda u + \tau$. If $\|u\|_\infty = \tilde{r}_\lambda$, then we have

$$\frac{f_0}{2} < \frac{f_m(\frac{\delta\tilde{r}_\lambda}{4})}{\phi(\frac{\delta\tilde{r}_\lambda}{4})} \leq \frac{w^*\psi(\frac{16}{\delta})}{\lambda h_*} \tag{5.2}$$

from (3.3). However, this is a contradiction for $\lambda > \lambda_0$. Hence $\|u\|_\infty \neq \tilde{r}_\lambda$ for $\lambda > \lambda_0$.

Let $\sigma \in (0,1]$ and $u \in C[0,1]$ be a solution of $u = \sigma T_\lambda u$ with $\|u\|_\infty \geq \tilde{r}_\lambda$. Then u satisfies (4.1). Since $\lim_{s\to\infty} \frac{f_M^*(s)}{\phi(s)} = 0$, we obtain $\|u\|_\infty \neq \widetilde{R}_\lambda$ $(> \tilde{r}_\lambda)$ for $\lambda > 0$, where $\widetilde{R}_\lambda$ is the constant found in the proof of Theorem 1.5. By Proposition 1.10 and Lemma 2.2, (1.1) has a positive solution v for $\lambda > \lambda_0$ such that $\tilde{r}_\lambda < \|v\|_\infty < \widetilde{R}_\lambda$. Further, we obtain $\|v\|_\infty \to \infty$ as $\lambda \to \infty$ from (3.3).

Next we show the multiplicity result for $\lambda \in (\max\{\lambda_*, \lambda_0\}, \lambda^*)$. Following the arguments in the proof of Theorem 1.2, we can show that (i) if $u \in C[0,1]$ is a solution of $u = \sigma T_\lambda u$ with $\sigma \in (0,1]$, then $\|u\|_\infty \neq \eta$ for $\lambda < \lambda^*$ and (ii) if $u \in C[0,1]$ is a solution of $u = T_\lambda u + \tau$ with $\tau \geq 0$, then $\|u\|_\infty \neq \frac{4\theta}{\delta}$ for $\lambda > \lambda_*$. Further, $\tilde{r}_\lambda < \eta < \frac{4\theta}{\delta}$ and $(\max\{\lambda_*, \lambda_0\}, \lambda^*) \neq \emptyset$ by (H7), (H8a) and (H9b), and we can choose $\widetilde{R}_\lambda \gg 1$ such that $\widetilde{R}_\lambda > \frac{4\theta}{\delta}$. Hence (1.1) has three positive solutions v_1, v_2 and v_3 for $\lambda \in (\max\{\lambda_*, \lambda_0\}, \lambda^*)$ such that $\tilde{r}_\lambda < \|v_1\|_\infty < \eta < \|v_2\|_\infty < \frac{4\theta}{\delta} < \|v_3\|_\infty < \widetilde{R}_\lambda$.

Now we show the nonexistence result for $\lambda \approx 0$. If u is a positive solution of (1.1), then u satisfies (4.2). However, this is a contradiction for $\lambda \approx 0$. Hence (1.1) has no positive solution for $\lambda \approx 0$. □

Proof of Theorem 1.9. We first consider the case $f_0 > f_\infty$. Then $\lambda_0 = \min\{\lambda_0, \lambda_\infty\}$ and $\lambda^\infty = \max\{\lambda^0, \lambda^\infty\}$. We choose $\rho_\lambda = \tilde{r}_\lambda$, which is the constant in the proof of Theorem 1.8.

We show the existence result for $\lambda \in (\lambda_0, \lambda^\infty) = (\min\{\lambda_0, \lambda_\infty\}, \max\{\lambda^0, \lambda^\infty\})$. Let $\tau \geq 0$ and $u \in C[0,1]$ be a solution of $u = T_\lambda u + \tau$. If $\|u\|_\infty = \tilde{r}_\lambda$, then (5.2) is satisfied. This implies $\|u\|_\infty \neq \tilde{r}_\lambda$ for $\lambda > \lambda_0$. Let $\sigma \in (0,1]$ and $u \in C[0,1]$ be a solution of $u = \sigma T_\lambda u$. Since $\lim_{s\to\infty} \frac{f_M^*(s)}{\phi(s)} = f_\infty$, there exists $R_\lambda^\diamond \gg 1$ such that $R_\lambda^\diamond > \tilde{r}_\lambda$ and $\frac{f_M^*(s)}{\phi(s)} < 2f_\infty$ for $s \geq R_\lambda^\diamond$. If $\|u\|_\infty = R_\lambda^\diamond$, then $\frac{w_*}{\lambda\|h\|_1\psi(\frac{1+a}{1-G})} \leq \frac{f_M^*(R_\lambda^\diamond)}{\phi(R_\lambda^\diamond)} < 2f_\infty$ from (4.1). However, this is a contradiction for $\lambda < \lambda^\infty$. Thus $\|u\|_\infty \neq R_\lambda^\diamond$ for $\lambda < \lambda^\infty$. Since $(\lambda_0, \lambda^\infty)$ is nonempty by (H10a), (1.1) has a positive solution for $\lambda \in (\lambda_0, \lambda^\infty)$.

Now we show the multiplicity results. We have (i) if $u \in C[0,1]$ is a solution of $u = \sigma T_\lambda u$ with $\sigma \in (0,1]$, then $\|u\|_\infty \neq \eta$ for $\lambda < \lambda^*$ and (ii) if $u \in C[0,1]$ is a solution of $u = T_\lambda u + \tau$ with $\tau \geq 0$, then $\|u\|_\infty \neq \frac{4\theta}{\delta}$ for $\lambda > \lambda_*$. Further, $\tilde{r}_\lambda < \min\{\eta, \frac{4\theta}{\delta}\}$ and we can choose $R_\lambda^\diamond \gg 1$ such that $R_\lambda^\diamond > \max\{\eta, \frac{4\theta}{\delta}\}$. Since $(\max\{\lambda_*, \lambda_0\}, \min\{\lambda^*, \lambda^\infty\})$ is nonempty by (H7), (H10a) and (H10b), (1.1) has three positive solutions v_1, v_2 and v_3 for $\lambda \in (\max\{\lambda_*, \lambda_0\}, \min\{\lambda^*, \lambda^\infty\})$ such that $\tilde{r}_\lambda < \|v_1\|_\infty < \eta < \|v_2\|_\infty < \frac{4\theta}{\delta} < \|v_3\|_\infty < R_\lambda^\diamond$.

Next we consider the case $f_0 < f_\infty$. Then $\lambda_\infty = \min\{\lambda_0, \lambda_\infty\}$ and $\lambda^0 = \max\{\lambda^0, \lambda^\infty\}$. We choose $\rho_\lambda = \overline{r}_\lambda$, which is the constant in the proof of Theorem 1.7.

We show the existence result for $\lambda \in (\lambda_\infty, \lambda^0) = (\min\{\lambda_0, \lambda_\infty\}, \max\{\lambda^0, \lambda^\infty\})$. Let $\sigma \in (0,1]$ and $u \in C[0,1]$ be a solution of $u = \sigma T_\lambda u$. If $\|u\|_\infty = \overline{r}_\lambda$, then (5.1) is satisfied. This implies $\|u\|_\infty \neq \overline{r}_\lambda$ for $\lambda < \lambda^0$. Let $\tau \geq 0$ and $u \in C[0,1]$ be a solution of $u = T_\lambda u + \tau$. If $\|u\|_\infty = \frac{4R_\lambda^*}{\delta}$, then (4.3) is satisfied. This implies $\|u\|_\infty \neq \frac{4R_\lambda^*}{\delta}$ for $\lambda > \lambda_\infty$. Since $(\lambda_\infty, \lambda^0)$ is nonempty by (H10a), (1.1) has a positive solution for $\lambda \in (\lambda_\infty, \lambda^0)$.

Since $(\max\{\lambda_*, \lambda_\infty\}, \min\{\lambda^*, \lambda^0\})$ is nonempty by (H7), (H10a) and (H10c), we can also show that (1.1) has three positive solutions v_1, v_2 and v_3 for $\lambda \in (\max\{\lambda_*, \lambda_\infty\}, \min\{\lambda^*, \lambda^0\})$ such that $\overline{r}_\lambda < \|v_1\|_\infty < \frac{4\theta}{\delta} < \|v_2\|_\infty < \eta < \|v_3\|_\infty < \frac{4R_\lambda^*}{\delta}$ following the above arguments.

The proofs of the nonexistence results for $\lambda \approx 0$ and for $\lambda \gg 1$ follow the arguments in the proofs of Theorem 1.5 and Theorem 1.1, respectively. □

6. Example

In this section, we discuss an example of a mixed nonlocal boundary value problem involving a nonhomogeneous operator and a singular nonlinear term. We consider the (p,q)-Laplacian problem

$$
\begin{gathered}
-(w(t)(|u'|^{p-2}u' + |u'|^{q-2}u'))' = \lambda h(t)\Big(\frac{A}{u^{\gamma_1}} + Bu^{\gamma_2} + Ce^{\frac{\gamma_3 u}{\gamma_3 + u}} + D\Big), \quad t \in (0,1), \\
u(0) - au'(0) = \int_0^1 g_0(s) k_0^0(s,u)ds + \sum_{i=1}^m \alpha_i k_i^0(\zeta_i, u(\zeta_i)), \\
u(1) + bu'(1) = \int_0^1 g_1(s) k_0^1(s,u)ds + \sum_{j=1}^n \beta_j k_j^1(\xi_j, u(\xi_j)),
\end{gathered} \tag{6.1}
$$

where $1 < p < q < \infty$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$, $A > 0$, $B \geq 0$, $C \geq 0$, $D \in \mathbb{R}$ and $C + D \geq 0$. We assume $a \geq 0$, $b \geq 0$, $0 \leq \alpha_i < 1$, $0 \leq \beta_j < 1$, w and h satisfy (H3), g_0 and g_1 satisfy (H4), and k_0^0, k_0^1, k_i^0, and k_j^1 satisfy (H5), where $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$.

The operator of this problem is $u \mapsto (|u'|^{p-2}u' + |u'|^{q-2}u')'$. It is a nonhomogeneous operator and $\phi(s) = |s|^{p-2}s + |s|^{q-2}s$ satisfies (H1) with $\psi(s) = \max\{s^{p-1}, s^{q-1}\}$. The nonlinear term is $f(s) = \frac{A}{s^{\gamma_1}} + Bs^{\gamma_2} + Ce^{\frac{\gamma_3 s}{\gamma_3 + s}} + D$. It is singular at 0 and $f_0 = \infty$ since $A > 0$. Further, it satisfies (H2) and (H6) with $\gamma = \gamma_1$. Hence, (6.1) satisfies (H1)–(H6), $f_0 = \infty$ and $\gamma = \gamma_1$ under the above conditions.

1. If $B > 0$ and $\gamma_2 > q - 1$, then $f_\infty = \infty$. Thus (6.1) has no positive solution for $\lambda \gg 1$ and has two positive solutions u_1 and u_2 for $\lambda \approx 0$ such that $\|u_1\|_\infty \to 0$ and $\|u_2\|_\infty \to \infty$ as $\lambda \to 0$.
2. If $\gamma_2 < q - 1$, then $f_\infty = 0$. Thus (6.1) has a positive solution u for $\lambda > 0$ such that $\|u\|_\infty \to 0$ as $\lambda \to 0$ and $\|u\|_\infty \to \infty$ as $\lambda \to \infty$. In addition, if $C > 0$, $\eta = 1$ and $\theta = \gamma_3$, then we have

$$
\begin{aligned}
\frac{f(\theta)}{\phi(\theta)} \Big/ \frac{f(\eta)}{\phi(\eta)} &= \frac{\frac{A}{\gamma_3^{\gamma_1}} + B\gamma_3^{\gamma_2} + Ce^{\frac{\gamma_3}{2}} + D}{\gamma_3^{p-1} + \gamma_3^{q-1}} \frac{2}{A + B + Ce^{\frac{\gamma_3}{\gamma_3+1}} + D} \\
&\geq \frac{Ce^{\frac{\gamma_3}{2}} + D}{2\gamma_3^{q-1}} \frac{2}{A + B + Ce + D} \\
&\geq \frac{Ce^{\frac{\gamma_3}{2}} + D}{\gamma_3^{q-1}(A + B + Ce + D)} \gg 1
\end{aligned} \tag{6.2}
$$

for $\gamma_3 \gg 1$. Thus (H7) and (H8a) are satisfied for $\gamma_3 \gg 1$. Further, we have

$$\begin{aligned} \lambda_* &= \frac{4^{\gamma_1} w^* (\frac{16}{\delta})^{q-1} (\gamma_3^{p-1} + \gamma_3^{q-1})}{h_* \delta^{\gamma_1} (\frac{A}{\gamma_3^{\gamma_1}} + B\gamma_3^{\gamma_2} + Ce^{\frac{\gamma_3}{2}} + D)}, \\ \lambda^* &= \frac{2w_* \delta^{\gamma_1}}{\|h_{\gamma_1}\|_1 (\frac{1+a}{1-G})^{q-1} (A + B + Ce^{\frac{\gamma_3}{\gamma_3+1}} + D)}. \end{aligned} \tag{6.3}$$

Hence if $\gamma_3 \gg 1$, then (6.1) has three positive solutions for $\lambda \in (\lambda_*, \lambda^*)$.

3. If $B > 0$ and $\gamma_2 = q - 1$, then $f_\infty = B \in (0, \infty)$ and $\lambda^\infty = \frac{w_*}{2B\|h\|_1 (\frac{1+a}{1-G})^{q-1}}$. Thus (6.1) has no positive solution for $\lambda \gg 1$ and has a positive solution u for $\lambda < \lambda^\infty$ such that $\|u\|_\infty \to 0$ as $\lambda \to 0$. In addition, if $C > 0$, $\eta = 1$ and $\theta = \gamma_3$, then we have

$$\begin{aligned} \frac{f(\theta)}{\phi(\theta)} &= \frac{\frac{A}{\gamma_3^{\gamma_1}} + B\gamma_3^{q-1} + Ce^{\frac{\gamma_3}{2}} + D}{\gamma_3^{p-1} + \gamma_3^{q-1}} \ge \frac{Ce^{\frac{\gamma_3}{2}} + D}{2\gamma_3^{q-1}} \gg 1, \\ \frac{f(\theta)}{\phi(\theta)} \Big/ \frac{f(\eta)}{\phi(\eta)} &\ge \frac{Ce^{\frac{\gamma_3}{2}} + D}{\gamma_3^{q-1}(A + B + Ce + D)} \gg 1 \end{aligned}$$

for $\gamma_3 \gg 1$ from (6.2). Thus (H7), (H8a) and (H9a) are satisfied for $\gamma_3 \gg 1$. Noting that λ_* and λ^* are the same as those in (6.3), if $\gamma_3 \gg 1$ then (6.1) has three positive solutions for $\lambda \in (\lambda_*, \min\{\lambda^*, \lambda^\infty\})$.

Acknowledgments. E. K. Lee was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea Government (NRF-2020R1F1A1A01048442). I. Sim was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea Government (NRF-2021R1I1A3A0403627013). B. Son was supported by a Summer Research Grant funded by Ohio Northern University.

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