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UPPER SEMICONTINUITY OF UNIFORM ATTRACTORS FOR SINGULAR PERTURBED SECOND ORDER NONAUTONOMOUS DELAY LATTICE SYSTEMS

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ABSTRACT. In this article, we consider the upper semicontinuity of the uniform attractors for the singular perturbed second order nonautonomous delay lattice systems driven by the almost periodic forces as the coefficient of second order derivative term tends to zero under the Hausdorff semidistance. First we prove the existence of uniform attractors for the second order and the corresponding first order nonautonomous delay lattice systems. Then we establish some prior uniform estimations of solutions. Finally we study the upper semicontinuity of the uniform attractors as the coefficient of second order derivative term tends to zero which showing the relationship between the uniform attractors for second order and the corresponding first order nonautonomous delay lattice systems.

1. INTRODUCTION

Let $k \in \mathbb{N}$ and

$$\ell^2 = \big\{u = (u_m)_{m\in\mathbb{Z}^k} : m = (m_1, \dots, m_k) \in \mathbb{Z}^k, u_m \in \mathbb{R}, \sum_{m\in\mathbb{Z}^k} u_m^2 < +\infty\big\},$$

be a Hilbert space endowed with the inner product and norm:

$$(u, v) = \sum_{m\in\mathbb{Z}^k} u_m v_m, \quad \|u\|^2 = (u, u), \quad u = (u_m)_{m\in\mathbb{Z}^k}, \quad v = (v_m)_{m\in\mathbb{Z}^k} \in \ell^2.$$

Let $C_b(\mathbb{R}, \ell^2)$ denote the space of continuous bounded functions from $\mathbb{R}$ into ℓ^2 and $g_0 = (g_{0,m})_{m\in\mathbb{Z}^k}$: $\mathbb{R} \to \ell^2$ be an almost periodic function in the Bohr sense,

$$\mathcal{H}(g_0) = \overline{\{g_0(\cdot + r) : r \in \mathbb{R}\}}^{C_b(\mathbb{R},\ell^2)}$$

(the closure in $C_b(\mathbb{R}, \ell^2)$).

In this article, we consider the family of second-order nonautonomous delay lattice systems with singular perturbation

$$\begin{gathered} \epsilon\ddot{u}_m + \dot{u}_m + \gamma(A\dot{u})_m + (Au)_m + \lambda_m u_m + f_m(u_j|j \in I_{mq}) \\ +h_m(u_m(t-\vartheta)) = g_m(t),\ t \ge \tau,\ g(\cdot) = (g_m(\cdot))_{m\in\mathbb{Z}^k} \in \mathcal{H}(g_0),\ \epsilon > 0, \\ u_{m,\tau}(\theta) = u_m(\tau+\theta) = u_{0,m\tau}(\theta), \quad \tau \in \mathbb{R}, \\ \dot{u}_{m,\tau}(\theta) = \dot{u}_m(\tau+\theta) = u_{1,m\tau}(\theta), \quad \theta \in [-\vartheta, 0],\ m \in \mathbb{Z}^k, \end{gathered} \tag{1.1}$$

where $m \in \mathbb{Z}^k$, $\lambda_m > 0$, ϵ, $\vartheta > 0$, $\gamma \ge 0$, u_m, $g_m(t)$, $f_m(u_j(t)|j \in I_{mq})$, $h_m(u_m(t-\vartheta)) \in \mathbb{R}$, $u = (u_m)_{m\in\mathbb{Z}^k}$, A is a linear coupled operator, $I_{mq} = \{j \in \mathbb{Z}^k : \|j-m\| = \max_{1\le l\le k} |j_l - m_l| \le q\}$, $q \in \mathbb{N}$, $f_m(u_j|j \in I_{mq})$ indicates that the state at the m-th lattice point can be related to the states at its surrounding $(2q+1)^k - 1$ lattice points (the relationship may be nonlinear).

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When $\epsilon = 0$, (1.1) becomes the following family of first order nonautonomous delay lattice systems

$$\begin{aligned}&\dot{u}_m + \gamma(A\dot{u})_m + (Au)_m + \lambda_m u_m + f_m(u_j|j \in I_{mq}) + h_m(u_m(t-\vartheta))\\&= g_m(t), \quad t \geq \tau, \quad g = (g_m)_{m\in\mathbb{Z}^k} \in \mathcal{H}(g_0),\\&u_{m,\tau}(\theta) = u_m(\tau+\theta) = u_{0,m\tau}(\theta), \quad \tau \in \mathbb{R}, \quad \theta \in [-\vartheta, 0],\ m \in \mathbb{Z}^k.\end{aligned} \tag{1.2}$$

The lattice systems (1.1)-(1.2) can be used as mathematical models for the various coupled oscillator systems (such as the system of coupled pendulum motions) and the dynamic network systems with infinite nodes etc.. The attractors of various different types of lattice systems (consisting of infinite dimensional ordinary differential equations) have been studied by many publications in the last more than 20 years from the work of Bate et al in 2001 [2], including the existence and related properties of the global attractor, pullback attractor, uniform attractor and random attractor, see [1, 5, 7, 8, 10, 12, 13, 14, 15, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28] and the references therein.

The relationship between the attractors of first-order and second-order lattice systems is of interesting topic. For the case of autonomous and nonautonomous lattice systems (1.1)-(1.2) without delay and the coupled term of first order derivatives (that is, $\vartheta \equiv 0$ and $\gamma = 0$), the relationship between the global attractors and uniform attractors of (1.1) and (1.2) as $\epsilon \to 0^+$ have been studied in [13, 25], respectively. In the case of $\vartheta \neq 0$ and $\gamma \neq 0$, the phase space of (1.1) and (1.2) are Banach spaces $C([-\vartheta, 0], \ell^2 \times \ell^2)$ and $C([-\vartheta, 0], \ell^2)$ consisting of continuous functions from a closed interval $[-\vartheta, 0]$ into the spaces $\ell^2 \times \ell^2$ and ℓ^2, respectively, which are different from the Hilbert phase spaces in [13, 25]. As we know, there is no results about the relationship between the uniform attractors of (1.1) and (1.2) until now.

Based on the works of [13, 25], here we consider the upper-semicontinuity of the uniform attractors for the singular perturbated second order nonautonomous delay lattice system (1.1) as $\epsilon \to 0^+$, which gives the relationship between the uniform attractors of (1.1) and (1.2). Since the Banach phase spaces cannot be decomposed into a direct sum of a finite dimensional space and an infinite dimensional space with a small norm, so in proving the key prior uniformly bounded estimations of solutions of systems, the asymptotic compactness of the solutions processes and the convergence of the solutions sequences et al., we have to use new techniques different from those in [13, 25]. Notice that the uniform attractors $\mathcal{A}_\epsilon^{\mathcal{H}(g_0)}$ of (1.1) and $\mathcal{A}_0^{\mathcal{H}(g_0)}$ of (1.2) are included in different spaces $C([-\vartheta, 0], \ell^2 \times \ell^2)$ and $C([-\vartheta, 0], \ell^2)$, respectively, to compare the relationship between them, we take them in the bigger space $C([-\vartheta, 0], \ell^2 \times \ell^2)$. For our purpose, we construct a compact set $\mathcal{B}_0^{\mathcal{H}(g_0)} \subset C([-\vartheta, 0], \ell^2 \times \ell^2)$ such that $\mathcal{A}_0^{\mathcal{H}(g_0)}$ is naturally embedded into $\mathcal{B}_0^{\mathcal{H}(g_0)}$ as the first component. It is worth mentioning that because of the lack of the structure of operator $(I + \gamma A)^{-1}$, the equivalent first order lattice equations of (1.2) may be not a locally coupled lattice system. Generally, the proof of the asymptotic compactness of the solutions process for a non-locally coupled lattice system is difficult. Fortunately, the linear boundedness of $(I + \gamma A)^{-1}$ here is just enough to solve this challenging problem.

In section 2, we present some spaces, some assumptions and the vector forms of (1.1) and (1.2). In section 3, we prove the existence of uniform attractors $\mathcal{A}_\epsilon^{\mathcal{H}(g_0)}$of (1.1) and $\mathcal{A}_0^{\mathcal{H}(g_0)}$ of (1.2). In section 4, we establish some prior uniform estimations for the solutions of (1.1). In section 5, we consider the upper semicontinuity of $\mathcal{A}_\epsilon^{\mathcal{H}(g_0)}$ as $\epsilon \to 0^+$.

2. Mathematical setting

Firstly, we present some concepts related with the uniform attractor for a family of processes. Let X be a Banach space with norm $\|\cdot\|_X$, $\mathcal{B}(X)$ be the union of all bounded sets of X and Σ be a parameter set.

Definition 2.1. A two-parameters family of mappings $\{U(t,\tau) : X \to X,\ t \geq \tau \in \mathbb{R}\}$ is said to be a continuous process on X, if (i) $U(t,s)U(s,\tau) = U(t,\tau)$, for all $t \geq s \geq \tau$; (ii) $U(\tau,\tau) = I$ (unit operator), for all $\tau \in \mathbb{R}$; (iii) for all $t \geq \tau \in \mathbb{R}$, $U(t,\tau)$ is continuous on X. $\{U^\sigma(t,\tau)\}_{t\geq\tau,\sigma\in\Sigma}$ is called a family of continuous processes in X with parameter $\sigma \in \Sigma$, if for each $\sigma \in \Sigma$, $\{U^\sigma(t,\tau)\}_{t\geq\tau}$ is a continuous process in X, where Σ is called a symbol space and $\sigma \in \Sigma$ is a symbol.

Definition 2.2. A subset D_0 of X is said to be uniformly (with respect to (w.r.t.) $\sigma \in \Sigma$) absorbing for a family of processes $\{U^\sigma(t,\tau)\}_{t\geq\tau,\sigma\in\Sigma}$, if for any $\tau \in \mathbb{R}$ and each bounded set $B \in \mathcal{B}(X)$, there exists $t_{\tau,B} \geq 0$ such that $\bigcup_{\sigma\in\Sigma} U^\sigma(t,\tau)B \subseteq D_0$ for all $t \geq \tau + t_{\tau,B}$.

Definition 2.3. A closed set $\mathcal{A}_\Sigma \subseteq X$ is said to be a uniform (w.r.t. $\sigma \in \Sigma$) attractor for a family of continuous processes $\{U^\sigma(t,\tau)\}_{t\geq\tau,\sigma\in\Sigma}$, if

(i) $\lim_{t\to+\infty} \sup_{\sigma\in\Sigma} \mathrm{d}_h(U^\sigma(t,\tau)B, \mathcal{A}_\Sigma) = 0$ for any $\tau \in \mathbb{R}$ and any bounded set $B \in \mathcal{B}(X)$, where "$\mathrm{d}_h(\cdot,\cdot)$" is the Hausdorff semidistance between two subsets of X;

(ii) $\mathcal{A}_\Sigma$ is the minimal set (for inclusion relation) among those sets satisfying (i).

Definition 2.4. A family of processes $\{U^\sigma(t,\tau)\}_{t\geq\tau,\sigma\in\Sigma}$ is said to be asymptotically compact in X if for any $\tau \in \mathbb{R}$, $B \in \mathcal{B}(X)$, each sequence $\{t_n\} \subset [0,+\infty)$ with $t_n \to +\infty$ as $n \to \infty$, each sequence $\{u_n\} \subset B$ and each sequence $\{\sigma_n\} \subset \Sigma$, the sequence $\{U^{\sigma_n}(t_n+\tau,\tau)u_n\}$ has a convergent subsequence in X.

We make the following assumptions on the quantities in (1.1)- (1.2):

(A1) A is a linear operator on ℓ^2 with decomposition $A = \sum_{j=1}^k A_j$, $A_j = B_j^* B_j = B_j B_j^*$, where the operators B_j are defined by
$$(B_j u)_m = \sum_{l=-m_0}^{l=m_0} d_{j,l} u_{m_{jl}}, \quad |d_{j,l}| \leq a_0 \text{ (constant)}, \quad u = (u_m)_{m\in\mathbb{Z}^k} \in \ell^2,$$
$m_{jl} = (m_1,\dots,m_{j-1}, m_j + l, m_{j+1},\dots,m_k) \in \mathbb{Z}^k$, $(B_j u, v) = (u, B_j^* v)$ for u, $v \in \ell^2$, $j = 1,\dots,k$.

(A2) $\forall m \in \mathbb{Z}^k$, $0 < \lambda_0 \leq \lambda_m \leq \lambda^0 < \infty$, where λ_0, λ^0 are two positive constants.

(A3) $g_0 = (g_{0,m})_{m\in\mathbb{Z}^k}$, $g_0' = (g_{0,m}')_{m\in\mathbb{Z}^k} : \mathbb{R} \to \ell^2$ are both almost periodic functions in the Bohr sense.

(A4) For all $m \in \mathbb{Z}^k$, $f_m(\cdot) \in C^1(\mathbb{R}^{(2q+1)k}, \mathbb{R})$, $f_m(u_j = 0 | j \in I_{mq}) = 0$ and there exist $\rho \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b = (b_m)_{m\in\mathbb{Z}^k} \in \ell^2$, such that
$$\sup_{m\in\mathbb{Z}^k} \max_{u_j\in[-r,r], j\in I_{mq}} |f'_{m,j}(u_j | j \in I_{mq})| \leq \rho(r),$$
$$f_m(u_j | j \in I_{mq}) u_m \geq G_m(u_j | j \in I_{mq}) \geq -b_m^2,$$
where $G_m(u_j|j \in I_{mq}) = \int_0^{u_m} f_m(r, u_j | j \in I_{mq} \setminus \{m\}) dr$, $f'_{m,j}(u_j|j \in I_{mq}) = \frac{\partial f_m}{\partial u_j}(u_j|j \in I_{mq})$ and $f_m(r, u_j|j \in I_{mq} \setminus \{m\})$ is the function $f_m(u_j|j \in I_{mq})$ in which u_m is replaced by r.

(A5) For all $m \in \mathbb{Z}^k$, $h_m \in C^1(\mathbb{R},\mathbb{R})$, $h_m(0) = 0$ and $h_m(s)$ is Lipschitz continuous in s:
$$|h_m(s_1) - h_m(s_2)| \leq L_h |s_1 - s_2|, \quad L_h \geq 0, \quad \forall s_1,\ s_2 \in \mathbb{R},\ m \in \mathbb{Z}^k,$$
where
$$0 \leq L_h \leq \begin{cases} \frac{\lambda_0}{2} e^{-\frac{1}{2}\tilde{\lambda}\vartheta}, & \epsilon = 0, \\ \frac{1}{2}\sqrt{\frac{\lambda_0 \varepsilon_0}{\epsilon}} e^{-\frac{\varepsilon_0}{8\epsilon}\vartheta}, & \epsilon > 0, \end{cases}$$
$$\tilde{\lambda} = \begin{cases} \frac{\lambda_0}{2}, & \gamma = 0, \\ \min\{\frac{2}{\gamma}, \frac{\lambda_0}{2}\}, & \gamma > 0, \end{cases}$$
$$\varepsilon_0 = \begin{cases} \frac{\epsilon\lambda_0}{1+3\epsilon\lambda_0}, & \gamma = 0, \\ \min\{\frac{\epsilon}{\gamma}, \frac{\epsilon\lambda_0}{1+3\epsilon\lambda_0}\}, & \gamma > 0. \end{cases}$$

(A6) For all $\epsilon > 0$, there exists a constant $\delta_\epsilon \geq 0$ such that
$$\Big|\frac{\partial G_m}{\partial u_j}(u_j|j \in I_{mq})\Big| = |G'_{m,j}(u_j|j \in I_{mq})| \leq \delta_\epsilon |u_m|, \quad m \neq j,\ m \in \mathbb{Z}^k,$$
where
$$0 \leq \delta_\epsilon \leq \min\Big\{\frac{\varepsilon_0\lambda_0}{4\epsilon(2\lambda_0^2(2q)^{2k}+1)}, \quad \frac{1}{4(2q)^{2k}}\Big\}.$$

Now we define some spaces. For each $\epsilon > 0$ and $u,\ v \in \ell^2$, we define the inner products

$$\begin{aligned}(u,v)_{\delta\lambda} &= \delta\sum_{j=1}^{k}(B_ju,B_jv)+(\lambda u,v)\\ &= \delta\sum_{j=1}^{k}(B_ju,B_jv)+\sum_{m\in\mathbb{Z}^k}\lambda_m u_m v_m,\quad \delta = 1-\frac{\varepsilon_0}{\epsilon}\gamma\in[0,1]\end{aligned}$$

and

$$(u,v)_{\delta\lambda\epsilon} = \epsilon^{-1}(u,v)_{\delta\lambda} = \epsilon^{-1}\delta\sum_{j=1}^{k}(B_ju,B_jv)+\epsilon^{-1}\sum_{m\in\mathbb{Z}^k}\lambda_m u_m v_m.$$

By (A1) and (A2), the three norms $\|\cdot\|$, $\|\cdot\|_{\delta\lambda}$, $\|\cdot\|_{\delta\lambda\epsilon}$ are equivalent to each other. Let

$$\ell^2_{\delta\lambda\epsilon} = \left(\ell^2, (\cdot,\cdot)_{\delta\lambda\epsilon}\right),\quad E=\ell^2\times\ell^2,\quad H=\ell^2_{\delta\lambda\epsilon}\times\ell^2,$$

then E, H are Hilbert spaces with the norms:

$$\|(u,v)^{\mathrm{T}}\|_E^2 = \|u\|^2+\|v\|^2,\quad \forall (u,v)^{\mathrm{T}}\in E,$$
$$\|(u,v)^{\mathrm{T}}\|_H^2 = \|u\|^2_{\delta\lambda\epsilon}+\|v\|^2 = \epsilon^{-1}\delta\|Bu\|^2+\epsilon^{-1}\sum_{m\in\mathbb{Z}^k}\lambda_m u_m^2+\|v\|^2,\quad \forall (u,v)^{\mathrm{T}}\in H.$$

For the positive delay number $\vartheta > 0$, write the Banach spaces $\ell^2_\vartheta = C([-\vartheta,0],\ell^2)$, $E_\vartheta = C([-\vartheta,0],E)$ and $H_\vartheta = C([-\vartheta,0],H)$ with norms, respectively, as follows:

$$\|u(\cdot)\|^2_{\ell^2_\vartheta} = \sup_{-\vartheta\le\theta\le0}\|u(\theta)\|^2,\quad \forall u(\cdot)\in\ell^2_\vartheta,$$
$$\|(u(\cdot),v(\cdot))^{\mathrm{T}}\|^2_{E_\vartheta} = \|u(\cdot)\|^2_{\ell^2_\vartheta}+\|v(\cdot)\|^2_{\ell^2_\vartheta},\quad \forall (u(\cdot),v(\cdot))^{\mathrm{T}}\in E_\vartheta,$$
$$\|(u(\cdot),v(\cdot))^{\mathrm{T}}\|^2_{H_\vartheta} = \sup_{-\vartheta\le\theta\le0}\|u(\theta)\|^2_{\delta\lambda\epsilon}+\|v(\cdot)\|^2_{\ell^2_\vartheta},\ \forall (u(\cdot),v(\cdot))^{\mathrm{T}}\in H_\vartheta.$$

By (A3) and the Bochner-Amerio criteria, the sets $\{g_0(\cdot+r)\}_{r\in\mathbb{R}}$, $\{g_0'(\cdot+r)\}_{r\in\mathbb{R}}$ are both precompact in $C_b(\mathbb{R},\ell^2)$ [4]. Thus $\mathcal{H}(g_0)$ and $\mathcal{H}(g_0') = \overline{\{g_0'(\cdot+r): r\in\mathbb{R}\}}^{C_b(\mathbb{R},\ell^2)}$ are compact in $C_b(\mathbb{R},\ell^2)$. We set

$$T(r): g\to T(r)g = g(\cdot+r),\quad \forall g\in\mathcal{H}(g_0),\ r\in\mathbb{R},$$

then $\{T(r)\}_{r\in\mathbb{R}}$ is a translation group acting on $\mathcal{H}(g_0)$, $(r,g)\to T(r)g$ is continuous and $T(r)\mathcal{H}(g_0) = \mathcal{H}(g_0)$ for all $r\in\mathbb{R}$.

Finally, we present the equivalent vector forms of systems (1.1)-(1.2). By (A1) and the Lax-Milgram theorem, the operator $(I+\gamma A)^{-1}$ exists and it is linear bounded from ℓ^2 into ℓ^2: $\|(I+\gamma A)^{-1}\|_{\mathcal{L}(\ell^2,\ell^2)}\le 1$.

For $\theta\in[-\vartheta,0]$, $t\in\mathbb{R}$, we write $u=(u_m)_{m\in\mathbb{Z}}$, $\lambda u = (\lambda_m u_m)_{m\in\mathbb{Z}}$, $f(u) = (f_m(u_j|j\in I_{mq}))_{m\in\mathbb{Z}}$, $h(u(t-\vartheta)) = (h_m(u_m(t-\vartheta)))_{m\in\mathbb{Z}}$, $g(t) = (g_m(t))_{m\in\mathbb{Z}}$, $u_t(\theta) = u(t+\theta) = (u_m(t+\theta))_{m\in\mathbb{Z}}$, $\dot u_t(\theta) = \dot u(t+\theta) = (\dot u_m(t+\theta))_{m\in\mathbb{Z}}$. Then (1.2) can be written as the following family of first order delay lattice systems with initial condition:

$$\begin{aligned}&\dot u = F_0(u_t(\theta),t),\quad t\ge\tau,\ \theta\in[-\vartheta,0],\ g\in\mathcal{H}(g_0),\\ &u_\tau(\theta) = u(\tau+\theta) = u_{0,\tau}(\theta),\quad \tau\in\mathbb{R},\ \theta\in[-\vartheta,0],\end{aligned}\tag{2.1}$$

where

$$F_0(u_t(\theta),t) = (I+\gamma A)^{-1}[-Au(t)-\lambda u(t)-f(u(t))-h\left(u(t-\vartheta)\right)+g(t)],\tag{2.2}$$
$$(I+\gamma A)\dot u+Au+\lambda u+f(u)+h\left(u(t-\vartheta)\right) = g(t),\quad t\ge\tau,\ g\in\mathcal{H}(g_0),\tag{2.3}$$

and (1.1) can be written as the following family of second order delay lattice systems with initial conditions:

$$\begin{aligned}&\epsilon\ddot u+\dot u+\gamma A\dot u+Au+\lambda u+f(u)+h\left(u(t-\vartheta)\right) = g(t),\quad t\ge\tau,\ g\in\mathcal{H}(g_0),\\ &u_\tau(\theta) = u(\tau+\theta) = u_{0,\tau}(\theta),\quad \dot u_\tau(\theta) = \dot u(\tau+\theta) = u_{1,\tau}(\theta),\quad \tau\in\mathbb{R},\ \theta\in[-\vartheta,0].\end{aligned}\tag{2.4}$$

For a fixed $g \in \mathcal{H}(g_0)$ and $\epsilon > 0$, let $u_\epsilon(t)$ be the solution of (2.4), set

$$v_\epsilon = \dot{u}_\epsilon + \frac{\varepsilon_0}{\epsilon} u_\epsilon, \quad u_{\epsilon,t}(\theta) = u_\epsilon(t+\theta), \quad v_{\epsilon,t}(\theta) = v_\epsilon(t+\theta), \quad \theta \in [-\vartheta, 0],\ t \in \mathbb{R}, \tag{2.5}$$

where ε_0 is defined in (A5). Then problem (2.4) is equivalent to the following vector forms:

$$\begin{gathered} \dot{\psi}_\epsilon(t) + H_\epsilon \psi_\epsilon(t) = F_\epsilon(\psi_{\epsilon,t}(\theta), t), \quad t \ge \tau,\ \theta \in [-\vartheta, 0],\ g \in \mathcal{H}(g_0),\ \tau \in \mathbb{R}, \\ \psi_\epsilon(\tau)(\theta) = \psi_\epsilon(\tau+\theta) = (u_\epsilon(\tau+\theta), \quad v_\epsilon(\tau+\theta))^{\mathrm{T}} = (u_{\epsilon,\tau}(\theta), v_{\epsilon,\tau}(\theta))^{\mathrm{T}}, \end{gathered} \tag{2.6}$$

where $\psi_\epsilon(t) = (u_\epsilon(t), v_\epsilon(t))^{\mathrm{T}}$, $\psi_{\epsilon,t} = (u_{\epsilon,t}, v_{\epsilon,t})^{\mathrm{T}}$,

$$\begin{aligned} H_\epsilon \psi_\epsilon(t) &= H_\epsilon \psi_{\epsilon,t}(0) \\ &= \begin{pmatrix} \frac{\varepsilon_0}{\epsilon} u_\epsilon(t) - v_\epsilon(t) \\ \frac{1}{\epsilon}[\lambda u_\epsilon(t) + (1 - \frac{1}{\epsilon}\gamma\varepsilon_0) A u_\epsilon(t) - \frac{1}{\epsilon}\varepsilon_0(1-\varepsilon_0) u_\epsilon(t) + (1-\varepsilon_0) v_\epsilon(t) + \gamma A v_\epsilon(t)] \end{pmatrix}, \\ F_\epsilon(\psi_{\epsilon,t}(\theta), t) &= \begin{pmatrix} 0 \\ -\frac{1}{\epsilon} f(u_{\epsilon,t}(0)) - \frac{1}{\epsilon} h\left(u_{\epsilon,t}(-\vartheta)\right) + \frac{1}{\epsilon} g(t) \end{pmatrix}. \end{aligned}$$

3. Existence of uniform attractors

We first consider the existence of uniform attractors for the family of continuous processes defined by the solutions of (2.1) and (2.6) on the spaces ℓ^2_ϑ and H_ϑ, respectively. Then based on the transformation (2.5), we obtain the existence of a uniform attractor for the family of continuous processes of solutions $\varphi_{\epsilon,t}(\cdot) = (u_{\epsilon,t}(\cdot), \dot{u}_{\epsilon,t}(\cdot))^{\mathrm{T}}$ of (2.4) in E_ϑ.

Theorem 3.1. *For the initial value problem* (2.1), *if* (A1)–(A5) *hold, then for each $g \in H(g_0)$, $\tau \in \mathbb{R}$, and $u_\tau(\cdot) \in \ell^2_\vartheta$,* (2.1) *has a unique solution $u_t(\cdot) = u(t, \tau, u_\tau(\cdot)) \in \ell^2_\vartheta$ existing on $t \in [\tau, +\infty)$, $u_t(\cdot)$ is continuous in $u_\tau(\cdot)$ and $u(\cdot) = u(\cdot, \tau, u_\tau(\theta)) \in C([\tau-\vartheta, +\infty), \ell^2) \cap C^1([\tau, +\infty), \ell^2)$, $\theta \in [-\vartheta, 0]$. Moreover, the solution maps:*

$$U_0^g(t,\tau) : \ell^2_\vartheta \ni u_\tau(\cdot) \to u_t(\cdot) = u(t, \tau, u_\tau(\cdot)) \in \ell^2_\vartheta, \quad t \ge \tau,\ \tau \in \mathbb{R}, \tag{3.1}$$

generates a continuous process $\{U_0^g(t,\tau)\}_{t\ge\tau}$ on ℓ^2_ϑ and the family of continuous processes $\{U_0^g(t,\tau)\}_{t\ge\tau, g\in\mathcal{H}(g_0)}$ possesses a unique compact uniform attractor $\mathcal{A}_0^{\mathcal{H}(g_0)}$:

$$\mathcal{A}_0^{\mathcal{H}(g_0)} = \cup_{g\in\mathcal{H}(g_0)} \mathcal{A}^g_{0,t} = \cup_{g\in\mathcal{H}(g_0)} \mathcal{A}^g_{0,0} \subset \ell^2_\vartheta, \quad \forall t \in \mathbb{R}, \tag{3.2}$$

where

$$\begin{aligned} \mathcal{A}^g_{0,t} = \Big\{ & u_t : u_t(\cdot) = u(t+\cdot) : [-\vartheta, 0] \to \ell^2 \text{ is the global solution of } (2.1), \\ & \|u_t(\cdot)\|_{\ell^2_\vartheta} \le r_0, \quad \forall t \in \mathbb{R} \Big\} \end{aligned} \tag{3.3}$$

with the invariance in the sense that $U_0^g(t,\tau)\mathcal{A}^g_{0,\tau} = \mathcal{A}^g_{0,t}$ for $t \ge \tau$, $\tau \in \mathbb{R}$ and $r_0 = 2\sqrt{\frac{\|g_0\|^2}{\tilde{\lambda}\lambda_0} + \frac{\|b\|^2}{\tilde{\lambda}}}$.

Proof. (i) By (A3), for any $g \in \mathcal{H}(g_0)$, g is almost periodic on $\mathbb{R}$ and $\mathcal{H}(g) = \mathcal{H}(g_0)$. By (A1)–(A5) and the linear boundedness of $(I + \gamma A)^{-1}$ from ℓ^2 into ℓ^2, it follows that for τ, $t \in \mathbb{R}$, $T > 0$, $\theta \in [-\vartheta, 0]$, $u_t(\theta) = u(t+\theta)$, $F_0(u_t(\cdot), t)$ is continuous from $\ell^2_\vartheta \times [\tau, \tau+T]$ into ℓ^2 and locally Lipschitz in $u_t(\cdot)$. Therefore, for any $u_\tau(\cdot) \in \ell^2_\vartheta$, (2.1) has a unique (locally) solution $u(\cdot) = u(\cdot, \tau, u_\tau(\theta)) \in C([\tau-\vartheta, T_{0,\max}), \ell^2) \cap C^1([\tau, T_{0,\max}), \ell^2)$, $\theta \in [-\vartheta, 0]$, $T_{0,\max} > \tau$ and $u_t(\cdot) = u(t, \tau, u_\tau(\cdot))$ is continuous in $u_\tau(\cdot)$ for $t \in [\tau, T_{0,\max})$ [9, 21]. $u(t)$ satisfies the initial value and integral equation:

$$\begin{gathered} u(\tau)(\theta) = u(\tau, \tau, u_\tau(\theta)) = u_\tau(\theta), \\ u(t) = u_\tau(0) + \int_\tau^t F_0(u_s(\theta), s) ds, \end{gathered}$$

for $\theta \in [-\vartheta, 0]$ and $t \in [\tau, T_{0,\max})$. Taking the inner product of $u(t) = u(t, \tau, u_\tau(\cdot)) \in \ell^2$ $(t \ge \tau)$ with (2.3) and by (A1)–(A5), we have

$$\frac{d}{dt}(\|u(t)\|^2 + \gamma \sum_{j=1}^k \|B_j u(t)\|^2) + \tilde{\lambda}(\|u(t)\|^2 + \gamma \sum_{j=1}^k \|B_j u(t)\|^2) + \frac{\lambda_0}{2}\|u(t)\|^2 \tag{3.4}$$

$$\le \frac{2L_h^2}{\lambda_0}\|u(t-\vartheta)\|^2 + \frac{2\|g_0\|^2}{\lambda_0} + 2\|b\|^2,\ t \ge \tau, \tag{3.5}$$

where $\|b\|^2 = \sum_{m\in\mathbb{Z}} b_m^2 < \infty$, and

$$\|g\|^2 = \sup_{t\in\mathbb{R}} \sum_{m\in\mathbb{Z}^k} g_m^2(t) \le \|g_0\|^2 = \sup_{t\in\mathbb{R}} \sum_{m\in\mathbb{Z}^k} g_{0,m}^2(t) < \infty.$$

Applying Gronwall's inequality on $[\tau, t]$ $(t \ge \tau)$ to (3.5), we obtain

$$\begin{aligned} \|u(t)\|^2 &\le (\|u(\tau)\|^2 + \gamma\sum_{j=1}^k \|B_j u(\tau)\|^2) e^{-\tilde\lambda(t-\tau)} \\ &\quad + \frac{2L_h^2}{\lambda_0\tilde\lambda} e^{\tilde\lambda\vartheta}\|u_\tau(\cdot)\|_{\ell^2_\vartheta}^2 e^{-\tilde\lambda(t-\tau)} + \frac{r_0^2}{2}, \quad t \ge \tau. \end{aligned} \tag{3.6}$$

Thus, for $\theta \in [-\vartheta, 0]$, set $t+\theta$ instead of t, it holds from (3.6) that for $t \ge \tau$,

$$\begin{aligned} \|u(t+\theta)\|^2 &\le \Big(\|u(\tau)\|^2 + \gamma\sum_{j=1}^k \|B_j u(\tau)\|^2\Big) e^{\tilde\lambda\vartheta} e^{-\tilde\lambda(t-\tau)} \\ &\quad + \frac{2L_h^2}{\lambda_0\tilde\lambda} e^{\tilde\lambda\vartheta}\|u_\tau(\cdot)\|_{\ell^2_\vartheta}^2 e^{\tilde\lambda\vartheta} e^{-\tilde\lambda(t-\tau)} + \frac{r_0^2}{2}, \quad t+\theta \ge \tau, \\ \|u(t+\theta)\|^2 &\le \|u_\tau(\cdot)\|_{\ell^2_\vartheta}^2, \quad t+\theta \le \tau. \end{aligned} \tag{3.7}$$

So $T_{0,\max} = +\infty$, and the solutions maps (3.1) generate a continuous process $\{U_0^g(t,\tau)\}_{t\ge\tau}$ on ℓ^2_ϑ. By the definition of $\{T(r)\}_{r\in\mathbb{R}}$, we have

$$U_0^g(t+r, \tau+r) = U_0^{T(r)g}(t,\tau), \quad \forall g \in \mathcal{H}(g_0),\ t \ge \tau,\ \tau,\ r \in \mathbb{R}.$$

By (3.7),

$$\begin{aligned} \|u_t(\cdot)\|_{\ell^2_\vartheta}^2 &\le \Big(\|u_\tau(\cdot)\|_{\ell^2_\vartheta}^2 + \gamma\sum_{j=1}^k \|B_j u(\tau)\|^2 + \frac{2L_h^2}{\lambda_0\tilde\lambda} e^{\tilde\lambda\vartheta}\|u_\tau(\cdot)\|_{\ell^2_\vartheta}^2\Big) e^{\tilde\lambda\vartheta} e^{-\tilde\lambda(t-\tau)} + \frac{r_0^2}{2} \\ &\doteq \tilde b^2(t,\tau,\|u_\tau(\cdot)\|_{\ell^2_\vartheta}^2), \quad t \ge \tau, \end{aligned} \tag{3.8}$$

where $\tilde b^2(t,\tau,u_\tau(\cdot))$ is continuous in t. Let $\tau \in \mathbb{R}$, $g^{(1)}$, $g^{(2)} \in \mathcal{H}(g_0)$, $u_\tau^{(1)}(\cdot)$, $u_\tau^{(2)}(\cdot) \in \ell^2_\vartheta$, $t \ge \tau$, $u^{(j)}(t) = u(t,\tau,g^{(j)},u_\tau^{(j)}(\cdot))$, $j = 1,2$,

$$\kappa(t,\tau,g^{(1)},g^{(2)},u_\tau^{(1)}(\cdot),u_\tau^{(2)}(\cdot)) = u(t,\tau,g^{(1)},u_\tau^{(1)}(\cdot)) - u(t,\tau,g^{(2)},u_\tau^{(2)}(\cdot)),$$

then

$$\begin{aligned} &(I+\gamma A)\dot\kappa + A\kappa + \lambda\kappa + f(u^{(1)}(t)) - f(u^{(2)}(t)) \\ &+ h(u^{(1)}(t-\vartheta)) - h(u^{(2)}(t-\vartheta)) \\ &= g^{(1)}(t) - g^{(2)}(t), \quad t \ge \tau, \\ &\kappa_\tau(\theta) = u_\tau^{(1)}(\theta) - u_\tau^{(2)}(\theta), \quad \tau \in \mathbb{R},\ \theta \in [-\vartheta, 0]. \end{aligned} \tag{3.9}$$

By (3.8),

$$\|f(u^{(1)}(t)) - f(u^{(2)}(t))\|^2 \le (2q+1)^{2k}\rho^2(\|\tilde b(t,\tau,\|u_\tau^{(1)}(\cdot)\|_{\ell^2_\vartheta}^2)\| + \|\tilde b(t,\tau,\|u_\tau^{(2)}(\cdot)\|_{\ell^2_\vartheta}^2)\|)\|\kappa(t)\|^2.$$

Taking the inner product of $\kappa(t) \in \ell^2$ $(t \ge \tau)$ with (3.9) and applying Gronwall's inequality, we have

$$\begin{aligned} \|\kappa(t)\|^2 &\le \Big(\|\kappa(\tau)\|^2 + \gamma\sum_{j=1}^k \|B_j\kappa(\tau)\|^2\Big) e^{-\int_\tau^t \widetilde C_0(l,\tau)dl} \\ &\quad + \frac{2L_h^2}{\lambda_0} e^{\tilde\lambda\vartheta}\|\kappa_\tau(\cdot)\|_{\ell^2_\vartheta}^2 \int_{\tau-\vartheta}^{\tau} e^{-\int_r^t \widetilde C_0(l,\tau)dl} dr \\ &\quad + \frac{4}{\lambda_0}\|g^{(1)} - g^{(2)}\|^2 \int_\tau^t e^{-\int_r^t \widetilde C_0(l,\tau)dl} dr, \quad t \ge \tau, \end{aligned} \tag{3.10}$$

where

$$\widetilde{C}_0(t,\tau) = \tilde{\lambda} - \frac{4}{\lambda_0}(2q+1)^{2k}\rho^2(\|\tilde{b}(t,\tau,\|u_\tau^{(1)}(\cdot)\|^2_{\ell^2_\vartheta})\| + \|\tilde{b}(t,\tau,\|u_\tau^{(2)}(\cdot)\|^2_{\ell^2_\vartheta})\|).$$

Setting $t+\theta$ instead of t, where $\theta \in [-\vartheta, 0]$, it holds from (3.10) that

$$\|\kappa(t+\theta)\|^2 \le \widetilde{C}_1(t,\tau)\left(\|\kappa_\tau(\cdot)\|^2_{\ell^2_\vartheta} + \|g^{(1)} - g^{(2)}\|^2\right), \quad t \ge \tau, \tag{3.11}$$

where

$$\widetilde{C}_1(t,\tau) = \max\big\{\widetilde{C}_{10}(t,\tau), \frac{4}{\lambda_0}e^{\tilde{\lambda}\vartheta}\int_\tau^t e^{-\int_r^t \widetilde{C}_0(l,\tau)dl}dr\big\},$$

$$\widetilde{C}_{10}(t,\tau) = (1+\gamma k(2m_0+1)^2a_0^2)e^{\tilde{\lambda}\vartheta}e^{-\int_\tau^t \widetilde{C}_0(l,\tau)dl} + \frac{2L_h^2}{\lambda_0}e^{2\tilde{\lambda}\vartheta}\int_{\tau-\vartheta}^{\tau} e^{-\int_r^t \widetilde{C}_0(l,\tau)dl}dr,$$

and (3.11) implies that $\{U_0^g(t,\tau)\}_{t\ge\tau, g\in\mathcal{H}(g_0)}$ is a family of continuous processes from $\ell^2_\vartheta \times \mathcal{H}(g_0)$ into ℓ^2_ϑ.

(ii) Let $B_0 = \{u(\cdot) \in \ell^2_\vartheta : \|u(\cdot)\|_{\ell^2_\vartheta} \le r_0\} \subset \ell^2_\vartheta$ (independent of $(\tau, g) \in \mathbb{R}\times\mathcal{H}(g_0)$), then by (3.8) , B_0 is a uniformly bounded closed absorbing ball of $\{U_0^g(t,\tau)\}_{t\ge\tau, g\in\mathcal{H}(g_0)}$ and there exists $T_{B_0} \ge 0$ such that for any $g \in H(g_0)$, $\tau \in \mathbb{R}$, $t - \tau \ge T_{B_0}$, $U_0^g(t,\tau)B_0 \subseteq B_0$. Additionally, by (2.2) and (3.8), it holds that there exist positive constants b_0, b_1 (independent of (g,t,τ)) such that for any $t \ge \tau$, $\tau \in \mathbb{R}$, $g \in \mathcal{H}(g_0)$,

$$\begin{aligned} \|U_0^g(t,\tau)B_0\|_{\ell^2_\vartheta} &= \sup_{u_t \in U_0^g(t,\tau)B_0} \|u_t(\cdot)\|_{\ell^2_\vartheta} \\ &= \sup_{u_t\in U_0(t,\tau)B_0}\sup_{\theta\in[-\vartheta,0]} \|u(t+\theta)\| \le b_0, \\ \sup_{u_t\in U_0(t,\tau)B_0}\sup_{\theta\in[-\vartheta,0]} &\|F_0(u_t(\theta),t)\| \le b_1. \end{aligned} \tag{3.12}$$

(iii) Choose a smooth increasing function $\varrho \in C^1(\mathbb{R}_+, [0,1])$ that satisfies

$$\begin{gathered} \varrho(s) = 0,\ 0 \le s \le 1, \quad 0 \le \varrho(s) \le 1,\ 1 \le s \le 2, \\ \varrho(s) = 1,\ s \ge 2, \quad \varrho'(s)| \le C_0,\ s \in \mathbb{R}_+, C_0 > 0. \end{gathered}$$

Let $g \in \mathcal{H}(g_0)$, $\tau \in \mathbb{R}$, $u_\tau(\cdot) \in B_0$,

$$u(t) = U_0^g(t,\tau)u_\tau(\cdot) = u(t,\tau,u_\tau(\cdot)) = (u_m(t,\tau,u_\tau(\cdot)))_{m\in\mathbb{Z}^k} \in \ell^2, \quad t \ge \tau.$$

By(3.12), $\|u(t)\| \le b_0$, $\|u(t-\vartheta)\| \le b_0$, $\|\dot{u}(t)\| \le b_1$ for $t \ge \tau$. Let K be a positive integer, $x_m = \varrho(\frac{||m||}{K})u_m$, $x = (x_m)_{m\in\mathbb{Z}^k}$. Taking the inner product of (2.3) with x in ℓ^2, we have

$$\begin{aligned} &\frac{d}{dt}\sum_{m\in\mathbb{Z}^k}\varrho(\frac{\|m\|}{K})(u_m^2(t) + \gamma\sum_{j=1}^k(B_ju(t))_m^2) \\ &+ \tilde{\lambda}\sum_{m\in\mathbb{Z}^k}\varrho(\frac{||m||}{K})(u_m^2(t) + \gamma\sum_{j=1}^k(B_ju(t))_m^2) + \frac{\lambda_0}{2}\sum_{m\in\mathbb{Z}^k}\varrho(\frac{||m||}{K})u_m^2(t) \\ &\le \frac{2L_h^2}{\lambda_0}\sum_{m\in\mathbb{Z}^k}\varrho(\frac{||m||}{K})u_m^2(t-\vartheta) + \frac{c_1}{K} + \frac{2}{\lambda_0}\sum_{\|m\|\ge 2K}g_m^2(t) + 2\sum_{\|m\|\ge 2K}b_m^2, \quad t \ge \tau, \end{aligned} \tag{3.13}$$

where $c_1 = \frac{kC_0m_0a_0^2(2m_0+1)^2}{2}(b_1^2 + 3b_0^2)$. Applying Gronwall's inequality on $[\tau, t]$ to (3.13), we have

$$\begin{aligned} \sum_{m\in\mathbb{Z}^k}\varrho(\frac{\|m\|}{K})u_m^2(t) &\le \Big(r_0^2 + \gamma k a_0^2(2m_0+1)^2r_0^2 + \frac{1}{\tilde{\lambda}}\frac{2L_h^2}{\lambda_0}e^{\tilde{\lambda}\vartheta}r_0^2\Big)e^{-\tilde{\lambda}(t-\tau)} \\ &\quad + \frac{1}{\tilde{\lambda}}\Big(\frac{2}{\lambda_0}\sup_{r\in\mathbb{R}}\sum_{\|m\|\ge K}g_m^2(r) + \frac{c_1}{K} + 2\sum_{\|m\|\ge K}b_m^2\Big), \quad t \ge \tau. \end{aligned} \tag{3.14}$$

Hance, set $t+\theta$ instead of t in (3.14), $\theta \in [-\vartheta, 0]$, we have

$$\sum_{m\in\mathbb{Z}^k} \varrho(\frac{\|m\|}{K}) u_m^2(t+\theta) \le r_1 e^{\tilde{\lambda}\vartheta} e^{-\tilde{\lambda}(t-\tau)} + \frac{1}{\tilde{\lambda}}\Big(\frac{2}{\lambda_0}\sup_{r\in\mathbb{R}}\sum_{\|m\|\ge K} g_m^2(r) + \frac{c_1}{K} + 2\sum_{\|m\|\ge K} b_m^2\Big),$$

where $r_1 = r_0^2 + \gamma k a_0^2(2m_0+1)^2 r_0^2 + \frac{1}{\tilde{\lambda}}\frac{2L_h^2}{\lambda_0} e^{\tilde{\lambda}\vartheta} r_0^2$. By (A3) and (A4), the compactness of $\mathcal{H}(g_0)$ in $C_b(\mathbb{R}, \ell^2)$, it follows that for any $\eta > 0$, there exists $K_0(\eta, g_0, r_0) \in \mathbb{N}$ (independent of $g \in \mathcal{H}(g_0)$) and $T_0(\eta, r_0) \ge T_{B_0} > 0$ such that

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^k} \varrho(\frac{\|m\|}{K}) u_m^2(t+\theta) \\
&\le r_1 e^{\tilde{\lambda}\vartheta} e^{-\tilde{\lambda}(t-\tau)} + \sup_{t\in\mathbb{R}}\sup_{g\in\mathcal{H}(g_0)} \frac{2}{\tilde{\lambda}\lambda_0}\sum_{\|m\|\ge K} g_m^2(t) + \frac{c_1}{\tilde{\lambda}K} + \frac{2}{\tilde{\lambda}}\sum_{\|m\|\ge K} b_m^2 \\
&\le \frac{\eta^2}{4}, \quad \forall K \ge K_0(\eta, g_0, r_0), \quad t \ge \tau + T_0(\eta, r_0), \quad \tau \in \mathbb{R}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sup_{\theta\in[-\vartheta,0]} \sum_{\|m\|>2K_0(\eta,g_0,r_0)} |(U_0^g(t,\tau)u_\tau)_m(\theta)|^2 &= \sup_{\theta\in[-\vartheta,0]} \sum_{\|m\|>2K_0(\eta,g_0,r_0)} u_m^2(t+\theta) \\
&\le \frac{\eta^2}{4}, \ \forall u_\tau \in B_0, \quad t \ge \tau + T_0(\eta, r_0).
\end{aligned}$$

(iv) For each fixed $\tau \in \mathbb{R}$, any sequence $\{t_n\}_{n=1}^{+\infty} \subset [\vartheta, +\infty)$ with $t_n \to +\infty$ as $n \to \infty$, any sequence $\{u_n\}_{n=1}^{+\infty} \subset B_0$ and any sequence $\{g_n\}_{n=1}^{+\infty} \subset \mathcal{H}(g_0)$, we use Arezla-Ascoli theorem to prove that the sequence $\{u_{t_n+\tau}^{(g_n)} = U_0^{g_n}(t_n+\tau, \tau)u_n\}_{n=1}^{+\infty}$ has a convergent subsequence in ℓ_ϑ^2. By (3.12), it follows that $\{u_{t_n+\tau}^{(g_n)}\}_{n=1}^{+\infty}$ is uniformly bounded in ℓ_ϑ^2:

$$\sup_{1\le n<+\infty} \|u_{t_n+\tau}^{(g_n)}\|_{\ell_\vartheta^2} = \sup_{1\le n<+\infty}\sup_{-\vartheta\le\theta\le 0} \|u^{(g_n)}(t_n+\tau+\theta)\| \le b_0. \tag{3.15}$$

Taking θ_1, $\theta_2 \in [-\vartheta, 0]$ with $\theta_1 \le \theta_2$, where $t_n + \theta_1 \ge 0$, by (3.12), we have

$$\|u_{t_n+\tau}^{(g_n)}(\theta_1) - u_{t_n+\tau}^{(g_n)}(\theta_2)\| = \|\int_{t_n+\tau+\theta_1}^{t_n+\tau+\theta_2} F_0(u_{t_n+\tau}^{(g_n)}(\theta), g_n(s))ds\| \le b_1|\theta_2 - \theta_1|, \ \forall n,$$

which implies the equicontinuity of $\{u_{t_n+\tau}^{(g_n)}\}_{n=1}^{+\infty}$ in ℓ_ϑ^2. For any $\eta > 0$, by (iii) and $t_n \to +\infty$ as $n \to \infty$, there exists $K_{0,\eta} \in \mathbb{N}$ such that for $n \ge K_{0,\eta}$, it follows that $t_n \ge T_0(\eta, B_0)$ and

$$\begin{aligned}
&\sup_{\theta\in[-\vartheta,0]} \sum_{\|m\|>2K_0(\eta,g_0,r_0)} |(U_0^{g_n}(t_n+\tau,\tau)u_n)_m(\theta)|^2 \\
&= \sup_{\theta\in[-\vartheta,0]} \sum_{\|m\|>2K_0(\eta,g_0,r_0)} |u_{t_n+\tau,m}^{(g_n)}(\theta)|^2 \le \frac{\eta^2}{4}.
\end{aligned}$$

By (3.15), the set

$$\Gamma_{0,t_n+\tau}^{(g_n)}(\theta) = \big\{\hat{u}_{t_n+\tau,m}^{(g_n)}(\theta) = (u_{t_n+\tau,m}^{(g_n)}(\theta))_{\|m\|\le 2K_0(\eta,g_0,r_0)} \in \mathbb{R}^{(4K_0(\eta,g_0,r_0)+1)k}\big\}$$

is precompact in $\mathbb{R}^{(4K_0(\eta,g_0,r_0)+1)k}$ and $\Gamma_{0,t_n+\tau}^{(g_n)}(\theta)$ can be covered by finite closed balls with radius $\frac{\eta}{2}$ centered at the points in $\Gamma_{0,t_n+\tau}^{(g_n)}(\theta) \subset \mathbb{R}^{(4K_0(\eta,g_0,r_0)+1)k}$. It follows that for any $\eta > 0$, $\{u_{t_n+\tau}^{(g_n)}\}_{n=1}^{+\infty}$ is precompact in ℓ^2. So $\{u_{t_n+\tau}^{(g_n)}\}_{n=1}^{+\infty}$ has a convergent subsequence in ℓ_ϑ^2. Since B_0 is absorbing in ℓ_ϑ^2, $\{U_0^g(t,\tau)\}_{t\ge\tau, g\in\mathcal{H}(g_0)}$ is uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) asymptotically compact in ℓ_ϑ^2.

(v) According to [4, 16] and (i)–(iv), $\{U_0^g(t,\tau)\}_{t\ge\tau, g\in\mathcal{H}(g_0)}$ possesses a unique compact uniform attractor $A_0^{\mathcal{H}(g_0)}$ satisfying (3.2)–(3.3). The proof is complete. □

Remark 3.2. In view of [3, 4, 27], for any $g \in H(g_0)$, $t \in \mathbb{R}$, $\{\mathcal{A}^g_{0,t}\}_{t\in\mathbb{R}}$ is the pullback attractor of $\{U^g_0(t,\tau)\}_{t\geq\tau}$ and

$$\begin{aligned}\mathcal{A}^g_{0,t} &= \{u_t | \{u_t(\cdot), t \in \mathbb{R}\} \text{ is a complete bounded trajectory of } \{U^g_0(t,\tau)\}_{t\geq\tau}\}\\ &= \cap_{r\geq 0}\overline{\cup_{s\geq r} U^g_0(t,t-s)B_0} \subset B_0 \subset \ell^2_\vartheta.\end{aligned}$$

that is, for all $g \in H(g_0)$, $t \in \mathbb{R}$, $\mathcal{A}^g_{0,t}$ is compact in ℓ^2_ϑ; for all $t \geq \tau \in \mathbb{R}$, $U^g_0(t,\tau)A^g_{0,\tau} = A^g_{0,t}$; for all $B \subset \mathcal{B}(\ell^2_\vartheta)$, $\lim_{s\to+\infty} \mathrm{d}_h(U^g_0(t,t-s)B, A^g_{0,t}) = 0$; moreover, if $u_t \in \mathcal{A}^g_{0,t}$, then there exists $u_\tau \in \mathcal{A}^g_{0,\tau}$ such that $U^g_0(t,\tau)u_\tau = u_t$ for $t \geq \tau \in \mathbb{R}$ and $\|u_t(\cdot)\|_{\ell^2_\vartheta} \leq C_u$ (constant) for all $t \in \mathbb{R}$.

We consider the system (2.6), for $\epsilon > 0$ and δ_ϵ in (A6), and can see that

$$\beta = 1 - 2\delta_\epsilon(2q)^{2k} \geq \frac{1}{2}, \quad \delta_1 = \frac{\varepsilon_0}{2} - \frac{\delta_\epsilon\epsilon(2\lambda_0^2(2q)^{2k}+1)}{\lambda_0} \in [\frac{\varepsilon_0}{4}, \frac{\varepsilon_0}{2}], \quad \mu = \frac{\delta_1}{\epsilon}.$$

Theorem 3.3. *For the initial value problem* (2.6) *and* $\epsilon > 0$, *if* (A1)–(A6) *hold, then for any* $g \in H(g_0)$, $\tau \in \mathbb{R}$ *and* $\psi_{\epsilon,\tau}(\cdot) = (u_{\epsilon,\tau}(\cdot), v_{\epsilon,\tau}(\cdot))^{\mathrm{T}} \in H_\vartheta$, (2.6) *has a unique solution*

$$\psi_{\epsilon,t}(\cdot) = \psi_\epsilon(t,\tau,\psi_{\epsilon,\tau}(\cdot)) = (u_\epsilon(t,\tau,\psi_{\epsilon,\tau}(\cdot)), v_\epsilon(t,\tau,\psi_{\epsilon,\tau}(\cdot)))^{\mathrm{T}} \in H, \ t \geq \tau,$$

$\psi_{\epsilon,t}(\cdot)$ *is continuous in* $\psi_{\epsilon,\tau}(\cdot)$ *and*

$$\psi_\epsilon(\cdot) = \psi_\epsilon(\cdot,\tau,\psi_{\epsilon,\tau}(\theta)) \in C([\tau-\vartheta,+\infty),H) \cap C^1([\tau,+\infty),H), \quad \theta \in [-\vartheta,0].$$

The solutions maps $U^g_\epsilon(t,\tau) : H_\vartheta \to H_\vartheta$, $\psi_{\epsilon,\tau}(\cdot) \to \psi_{\epsilon,t}(\cdot) = \psi_\epsilon(t,\tau,\psi_{\epsilon,\tau}(\cdot))$, $t \geq \tau$, *generate a continuous process* $\{U^g_\epsilon(t,\tau)\}_{t\geq\tau}$ *on* H_ϑ, *and* $\{U^g_\epsilon(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ *possesses a unique compact uniform attractor* $\mathcal{K}^{\mathcal{H}(g_0)}_\epsilon \subset H_\vartheta$ *defined by*

$$\mathcal{K}^{\mathcal{H}(g_0)}_\epsilon = \cup_{g\in\mathcal{H}(g_0)}\mathcal{K}^g_{\epsilon,t} = \cup_{g\in\mathcal{H}(g_0)}\mathcal{K}^g_{\epsilon,0} \subset H_\vartheta, \quad \forall t \in \mathbb{R},$$

where

$$\begin{aligned}\mathcal{K}^g_{\epsilon,t} = \Big\{&\psi_{\epsilon,t}(\cdot) = \psi_\epsilon(t+\cdot) : [-\vartheta,0] \to H \text{ is the global solution of } (2.6),\\ &\|\psi_{\epsilon,t}(\cdot)\|_{H_\vartheta} \leq r_\epsilon t \in \mathbb{R}\Big\}.\end{aligned}$$

with $U^g_\epsilon(t,\tau)\mathcal{K}^g_{\epsilon,\tau} = \mathcal{K}^g_{\epsilon,t}$ *for* $t \geq \tau$, $g \in H(g_0)$, *and* $r_\epsilon = 2\sqrt{\frac{1}{\mu\epsilon\beta}\|g_0\|^2 + \frac{\lambda_0}{\mu\epsilon}\|b\|^2}$.

Proof. (i) By (A1)–(A6), it holds that for any $\tau \in \mathbb{R}$, $g \in \mathcal{H}(g_0)$ and any $\psi_{\epsilon,\tau}(\cdot) \in H_\vartheta$, the system (2.6) has a unique local solution $\psi_{\epsilon,t}(\cdot) = \psi_\epsilon(t,\tau,\psi_{\epsilon,\tau}(\cdot)) = (u_\epsilon(t,\tau,\psi_{\epsilon,\tau}(\cdot), v_\epsilon(t,\tau,\psi_{\epsilon,\tau}(\cdot))^{\mathrm{T}}$ for $t \in [\tau, T_{1,\max})$, $\psi_{\epsilon,t}(\cdot)$ is continuous in $\psi_{\epsilon,\tau}(\cdot)$ and $\psi_\epsilon(\cdot) = \psi_\epsilon(\cdot,\tau,\psi_{\epsilon,\tau}(\theta)) \in C([\tau-\vartheta, T_{1,\max}), H) \cap C^1([\tau, T_{1,\max}), H)$, $\theta \in [-\vartheta, 0]$, where $\psi_\epsilon(t)$ satisfies the following initial value and integral equation

$$\begin{aligned}&\psi_\epsilon(\tau)(\theta) = \psi_{\epsilon,\tau}(\theta), \quad \theta \in [-\vartheta,0],\\ &\psi_\epsilon(t) = \psi_{\epsilon,\tau}(0) + \int_\tau^t (F_\epsilon(\psi_{\epsilon,s}(\theta), g(s)) - H_\epsilon\psi_{\epsilon,s}(0))ds, \quad t \in [\tau, T_{1,\max}).\end{aligned} \tag{3.16}$$

Taking the inner product of $\psi_\epsilon(t)$ with (2.6) ($t \geq T_{1,\max}$) in H, by (A1)–A6), we have

$$\begin{aligned}&\frac{d}{dt}[\|\psi_\epsilon(t)\|^2_H + \frac{2}{\epsilon}\sum_{m\in\mathbb{Z}^k}(G_m(u_{\epsilon,j}(t,\tau)|j \in I_{mq}) + b_m^2)] + \frac{\varepsilon_0\lambda_0}{2\epsilon^2}\|u_\epsilon(t)\|^2\\ &+ \mu[\|\psi_\epsilon(t,\tau)\|^2_H + \frac{2}{\epsilon}\sum_{m\in\mathbb{Z}^k}(G_m(u_{\epsilon,j}(t,\tau)|j \in I_{mq}) + b_m^2)]\\ &\leq \frac{2L_h^2}{\epsilon\beta}\|u_\epsilon(t-\vartheta)\|^2 + \frac{2}{\epsilon\beta}\|g_0\|^2 + \frac{2\lambda_0}{\epsilon}\|b\|^2, \quad t \geq \tau.\end{aligned} \tag{3.17}$$

Applying Gronwall's inequality to (3.17) on $[\tau, t]$ ($t \geq \tau$), we obtain

$$\begin{aligned}\|\psi_\epsilon(t)\|^2_H \leq &\Big(\|\psi_\epsilon(\tau)\|^2_H + \frac{2(2q+1)^k\rho(\|u_\epsilon(\tau)\|)\|u_\epsilon(\tau)\|^2}{\epsilon} + \frac{2\|b\|^2}{\epsilon}\Big)e^{\mu\vartheta}e^{-\mu(t-\tau)}\\ &+ \frac{4L_h^2}{\varepsilon_0}e^{\mu\vartheta}\|u_{\epsilon,\tau}(\cdot)\|^2_{\ell^2_\vartheta}e^{-\mu(t-\tau)} + \frac{2}{\mu\epsilon\beta}\|g_0\|^2 + \frac{2\lambda_0}{\mu\epsilon}\|b\|^2, \quad t \geq \tau.\end{aligned} \tag{3.18}$$

Set $t+\theta$ instead of t in (3.18), where $\theta\in[-\vartheta,0]$, it holds that for $t\geq\tau$,

$$\begin{aligned}\|\psi_\epsilon(t+\theta)\|_H^2 &\leq \Big(\|\psi_\epsilon(\tau)\|_H^2+\frac{2(2q+1)^k\rho(\|u_\epsilon(\tau)\|)\|u_\epsilon(\tau)\|^2}{\epsilon}+\frac{2\|b\|^2}{\epsilon}\Big)e^{\mu\vartheta}e^{-\mu(t-\tau)}\\ &\quad+\frac{4L_h^2}{\varepsilon_0}e^{\mu\vartheta}\|u_{\epsilon,\tau}(\cdot)\|_{\ell^2_\vartheta}^2e^{-\mu(t-\tau)}+\frac{r_\epsilon^2}{2},\quad t+\theta\geq\tau,\end{aligned}\tag{3.19}$$

and for $t+\theta\leq\tau$, $\|\psi_\epsilon(t+\theta)\|_H^2\leq\|\psi_{\epsilon,\tau}(\cdot)\|_{H_\vartheta}^2$. Thus, $T_{1,\max}=+\infty$, the solution $\psi_\epsilon(\cdot)\in C([\tau-\vartheta,+\infty),H)\cap C^1((\tau,+\infty),H)$, $\theta\in[-\vartheta,0]$ and the solutions map $U_\epsilon^g(t,\tau)$ $(t\geq\tau)$ generates a continuous process $\{U_\epsilon^g(t,\tau)\}_{t\geq\tau}$ on H_ϑ. Moreover,

$$\begin{aligned}\|\psi_{\epsilon,t}(\cdot)\|_{H_\vartheta}^2&=\sup_{\theta\in[-\vartheta,0]}\|\psi_\epsilon(t+\theta)\|_H^2\\ &=\sup_{\theta\in[-\vartheta,0]}\Big(\frac{\delta}{\epsilon}\sum_{j=1}^k\|B_ju_\epsilon(t+\theta)\|^2+\frac{1}{\epsilon}\|u_\epsilon(t+\theta)\|_\lambda^2+\|v_\epsilon(t+\theta)\|^2\Big)\\ &\leq\Big(\|\psi_{\epsilon,\tau}(\cdot)\|_{H_\vartheta}^2+\frac{2(2q+1)^k\rho(\|u_\epsilon(\tau)\|)\|u_\epsilon(\tau)\|^2}{\epsilon}+\frac{2\|b\|^2}{\epsilon}\Big)e^{\mu\vartheta}e^{-\mu(t-\tau)}\\ &\quad+\frac{4L_h^2}{\varepsilon_0}e^{\mu\vartheta}\|u_{\epsilon,\tau}(\cdot)\|_{\ell^2_\vartheta}^2e^{-\mu(t-\tau)}+\frac{r_\epsilon^2}{2}\doteq\tilde b_\epsilon^2(t,\tau,\|\psi_{\epsilon,\tau}(\cdot)\|_{H_\vartheta}^2),\quad t\geq\tau,\end{aligned}\tag{3.20}$$

where $\tilde b_\epsilon^2(t,\tau,\|\psi_{\epsilon,\tau}(\cdot)\|_{H_\vartheta}^2)$ is continuous in t. Let $\tau\in\mathbb{R}$, $g^{(1)}$, $g^{(2)}\in\mathcal{H}(g_0)$, $\psi_{\epsilon,\tau}^{(1)}(\cdot)$, $\psi_{\epsilon,\tau}^{(2)}(\cdot)\in H_\vartheta$, $\psi_\epsilon^{(j)}(t)=\psi_\epsilon(t,\tau,g^{(j)},\psi_{\epsilon,\tau}^{(j)}(\cdot))$, $j=1,2$,

$$\kappa_\epsilon(t)=\kappa_\epsilon(t,\tau,g^{(1)},g^{(2)},\psi_{\epsilon,\tau}^{(1)}(\cdot),\psi_{\epsilon,\tau}^{(2)}(\cdot))=\psi_\epsilon^{(1)}(t)-\psi_\epsilon^{(2)}(t),\quad t\geq\tau,$$

then

$$\begin{gathered}\dot\kappa_\epsilon(t)+H_\epsilon\kappa_\epsilon(t)=F_\epsilon(\psi_{\epsilon,t}^{(1)}(\theta),g^{(1)}(t))-F_\epsilon(\psi_{\epsilon,t}^{(1)}(\theta),g^{(1)}(t)),\quad t\geq\tau,\\ \kappa_\epsilon(\tau)(\theta)=\psi_{\epsilon,\tau}^{(1)}(\theta)-\psi_{\epsilon,\tau}^{(2)}(\theta),\quad\theta\in[-\vartheta,0],\ \tau\in\mathbb{R}.\end{gathered}\tag{3.21}$$

Taking the inner product of $\kappa_\epsilon(t)\in H$ $(t\geq\tau)$ with (3.21), we have

$$\begin{aligned}&\frac{d}{dt}\|\kappa_\epsilon(t)\|_H^2+\widetilde C_\epsilon(t,\tau)\|\kappa_\epsilon(t)\|_H^2+\frac{\varepsilon_0\lambda_0}{2\epsilon^2}\|u_\epsilon^{(1)}(t)-u_\epsilon^{(2)}(t)\|^2\\ &\leq\frac{2L_h^2}{\epsilon}\|u_\epsilon^{(1)}(t-\vartheta))-u_\epsilon^{(2)}(t-\vartheta)\|^2+\frac{4}{\epsilon}\|g^{(1)}-g^{(2)}\|^2,\quad t\geq\tau,\end{aligned}\tag{3.22}$$

where

$$\widetilde C_\epsilon(t,\tau)=\frac{\varepsilon_0}{2\epsilon}-\frac{4}{\lambda_0}(2q+1)^{2k}\rho^2\Big(\sqrt{\frac{\epsilon}{\lambda_0}}(\tilde b_\epsilon(t,\tau,\|\psi_{\epsilon,\tau}^{(1)}(\cdot)\|_{H_\vartheta}^2)+\tilde b_\epsilon(t,\tau,\|\psi_{\epsilon,\tau}^{(1)}(\cdot)\|_{H_\vartheta}^2))\Big).$$

Applying Gronwall's inequality to (3.22) on $[\tau,t]$ $(t\geq\tau)$, we obtain

$$\begin{aligned}\|\kappa_\epsilon(t)\|^2&\leq\|\kappa_\epsilon(\tau)\|_H^2e^{-\int_\tau^t\widetilde C_\epsilon(l,\tau)dl}\\ &\quad+\frac{4L_h^2}{\varepsilon_0}e^{\mu\vartheta}\|u_{\epsilon,\tau}^{(1)}(\cdot)-u_{\epsilon,\tau}^{(2)}(\cdot)\|_{\ell^2_\vartheta}^2\int_{\tau-\vartheta}^\tau e^{-\int_r^t\widetilde C_\epsilon(l,\tau)dl}dr\\ &\quad+\frac{4}{\epsilon}\|g^{(1)}-g^{(2)}\|^2\int_\tau^t e^{-\int_r^t\widetilde C_\epsilon(l,\tau)dl}dr,\quad t\geq\tau.\end{aligned}$$

Thus

$$\|\kappa_\epsilon(t+\theta)\|^2\leq\widetilde C_{\epsilon,1}(t,\tau)\left(\|\kappa_{\epsilon,\tau}(\cdot)\|_{H_\vartheta}^2+\|g^{(1)}-g^{(2)}\|^2\right),\quad t\geq\tau,\tag{3.23}$$

where

$$\widetilde C_{\epsilon,1}(t,\tau)=\max\Big\{e^{\mu\vartheta}+\frac{4\epsilon L_h^2}{\lambda_0\varepsilon_0}e^{2\mu\vartheta}\int_{\tau-\vartheta}^\tau e^{-\int_r^t\widetilde C_\epsilon(l,\tau)dl}dr,\ \frac{4}{\lambda_0}e^{\mu\vartheta}\int_\tau^t e^{-\int_r^t\widetilde C_\epsilon(l,\tau)dl}dr\Big\},$$

and (3.23) implies that $\{U_\epsilon^g(t,\tau)\}_{t\geq\tau,g\in\mathcal{H}(g_0)}$ is continuous from $H_\vartheta\times\mathcal{H}(g_0)$ into H_ϑ.

(ii) From (3.20), the family $\{U_\epsilon^g(t,\tau)\}_{t\geq\tau,g\in\mathcal{H}(g_0)}$ has a (g,τ)-uniformly bounded closed absorbing set $B_\epsilon=\{\psi\in H_\vartheta:\|\psi\|_{H_\vartheta}\leq r_\epsilon\}\subset H_\vartheta$ and for any $g\in H(g_0)$, $\tau\in\mathbb{R}$, there exists $T_{B_\epsilon}\geq0$ (independent of (g,τ)) such that $\cup_{g\in\mathcal{H}(g_0)}U_\epsilon^g(t,\tau)B_\epsilon\subseteq B_\epsilon$ for $t\geq\tau+T_{B_\epsilon}$. Moreover, by (3.20) again, there exists a positive constants $b_{0,\epsilon}$, $b_{1,\epsilon}$ (independent of (g,t,τ)) such that for $t \geq \tau$, $g \in \mathcal{H}(g_0)$,

$$\begin{aligned} \|U_\epsilon^g(t,\tau)B_\epsilon\|_{H_\vartheta} &= \sup_{\psi_{\epsilon,t}\in U_\epsilon^g(t,\tau)B_\epsilon} \|\psi_{\epsilon,t}(\cdot)\|_{H_\vartheta} \\ &= \sup_{\psi_t\in U_\epsilon^g(t,\tau)B_\epsilon} \sup_{\theta\in[-\vartheta,0]} \|\psi_\epsilon(t+\theta)\|_{H_\vartheta} \leq b_{0,\epsilon}, \\ \sup_{\psi_{\epsilon,t}\in U_\epsilon^g(t,\tau)B_\epsilon} & \sup_{\theta\in[-\vartheta,0]} \|F_\epsilon(\psi_{\epsilon,t}(\theta),t) - H_\epsilon\psi_\epsilon(t)\| \leq b_{1,\epsilon}. \end{aligned} \tag{3.24}$$

(iii) Fix $g \in \mathcal{H}(g_0)$, $\tau \in \mathbb{R}$, $\psi_{\epsilon,\tau}(\cdot) \in B_\epsilon$, and let

$$\begin{aligned} \psi_\epsilon(t) &= U_\epsilon^g(t,\tau)\psi_{\epsilon,\tau}(\cdot) = \psi_\epsilon(t,\tau,\psi_{\epsilon,\tau}(\cdot)) \\ &= (u_{\epsilon,m}(t,\tau,\psi_{\epsilon,\tau}(\cdot)), v_{\epsilon,m}(t,\tau,\psi_{\epsilon,\tau}(\cdot)))^{\mathrm{T}}_{m\in\mathbb{Z}} \in H_\vartheta, \quad t \geq \tau, \end{aligned}$$

be the solution of equation (2.6). Let $K \in \mathbb{N}$, $w_{\epsilon,m} = \varrho(\frac{|m|}{K})u_{\epsilon,m}$, $z_{\epsilon,m} = \varrho(\frac{|m|}{K})v_{\epsilon,m}$, $\tilde{y}_\epsilon = (w_\epsilon, z_\epsilon)^{\mathrm{T}} = ((w_{\epsilon,m})_{m\in\mathbb{Z}^k}, (z_{\epsilon,m})_{m\in\mathbb{Z}^k})^{\mathrm{T}}$. Taking inner product $(\cdot,\cdot)_H$ of (2.6) with $\tilde{y}_\epsilon$, we have

$$\begin{aligned} &\frac{d}{dt}\Big[\sum_{m\in\mathbb{Z}^k} \varrho(\frac{\|m\|}{K})|\psi_{\epsilon,m}|_H^2 + \frac{2}{\epsilon}\sum_{m\in\mathbb{Z}^k} \varrho(\frac{\|m\|}{K})(G_m(u_{\epsilon,j}|j\in I_{mq}) + b_m^2)\Big] \\ &+ \frac{\varepsilon_0\lambda_0}{2\epsilon^2}\sum_{m\in\mathbb{Z}^k} \varrho(\frac{||m||}{K})u_{\epsilon,m}^2(t) \\ &+ \mu\Big[\sum_{m\in\mathbb{Z}^k} \varrho(\frac{\|m\|}{K})|\psi_{\epsilon,m}|_H^2 + \frac{2}{\epsilon}\sum_{m\in\mathbb{Z}^k} \varrho(\frac{\|m\|}{K})(G_m(u_{\epsilon,j}|j\in I_{mq}) + b_m^2)\Big] \\ &\leq \frac{2L_h^2}{\epsilon\beta}\sum_{m\in\mathbb{Z}^k} \varrho(\frac{||m||}{K})u_{\epsilon,m}^2(t-\vartheta) + \frac{\delta_3 b_{0,\epsilon}^2}{\epsilon K} \\ &\quad + \frac{2}{\epsilon\beta}\sum_{\|m\|\geq K} g_m^2(t) + \frac{2\lambda_0}{\epsilon}\sum_{\|m\|\geq K} b_m^2, \quad t \geq \tau, \end{aligned} \tag{3.25}$$

where $c_2 = C_0 m_0 a_0^2(2m_0+1)^2$, $\delta_2 = 2+2\epsilon\lambda_0+\frac{\epsilon}{\lambda_0}$, $\delta_3 = 2(\lambda_0\delta c_2\delta_2 k+\gamma c_2 k+\delta c_2\delta_2 k)+\delta_\epsilon\delta_2(2q)^{2k}C_0 q$. Applying Gronwall's inequality on $[\tau,t]$ to (3.25), we obtain

$$\begin{aligned} &\sum_{m\in\mathbb{Z}^k} \varrho(\frac{\|m\|}{K})|\psi_{\epsilon,m}(t)|_H^2 \\ &\leq (r_\epsilon^2 + \frac{2}{\epsilon}(\frac{\epsilon(2q+1)^k}{\lambda_0}\rho(r_\epsilon\sqrt{\frac{\epsilon}{\lambda_0}})r_\epsilon^2 + \|b\|^2) + \frac{4L_h^2}{\varepsilon_0}e^{\mu\vartheta}\|u_{\epsilon,\tau}(\cdot)\|_{\ell_\vartheta^2}^2)e^{-\mu(t-\tau)} \\ &\quad + \frac{4\delta_3 b_{0,\epsilon}^2}{\varepsilon_0 K} + \frac{16}{\varepsilon_0}\sup_{r\in\mathbb{R}}\sum_{\|m\|>K} g_m^2(r) + \frac{8\lambda_0}{\varepsilon_0}\sum_{\|m\|>K} b_m^2, \quad t \geq \tau. \end{aligned} \tag{3.26}$$

Hance, set $t+\theta$ instead of t in (3.26), where $\theta \in [-\vartheta, 0]$, we have

$$\begin{aligned} &\sum_{\|m\|>2K} |\psi_{\epsilon,m}(t+\theta)|_H^2 \\ &\leq \Big(r_\epsilon^2 + \frac{2(2q+1)^k}{\lambda_0}\rho(r_\epsilon\sqrt{\frac{\epsilon}{\lambda_0}})r_\epsilon^2 + \frac{2}{\epsilon}\|b\|^2 + \frac{4L_h^2}{\varepsilon_0}e^{\mu\vartheta}r_\epsilon^2\Big)e^{\mu\vartheta}e^{-\mu(t-\tau)} \\ &\quad + \frac{4\delta_3}{\varepsilon_0 K}b_{0,\epsilon}^2 + \frac{16}{\varepsilon_0}\sup_{r\in\mathbb{R}}\sum_{\|m\|>K} g_m^2(r) + \frac{8\lambda_0}{\varepsilon_0}\sum_{\|m\|>K} b_m^2, \quad t \geq \tau. \end{aligned} \tag{3.27}$$

Thus, by (3.27), for any $\eta > 0$, there exist $T_\epsilon(\eta, r_\epsilon) \geq T_{B_\epsilon} > 0$ and $K_\epsilon(\eta, g_0, r_\epsilon) \in \mathbb{N}$ (independent of g) such that for $t \geq \tau + T_\epsilon(\eta, r_\epsilon)$, $\tau \in \mathbb{R}$,

$$\sup_{g\in\mathcal{H}(g_0)} \sup_{\psi_{\epsilon,\tau}\in B_\epsilon} \sup_{\theta\in[-\vartheta,0]} \sum_{\|m\|>2K_\epsilon(\eta,g_0,r_\epsilon)} |(U_\epsilon^g(t,\tau)\psi_{\epsilon,\tau})_m(\theta)|_H^2 \leq \frac{\eta^2}{4}.$$

(iv) For any fixed $\tau \in \mathbb{R}$, any sequence $\{t_n\}_{n=1}^{+\infty} \subset [\vartheta, +\infty)$ with $t_n \to +\infty$ as $n \to \infty$, any sequence $\{\psi_{\epsilon,n}\}_{n=1}^{+\infty} \in B_\epsilon$ and any sequence $\{g_n\}_{n=1}^{+\infty} \in \mathcal{H}(g_0)$, we show that the sequence $\{\psi_{\epsilon,t_n+\tau}^{(g_n)} = U_\epsilon^{g_n}(t_n+\tau,\tau)\psi_{\epsilon,n}\}_{n=1}^{+\infty}$ has a convergent subsequence in H_ϑ. By (3.24), it follows that $\{\psi_{\epsilon,t_n+\tau}^{(g_n)}\}_{n=1}^{+\infty}$ is uniformly bounded in H_ϑ. Taking $\theta_1, \theta_2 \in [-\vartheta, 0]$ with $\theta_1 \le \theta_2$, where $t_n + \theta_1 \ge 0$, by (3.16) and (3.24), we have

$$
\begin{aligned}
\|\psi_{\epsilon,t_n+\tau}^{(g_n)}(\theta_1) - \psi_{\epsilon,t_n+\tau}^{(g_n)}(\theta_2)\| &= \|\int_{t_n+\tau+\theta_1}^{t_n+\tau+\theta_2} (F_\epsilon(\psi_{\epsilon,t_n+\tau}^{(g_n)}(\theta), s) - H_\epsilon \psi_{\epsilon,t_n+\tau}^{(g_n)}(0)) ds\| \\
&\le b_{1,\epsilon}|\theta_2 - \theta_1|,
\end{aligned}
$$

which implies that $\{\psi_{\epsilon,t_n+\tau}^{(g_n)}\}_{n=1}^{+\infty}$ is equicontinuous in H_ϑ. For any $\eta > 0$, by (iii) and $t_n \to +\infty$ as $n \to \infty$, there exists $K_{\epsilon,\eta} \in \mathbb{N}$ such that for $n \ge K_{\epsilon,\eta}$, $t_n \ge \tau + T_\epsilon(\eta, r_\epsilon)$ and

$$
\begin{aligned}
&\sup_{\theta \in [-\vartheta,0]} \sum_{\|m\| > 2K_0(\eta,g_0,r_0)} |(U_\epsilon^{g_n}(t_n+\tau,\tau)\psi_{\epsilon,n})_m(\theta)|^2 \\
&= \sup_{\theta \in [-\vartheta,0]} \sum_{\|m\| > 2K_0(\eta,g_0,r_0)} |\psi_{\epsilon,t_n+\tau}^{(g_n)}(\theta)|_H^2 \le \frac{\eta^2}{4}.
\end{aligned}
$$

It follows that $\{\psi_{\epsilon,t_n+\tau}^{(g_n)}(\theta)\}_{n=1}^{+\infty}$ is precompact in H. By Arezla-Ascoli theorem, $\{\psi_{\epsilon,t_n+\tau}^{(g_n)}\}_{n=1}^{+\infty}$ has a convergent subsequence in H_ϑ, that is, $\{U_\epsilon^g(t,\tau)\}_{t\ge\tau, g\in\mathcal{H}(g_0)}$ is uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) asymptotically compact in $B_\epsilon \subset H_\vartheta$.

(v) It is the results from (i)–(iv) and [4, 16, 27]. The proof is complete. □

By the transformation (2.5), if $\psi_\epsilon(t) = \psi_{\epsilon,t}(\cdot) = (u_\epsilon(t), v_\epsilon(t))^{\mathrm{T}} \in H_\vartheta$, where $v_\epsilon = \dot{u}_\epsilon + \frac{\varepsilon_0}{\epsilon} u_\epsilon$, is a solution of (2.6), then $\varphi_\epsilon(t) = \varphi_{\epsilon,t}(\cdot) = (u_{\epsilon,t}(\cdot), \dot{u}_{\epsilon,t}(\cdot))^{\mathrm{T}}$ is the a solution of the following system (3.28) in $E_\vartheta = C([-\vartheta, 0], E)$:

$$
\begin{gathered}
\dot{\varphi}_\epsilon(t) = \tilde{F}(\varphi_{\epsilon,t}(\theta), g(t)), \quad t \ge \tau,\ g \in \mathcal{H}(g_0),\ \tau \in \mathbb{R}, \\
\varphi_{\epsilon,\tau}(\theta) = (u_{\epsilon,\tau}(\theta), \dot{u}_{\epsilon,\tau}(\theta))^{\mathrm{T}} = (u_{\epsilon,\tau}(\tau+\theta), \dot{u}_\epsilon(\tau+\theta))^{\mathrm{T}}, \quad \theta \in [-\vartheta, 0],
\end{gathered} \tag{3.28}
$$

where

$$
\begin{aligned}
\tilde{F}(\varphi_{\epsilon,t}(\theta), g(t)) &= \begin{pmatrix} \dot{u}_\epsilon(t) \\ -\frac{1}{\epsilon}\dot{u}_\epsilon(t) - \frac{1}{\epsilon}\gamma A\dot{u}_\epsilon(t) - \frac{1}{\epsilon}Au_\epsilon(t) - \frac{1}{\epsilon}\lambda u_\epsilon(t) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ -\frac{1}{\epsilon}f(u_\epsilon(t)) - \frac{1}{\epsilon}h\left(u_\epsilon(t-\vartheta)\right) + \frac{1}{\epsilon}g(t) \end{pmatrix}
\end{aligned}
$$

and $\|\varphi_\epsilon(t)\|_E^2 = \|u_\epsilon(t)\|^2 + \|\dot{u}_\epsilon(t)\|^2 \le \delta_2 \|\psi_\epsilon(t)\|_H^2$.

From Theorem 3.3 and (2.5), we have the following result.

Theorem 3.4. *For the system* (3.28) *and* $\epsilon > 0$, *if* (A1)–(A6) *hold, then for any* $g \in H(g_0)$, $\tau \in \mathbb{R}$ *and* $\varphi_{\epsilon,\tau}(\cdot) = (u_{\epsilon,\tau}(\cdot), \dot{u}_{\epsilon,\tau}(\cdot))^{\mathrm{T}} \in H_\vartheta$, (3.28) *has a unique solution*

$$
\varphi_{\epsilon,t}(\cdot) = \varphi_\epsilon(t,\tau,\varphi_{\epsilon,\tau}(\cdot)) = (u_\epsilon(t,\tau,\varphi_{\epsilon,\tau}(\cdot)), \dot{u}_\epsilon(t,\tau,\varphi_{\epsilon,\tau}(\cdot)))^{\mathrm{T}} \in E_\vartheta,\ t \ge \tau,
$$

$\varphi_{\epsilon,t}(\cdot)$ *is continuous in* $\varphi_{\epsilon,\tau}(\cdot)$ *and*

$$
\varphi_\epsilon(\cdot) = \varphi_\epsilon(\cdot,\tau,\varphi_{\epsilon,\tau}(\theta)) \in C([\tau-\vartheta, +\infty), E) \cap C^1([\tau, +\infty), E), \quad \theta \in [-\vartheta, 0].
$$

The solution maps

$$
V_\epsilon^g(t,\tau) : E_\vartheta \to E_\vartheta,\ \varphi_{\epsilon,\tau}(\cdot) \to \varphi_{\epsilon,t}(\cdot) = \varphi_\epsilon(t,\tau,\varphi_{\epsilon,\tau}(\cdot)), \quad t \ge \tau,
$$

generates a continuous process $\{V_\epsilon^g(t,\tau)\}_{t\ge\tau}$ *on* E_ϑ, $V_\epsilon^g(t,\tau) = D_\epsilon^{-1}U_\epsilon^g(t,\tau)D_\epsilon$, *where* $D_\epsilon : (a,b)^{\mathrm{T}} \to (a, b + \frac{\varepsilon_0}{\epsilon}a)^{\mathrm{T}}$ *is a reversible operator from* E *into* E.

$\{V_\epsilon^g(t,\tau)\}_{t\ge\tau, g\in\mathcal{H}(g_0)}$ *possesses a unique compact uniform attractor* $\mathcal{A}_\epsilon^{\mathcal{H}(g_0)} \subset H_\vartheta$ *given by*

$$
\mathcal{A}_\epsilon^{\mathcal{H}(g_0)} = \cup_{g\in\mathcal{H}(g_0)} \mathcal{A}_{\epsilon,t}^g = \cup_{g\in\mathcal{H}(g_0)} \mathcal{A}_{\epsilon,0}^g = D_\epsilon^{-1}\mathcal{K}_\epsilon^{\mathcal{H}(g_0)} \subset H_\vartheta, \quad \forall t \in \mathbb{R}, \tag{3.29}
$$

where

$$\mathcal{A}^g_{\epsilon,t} = \Big\{\{\varphi_{\epsilon,t}\ \varphi_{\epsilon,t}(\cdot) = \varphi_\epsilon(t+\cdot) : [-\vartheta, 0] \to E \text{ is the global solution of } (3.28), \\ \|\varphi_{\epsilon,t}(\cdot)\|_{E_\vartheta} \le \tilde{r}_\epsilon \text{ for } t \in \mathbb{R}\Big\}.$$

with $V^g_\epsilon(t,\tau)\mathcal{A}^g_{\epsilon,\tau} = \mathcal{A}^g_{\epsilon,t}$ *for* $t \ge \tau$, $g \in H(g_0)$ *and* $\tilde{r}_\epsilon = \frac{2}{\sqrt{\epsilon\mu}}\sqrt{\delta_2\|g_0\|^2 + \lambda_0\delta_2\|b\|^2}$.

4. Prior uniform estimations of solutions

To investigate the upper semicontinuity of uniform attractors $\mathcal{A}^{\mathcal{H}(g_0)}_\epsilon$ and the relationship between $\mathcal{A}^{\mathcal{H}(g_0)}_\epsilon$ and $\mathcal{A}^{\mathcal{H}(g_0)}_0$ as $\epsilon \to 0^+$, in this section, we establish some prior uniform estimations for the solutions of (3.28) with respect to finite ϵ. Let the conditions (A1)–(A6) hold and $\overline{\epsilon} > 0$ be a given positive constant.

Lemma 4.1. *For each* $\epsilon \in (0, \overline{\epsilon}]$, $g \in H(g_0)$, $t \in \mathbb{R}$ *and a constant* $q_1 \ge 0$, *let* $s \ge 0$,

$$\varphi_\epsilon(t) = \varphi_{\epsilon,t}(\cdot) = \varphi_\epsilon(t, t-s, \varphi_{\epsilon,t-s}(\cdot)) \\ = (u_{\epsilon,t}(\cdot), \dot{u}_{\epsilon,t}(\cdot))^{\mathrm{T}} = V^g_\epsilon(t, t-s)\varphi_{\epsilon,t-s}(\cdot) \in E_\vartheta,$$

be a solution of (3.28) *with the initial value* $\varphi_{\epsilon,t-s}(\cdot) \in E_\vartheta$ *satisfying*

$$\begin{aligned} \epsilon\|\dot{u}_{\epsilon,t-s}(\cdot)\|^2_{\ell^2_\vartheta} + \|u_{\epsilon,t-s}(\cdot)\|^2_{\ell^2_\vartheta} &= \sup_{-\vartheta\le\theta\le 0}\big(\epsilon\|\dot{u}_\epsilon(t-s+\theta\|^2 + \|u_\epsilon(t-s+\theta)\|^2\big) \\ &\le q_1, \quad s \ge 0. \end{aligned} \tag{4.1}$$

Then there exist positive constants $M_1 = M_1(\overline{\epsilon})$, $\bar{\mu} = \bar{\mu}(\overline{\epsilon})$, $C_1(q_1, \overline{\epsilon})$, $\tilde{K}_1(\overline{\epsilon}, q_1)$, $\tilde{K}_2(\overline{\epsilon}, q_1)$, $M_2 = M_2(\overline{\epsilon}, q_1)$, $M_3 = M_3(\overline{\epsilon}, q_1) > 0$ *(independent of* (g, t, ϵ)*) and* $C_2(q_1, \epsilon)$, $C_3(q_1, \epsilon) > 0$ *(depending on* ϵ*) such that for any* $s \ge 0$, $t \in \mathbb{R}$,

$$\begin{aligned} \epsilon\|\dot{u}_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} + \|u_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} &= \sup_{-\vartheta\le\theta\le 0}\big(\epsilon\|\dot{u}_\epsilon(t+\theta)\|^2 + \|u_\epsilon(t+\theta)\|^2\big) \\ &\le M_1 + C_1(q_1, \overline{\epsilon})e^{-\bar{\mu}s}, \\ \int_t^{t+1}\|\dot{u}_\epsilon(r)\|^2 dr \le \tilde{K}_1(\overline{\epsilon}, q_1), &\quad \int_{t-\vartheta}^t \|\dot{u}_\epsilon(r)\|^2 dr \le \tilde{K}_2(\overline{\epsilon}, q_1), \\ \epsilon\|\ddot{u}_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} + \|\dot{u}_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} &= \sup_{-\vartheta\le\theta\le 0}\big(\epsilon\|\ddot{u}_\epsilon(t+\theta)\|^2 + \|\dot{u}_\epsilon(t+\theta)\|^2\big) \\ &\le M_2 + C_2(q_1, \epsilon)e^{-\bar{\mu}s}, \end{aligned} \tag{4.2}$$

$$\epsilon\|\ddot{u}_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} + \|\dot{u}_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} + \|u_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} \le M_3 + C_3(q_1, \epsilon)e^{-\bar{\mu}s}. \tag{4.3}$$

Proof. For $g \in \mathcal{H}(g_0)$, $t \in \mathbb{R}$, let $v_\epsilon = \dot{u}_\epsilon + \frac{\varepsilon_0}{\epsilon}u_\epsilon$, where $\varphi_{\epsilon,t}(\cdot) = (u_{\epsilon,t}(\cdot), \dot{u}_{\epsilon,t}(\cdot))^{\mathrm{T}}$ is the solution of problem (3.28) with initial data $\varphi_{\epsilon,t-s}(\cdot) \in E_\vartheta$ satisfying (4.1), then $\psi_\epsilon(t) = \psi_{\epsilon,t}(\cdot) = \psi_\epsilon(t, t-s, D_\epsilon\varphi_{\epsilon,t-s}(\cdot)) = (u_\epsilon(t), v_\epsilon(t))^{\mathrm{T}} \in H_\vartheta$ $(s \ge 0)$ is a solution of problem (2.6). It follows from (3.20) that

$$\begin{aligned} &\sup_{\theta\in[-\vartheta,0]}\Big(\frac{\lambda_0}{\epsilon}\|u_\epsilon(t+\theta)\|^2 + \|v_\epsilon(t+\theta)\|^2\Big) \\ &\le \Big(\|\psi_{\epsilon,t-s}(\cdot)\|^2_{H_\vartheta} + \frac{2(2q+1)^k\rho(\|u_\epsilon(t-s)\|)\|u_\epsilon(t-s)\|^2}{\epsilon} + \frac{2\|b\|^2}{\epsilon}\Big)e^{\mu\vartheta}e^{-\mu s} \\ &\quad + \frac{4L_h^2}{\varepsilon_0}e^{\mu\vartheta}\|u_{\epsilon,t-s}(\cdot)\|^2_{\ell^2_\vartheta}e^{\mu\vartheta}e^{-\mu s} + \frac{2}{\mu\epsilon\beta}\|g_0\|^2 + \frac{2\lambda_0}{\mu\epsilon}\|b\|^2, \quad s \ge 0. \end{aligned} \tag{4.4}$$

Multiplying both sides of (4.4) by ϵ, we have

$$\begin{aligned} &\sup_{\theta\in[-\vartheta,0]}\big(\lambda_0\|u_\epsilon(t+\theta)\|^2 + \epsilon\|v_\epsilon(t+\theta)\|^2\big) \\ &\le \big(\epsilon\|\psi_{\epsilon,t-s}(\cdot)\|^2_{H_\vartheta} + 2(2q+1)^k\rho(\|u_\epsilon(t-s)\|)\|u_\epsilon(t-s)\|^2 + 2\|b\|^2\big)e^{\mu\vartheta}e^{-\mu s} \\ &\quad + \frac{4L_h^2\epsilon}{\varepsilon_0}e^{\mu\vartheta}\|u_{\epsilon,t-s}(\cdot)\|^2_{\ell^2_\vartheta}e^{\mu\vartheta}e^{-\mu s} + \frac{2}{\mu\beta}\|g_0\|^2 + \frac{2\lambda_0}{\mu}\|b\|^2, \quad s \ge 0. \end{aligned} \tag{4.5}$$

Since $(1+3\epsilon\lambda_0)^2 \geq 12\epsilon\lambda_0$,and $\frac{\varepsilon_0^2}{\epsilon} \leq \frac{\lambda_0}{4} \leq \frac{\lambda_0}{2}$, we have

$$\sup_{\theta\in[-\vartheta,0]} \big(\lambda_0\|u_\epsilon(t+\theta)\|^2 + \epsilon\|v_\epsilon(t+\theta)\|^2\big) \geq \frac{\lambda_0}{2}\|u_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} + \frac{\epsilon}{2}\|\dot{u}_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta},$$

$$\epsilon\|v_\epsilon\|^2 \leq 2\epsilon\|\dot{u}_\epsilon\|^2 + \frac{\lambda_0}{2}\|u_\epsilon\|^2,$$

$$\epsilon\|\psi_{\epsilon,t-s}(\cdot)\|^2_{H_\vartheta} \leq (a_0^2(2m_0+1)^2k + \lambda^0 + 2 + \frac{\lambda_0}{2})q_1.$$

By (A5)–(A6), we have

$$\begin{gathered} \frac{\varepsilon_0}{4\epsilon} \geq \begin{cases} \frac{\lambda_0}{4(1+3\bar{\epsilon}\lambda_0)}, & \gamma = 0,\\ \min\{\frac{1}{4\gamma}, \frac{\lambda_0}{4(1+3\bar{\epsilon}\lambda_0)}\}, & \gamma > 0, \end{cases} \quad \doteq \bar{\mu} > 0,\\ \overline{\mu} \leq \frac{\varepsilon_0}{4\epsilon} \leq \mu = \frac{\delta_1}{\epsilon} \leq \frac{\varepsilon_0}{2\epsilon} \leq \frac{\lambda_0}{2}, \quad e^{\mu\vartheta} \leq e^{\frac{\lambda_0}{2}\vartheta}, \quad e^{-\mu s} \leq e^{-\bar{\mu}s}, \quad \forall \epsilon \in (0,\bar{\epsilon}],\ s \geq 0,\\ \frac{\epsilon}{\varepsilon_0} \leq \max\big\{\gamma, \frac{1+3\bar{\epsilon}\lambda_0}{\lambda_0}\big\} = \check{\mu}, \quad \frac{1}{\mu} \leq \frac{4\epsilon}{\varepsilon_0} \leq 4\check{\mu}, \quad \frac{1}{\mu\beta} \leq 8\check{\mu}. \end{gathered} \tag{4.6}$$

By (4.5) and (4.6), we have

$$\frac{\lambda_0}{2}\|u_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} + \frac{\epsilon}{2}\|\dot{u}_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} \leq \tilde{C}_1(q_1,\bar{\epsilon})e^{-\bar{\mu}s} + M_0(\bar{\epsilon}), \quad s \geq 0,$$

where

$$\begin{aligned} \tilde{C}_1(q_1,\bar{\epsilon}) = &\bigg((a_0^2(2m_0+1)^2k + \lambda^0 + 2 + \frac{\lambda_0}{2} + 2(2q+1)^k\rho(\sqrt{q_1})\bigg)e^{\frac{\lambda_0}{2}\vartheta}\\ &+ 4L_h^2 e^{\frac{\lambda_0}{2}\vartheta}\check{\mu})q_1 + 2\|b\|^2 e^{\frac{\lambda_0}{2}\vartheta},\\ M_0(\bar{\epsilon}) = &\,16\check{\mu}\|g_0\|^2 + 8\check{\mu}\lambda_0\|b\|^2. \end{aligned}$$

Hance

$$\|u_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} + \epsilon\|\dot{u}_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} \leq C_1(q_1,\bar{\epsilon})e^{-\bar{\mu}s} + M_1, \quad \forall s \geq 0,\ t \in \mathbb{R}, \tag{4.7}$$

where

$$C_1(q_1) = \frac{2\tilde{C}_1(q_1,\bar{\epsilon})}{\min\{\lambda_0,1\}}, \quad M_1 = \frac{2M_0(\bar{\epsilon})}{\min\{\lambda_0,1\}}.$$

In particular,

$$\|u_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} + \epsilon\|\dot{u}_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} \leq C_1(q_1,\bar{\epsilon}) + M_1, \quad \forall t \in \mathbb{R},\ \epsilon \in (0,\bar{\epsilon}]. \tag{4.8}$$

Then

$$\begin{aligned} &\|-Au_\epsilon(t) - \lambda u_\epsilon(t) - f(u_\epsilon(t)) - h\left(u_\epsilon(t-\vartheta)\right) + g(t)\|^2\\ &\leq 5[a_0^4(2m_0+1)^4 + (\lambda^0)^2 + (2q+1)^{2k}\rho^2(\sqrt{C_1(q_1,\bar{\epsilon}) + M_1})](C_1(q_1,\bar{\epsilon}) + M_1)\\ &\quad + 5L_h^2(C_1(q_1,\bar{\epsilon}) + M_1) + 4\|g_0\|^2 \doteq \tilde{K}_3(\bar{\epsilon},q_1), \quad s \geq 0. \end{aligned}$$

Taking the inner product of (2.4) with $\dot{u}_\epsilon$, we have

$$\epsilon\frac{d}{dt}\|\dot{u}_\epsilon(t)\|^2 + \|\dot{u}_\epsilon(t)\|^2 \leq \tilde{K}_3(\bar{\epsilon},q_1), \quad t \in \mathbb{R},\ s \geq 0. \tag{4.9}$$

Integrating both sides of (4.9) over $[t,t+1]$ and $[t-\vartheta,t]$, respectively, we have

$$\epsilon(\|\dot{u}_\epsilon(t+1)\|^2 - \|\dot{u}_\epsilon(t)\|^2) + \int_t^{t+1}\|\dot{u}_\epsilon(r)\|^2 dr \leq \tilde{K}_3(\bar{\epsilon},q_1),\ t \in \mathbb{R},$$

and

$$\epsilon(\|\dot{u}_\epsilon(t)\|^2 - \|\dot{u}_\epsilon(t-\vartheta)\|^2) + \int_{t-\vartheta}^t \|\dot{u}_\epsilon(r)\|^2 dr \leq \vartheta\tilde{K}_3(\bar{\epsilon},q_1),\ t \in \mathbb{R}.$$

Then for $t \in \mathbb{R}$, $\epsilon \in (0, \bar{\epsilon}]$, we have

$$\begin{aligned} \int_t^{t+1} \|\dot{u}_\epsilon(r)\|^2 dr &\le \tilde{K}_3(\bar{\epsilon}, q_1) + \epsilon\|\dot{u}_\epsilon(t)\|^2 \\ &\le \tilde{K}_3(\bar{\epsilon}, q_1) + C_1(q_1) + M_1 \doteq \tilde{K}_1(\bar{\epsilon}, q_1), \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \int_{t-\vartheta}^{t} \|\dot{u}_\epsilon(r)\|^2 dr &\le \vartheta\tilde{K}_3(\bar{\epsilon}, q_1) + \epsilon\|\dot{u}_\epsilon(t-\vartheta)\|^2 \\ &\le \vartheta\tilde{K}_3(\bar{\epsilon}, q_1) + C_1(q_1) + M_1 \doteq \tilde{K}_2(\bar{\epsilon}, q_1). \end{aligned} \tag{4.11}$$

(ii) Set $\|g'\|^2 = \sup_{t\in\mathbb{R}} \sum_{m\in\mathbb{Z}} g_m'^2(t) < \infty$, $\zeta_\epsilon(t) = \dot{u}_\epsilon(t)$. We differentiate equation (2.4), with respect to t, to obtain

$$\begin{aligned} &\epsilon\ddot{\zeta}_\epsilon + \dot{\zeta}_\epsilon + \gamma A\dot{\zeta}_\epsilon + A\zeta + \lambda\zeta_\epsilon + \Big(\sum_{j\in I_{mq}} f'_{m,j}(u_{\epsilon,j}|j \in I_{mq})\zeta_{\epsilon,j}\Big)_{m\in\mathbb{Z}^k} \\ &+ \left(h'_m\left(u_{\epsilon,m}(t-\vartheta)\right)\zeta_{\epsilon,m}(t-\vartheta)\right)_{m\in\mathbb{Z}^k} = g'(t), \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} \dot{\zeta}_\epsilon(t-s) &= \ddot{u}_\epsilon(t-s) \\ &= \frac{1}{\epsilon}(g(t-s) - h\left(u_\epsilon\left(t-s-\vartheta\right)\right) - f(u_\epsilon(t-s)) - \lambda u_\epsilon(t-s) \\ &\quad - Au_\epsilon(t-s) - \gamma A\dot{u}_\epsilon(t-s) - \dot{u}_\epsilon(t-s)), \\ \zeta_{\epsilon,t-s}(\theta) &= \dot{u}_{\epsilon,t-s}(\theta), \quad s \ge 0, \quad t \in \mathbb{R}, \quad \theta \in [-\vartheta, 0]. \end{aligned}$$

Then

$$\begin{aligned} &\sup_{-\vartheta\le\theta\le0} \|\zeta_{\epsilon,t-s}(\theta)\|^2 + \epsilon\|\dot{\zeta}(t-s)\|^2 \\ &\le \frac{q_1}{\epsilon}\left(2 + 7[\|g\|^2 + L_h^2 + (2q+1)^{2k}\rho^2(\sqrt{q_1}) + (\lambda^0)^2 + a_0^4(2m_0+1)^4k\right) \\ &\quad + \frac{q_1}{\epsilon}\left(\gamma^2 k a_0^4(2m_0+1)^4 + 1\right) \doteq q_2(q_1, \epsilon). \end{aligned}$$

Let

$$\tilde{v}_\epsilon = \dot{\zeta}_\epsilon + \frac{\varepsilon_0}{\epsilon}\zeta_\epsilon, \quad \tilde{\psi}_\epsilon = (\zeta_\epsilon, \tilde{v}_\epsilon)^T.$$

Then problem (4.12) can be written as

$$\dot{\tilde{\psi}}_\epsilon + H_\epsilon\tilde{\psi}_\epsilon = \tilde{F}_\epsilon(\tilde{\psi}_\epsilon, g', t), \quad t \in \mathbb{R},\ s \ge 0, \tag{4.13}$$

where

$$\begin{aligned} H_\epsilon\tilde{\psi}_\epsilon &= \begin{pmatrix} \frac{\varepsilon_0}{\epsilon}\zeta_\epsilon - \tilde{v}_\epsilon \\ \frac{1}{\epsilon}\lambda\zeta_\epsilon + \frac{1}{\epsilon}(1 - \frac{1}{\epsilon}\gamma\varepsilon_0)A\zeta_\epsilon - \frac{1}{\epsilon^2}\varepsilon_0(1-\varepsilon_0)\zeta_\epsilon + \frac{1}{\epsilon}(1-\varepsilon_0)\tilde{v}_\epsilon + \frac{1}{\epsilon}\gamma A\tilde{v}_\epsilon \end{pmatrix}, \\ \tilde{F}_\epsilon(\tilde{\psi}_\epsilon, g', t) &= \begin{pmatrix} 0 \\ -\frac{1}{\epsilon}(\sum_{j\in I_{mq}} f'_{m,j}(u_{\epsilon,j}|j \in I_{mq})\zeta_{\epsilon,j})_{m\in\mathbb{Z}^k} \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 \\ \frac{1}{\epsilon}[(h'_m\left(u_{\epsilon,m}(t-\vartheta)\right)\zeta_{\epsilon,m}(t-\vartheta))_{m\in\mathbb{Z}^k} + g'(t)] \end{pmatrix}. \end{aligned}$$

By computations,

$$2(H_\epsilon\tilde{\psi}_\epsilon, \psi_\epsilon)_H \ge \frac{\varepsilon_0}{\epsilon}\|\tilde{\psi}_\epsilon\|_H^2 + \frac{1}{\epsilon}\|\tilde{v}_\epsilon\|^2. \tag{4.14}$$

Taking the inner product of (4.13) with $\tilde{\psi}_\epsilon$ in H, by (4.14), we get that for $t \in \mathbb{R}$, $s \ge 0$,

$$\begin{aligned} &\frac{d}{dt}\|\tilde{\psi}_\epsilon(t)\|_H^2 + \frac{\varepsilon_0}{2\epsilon}\|\tilde{\psi}_\epsilon(t)\|_H^2 \\ &\le \frac{3}{\epsilon}\left((2q+1)^{2k}\rho^2(\sqrt{C_1(q_1)+M_1})\|\dot{u}_\epsilon(t)\|^2 + L_h^2\|\dot{u}_\epsilon(t-\vartheta)\|^2 + \|g_0'\|^2\right). \end{aligned} \tag{4.15}$$

By (4.10) and (4.11), for $t \in \mathbb{R}$, we have

$$\begin{aligned}\int_t^{t+1} \|\dot{u}_\epsilon(r-\vartheta)\|^2 dr &\le \int_t^{t+1} \|\dot{u}_\epsilon(r)\|^2 dr + \int_{t-\vartheta}^{t} \|\dot{u}_\epsilon(r)\|^2 dr \\ &\le \tilde{K}_1(\bar{\epsilon}, q_1) + \tilde{K}_2(\bar{\epsilon}, q_1), \quad \epsilon \in (0, \bar{\epsilon}],\end{aligned}$$

and

$$\begin{aligned}&\int_t^{t+1} \frac{3}{\epsilon}((2q+1)^{2k}\rho^2(\sqrt{C_1(q_1)+M_1})\|\dot{u}_\epsilon(r)\|^2 + L_h^2\|\dot{u}_\epsilon(r-\vartheta)\|^2 + \|g_0'\|^2)dr \\ &\le \frac{3}{\epsilon}((2q+1)^{2k}\rho^2(\sqrt{C_1(q_1)+M_1})\tilde{K}_1(\bar{\epsilon}, q_1) + L_h^2(\tilde{K}_1(\bar{\epsilon}, q_1) + \tilde{K}_2(\bar{\epsilon}, q_1)) + \|g_0'\|^2) \\ &\doteq \frac{1}{\epsilon}\tilde{K}_4(\bar{\epsilon}, q_1), \quad \epsilon \in (0, \bar{\epsilon}].\end{aligned}$$

By applying Gronwall's inequality on $[t-s, t]$ $(s \ge 0)$ to (4.15), we have

$$\|\tilde{\psi}_\epsilon(t)\|_H^2 \le \|\tilde{\psi}_\epsilon(t-s)\|_H^2 e^{-\bar{\mu}s} + \frac{1}{\epsilon}\tilde{K}_4(\bar{\epsilon}, q_1)(1+2\check{\mu}), \quad t \in \mathbb{R},\ s \ge 0.$$

where

$$\begin{aligned}\epsilon\|\tilde{\psi}_{\epsilon,t-s}(\cdot)\|_{H_\vartheta}^2 &\le (a_0^2(2m_0+1)^2k + \lambda^0)\|\zeta_{\epsilon,t-s}(\cdot)\|_{\ell_\vartheta^2}^2 + 2\epsilon\|\dot{\zeta}_{\epsilon,t-s}(\cdot)\|_{\ell_\vartheta^2}^2 + \frac{\lambda_0}{2}\|\zeta_{\epsilon,t-s}(\cdot)\|_{\ell_\vartheta^2}^2 \\ &\le (a_0^2(2m_0+1)^2k + \lambda^0 + 2 + \frac{\lambda_0}{2})q_2(q_1, \epsilon).\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\lambda_0}{2}\|\zeta_{\epsilon,t}(\cdot)\|_{\ell_\vartheta^2}^2 + \frac{\epsilon}{2}\|\dot{\zeta}_{\epsilon,t}(\cdot)\|_{\ell_\vartheta^2}^2 \\ &\le \sup_{\theta\in[-\vartheta,0]} \epsilon\|\tilde{\psi}_\epsilon(t+\theta)\|_H^2 \\ &\le (a_0^2(2m_0+1)^2k + \lambda^0 + 2 + \frac{\lambda_0}{2})q_2(q_1, \epsilon)e^{-\bar{\mu}s} + \tilde{K}_4(\bar{\epsilon}, q_1)(1+2\check{\mu}), \quad s \ge 0.\end{aligned}$$

Then

$$\epsilon\|\ddot{u}_{\epsilon,t}(\cdot)\|_{\ell_\vartheta^2}^2 + \|\dot{u}_{\epsilon,t}(\cdot)\|_{\ell_\vartheta^2}^2 \le M_2 + C_2(q_1, \epsilon)e^{-\bar{\mu}s}, \quad t \in \mathbb{R},\ s \ge 0, \tag{4.16}$$

where

$$M_2 = \frac{2\tilde{K}_4(\bar{\epsilon}, q_1)(1+2\check{\mu})}{\min\{\lambda_0, 1\}}, \quad C_2(q_1, \epsilon) = \frac{4 + 2a_0^2(2m_0+1)^2k + 2\lambda^0 + \lambda_0}{\min\{\lambda_0, 1\}}q_2(q_1, \epsilon).$$

Combining (4.7) and (4.16), we conclude (4.3). The proof is complete. □

Lemma 4.2. *For any $g \in \mathcal{H}(g_0)$, $t \in \mathbb{R}$, $s \ge 0$, $\epsilon \in (0, \bar{\epsilon}]$, let*

$$\varphi_{\epsilon,t}(\cdot) = \varphi_\epsilon(t, t-s, \varphi_{\epsilon,t-s}(\cdot)) = (u_{\epsilon,t}(\cdot), \dot{u}_{\epsilon,t}(\cdot))^{\mathrm{T}} = V_\epsilon^g(t, t-s)\varphi_{\epsilon,t-s}(\cdot) \in E_\vartheta,$$

be the solution of problem (3.28) with the initial value $\varphi_{\epsilon,t-s}(\cdot) \in \mathcal{A}_{\epsilon,t-s}^g \subseteq \tilde{B}_\epsilon$, where $\tilde{B}_\epsilon = \{\psi \in H_\vartheta : \|\psi\|_{H_\vartheta} \le \tilde{r}_\epsilon\} \subset H_\vartheta$. Then

(i) *there exists a positive constant $M_4 > 0$ (independent of (g, ϵ)) such that*

$$\epsilon\|\ddot{u}_{\epsilon,t}(\cdot)\|_{\ell_\vartheta^2}^2 + \|\dot{u}_{\epsilon,t}(\cdot)\|_{\ell_\vartheta^2}^2 + \|u_{\epsilon,t}(\cdot)\|_{\ell_\vartheta^2}^2 \le 2M_4, \quad \forall t \in \mathbb{R},\ \epsilon \in (0, \bar{\epsilon}]. \tag{4.17}$$

(ii) *For any $\eta > 0$, there exists a $I_2(\eta) = I_2(\eta, g_0) \in \mathbb{N}$ (independent of (g, ϵ)) such that*

$$\sup_{-\vartheta\le\theta\le0} \sum_{\|m\|>2I_2(\eta)} |u_{\epsilon,t,m}(\theta)|^2 \le \eta^2, \quad \forall t \in \mathbb{R},\ \epsilon \in (0, \bar{\epsilon}].$$

Proof. (i) Since $V_\epsilon^g(t, t-s)\mathcal{A}_{\epsilon,t-s}^g = \mathcal{A}_{\epsilon,t}^g$ and $\varphi_{\epsilon,t-s}(\cdot) \in \mathcal{A}_{\epsilon,t-s}^g \subseteq \tilde{B}_\epsilon$, we have

$$\varphi_{\epsilon,t}(\cdot) = \varphi_\epsilon(t+\cdot) = (u_{\epsilon,t}(\cdot), \dot{u}_{\epsilon,t}(\cdot))^{\mathrm{T}} = V_\epsilon^g(t, t-s)\varphi_{\epsilon,t-s}(\cdot) \in \mathcal{A}_{\epsilon,t}^g \subseteq \tilde{B}_\epsilon,$$

and

$$\psi_{\epsilon,t}(\cdot) = (u_{\epsilon,t}(\cdot), \dot{u}_{\epsilon,t}(\cdot) + \frac{\varepsilon_0}{\epsilon}u_{\epsilon,t}(\cdot))^{\mathrm{T}} = U_\epsilon^g(t, t-s)\psi_{\epsilon,t-s}(\cdot)$$

$$= D_\epsilon V_\epsilon^g(t,t-s)\varphi_{\epsilon,t-s}(\cdot) \in \mathcal{K}_{\epsilon,t}^g \subseteq B_\epsilon, \quad \forall t\in\mathbb{R}, \quad s\ge 0,$$

is the solution of (2.6). Again,

$$\begin{aligned}
&\epsilon\|\dot u_{\epsilon,t-s}(\cdot)\|^2_{\ell^2_\vartheta} + \|u_{\epsilon,t-s}(\cdot)\|^2_{\ell^2_\vartheta}\\
&\le \frac{2\epsilon}{\min\{\lambda_0,1\}}\Big(\|u_{\epsilon,t-s}(\cdot)\|^2_{\delta\lambda\epsilon} + \|\dot u_{\epsilon,t-s}(\cdot) + \frac{\varepsilon_0}{\epsilon}u_{\epsilon,t-s}(\cdot)\|^2_{\ell^2_\vartheta}\Big)\\
&\le \frac{2\epsilon}{\min\{\lambda_0,1\}}\Big(\frac{4}{\mu\epsilon\beta}\|g_0\|^2 + \frac{4\lambda_0}{\mu\epsilon}\|b\|^2\Big)\\
&\le \frac{32\check\mu}{\min\{\lambda_0,1\}}\left(2\|g_0\|^2 + \lambda_0\|b\|^2\right) \doteq q_4(\bar\epsilon) = q_4 \text{ (independent of } (g,\epsilon)).
\end{aligned}$$

Therefore, by Lemma 4.1, there exist positive constants $M_4 = M_4(\bar\epsilon)$, $\bar\mu = \bar\mu(\bar\epsilon)$ (independent of (g,ϵ)) and a finite positive constant $C_4(q_4,\epsilon)$ (depending on ϵ) such that for any $t\in\mathbb{R}$,

$$\epsilon\|\ddot u_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} + \|\dot u_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} + \|u_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} \le M_4 + C_4(q_4,\epsilon)e^{-\bar\mu s}, \quad \forall s\ge 0,\ \epsilon\in(0,\bar\epsilon].$$

So, for each fixed $t\in\mathbb{R}$ and $\epsilon\in(0,\bar\epsilon]$, there must exists a large number $\tau_\epsilon>0$ (depending on ϵ) such that $C_4(q_4,\epsilon)e^{-\bar\mu s}\le M_4$ for all $s\ge\tau_\epsilon$, thus

$$\epsilon\|\ddot u_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} + \|\dot u_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} + \|u_{\epsilon,t}(\cdot)\|^2_{\ell^2_\vartheta} \le 2M_4, \quad \forall t\in\mathbb{R},\ \epsilon\in(0,\bar\epsilon], \tag{4.18}$$

which implies that for any solution $\varphi_{\epsilon,t}(\cdot)$ of (3.28) in $\mathcal{A}^g_{\epsilon,t}(\cdot)$, (4.17) holds.

(ii) Similar to the proof of (3.27), it follows from (4.18) that there exists positive constants $q_5(M_4,\bar\epsilon)>0$, $q_6(M_4,\bar\epsilon)>0$ (independent of ϵ) such that for $K\in\mathbb{N}$, $t\in\mathbb{R}$, $s\ge0$, $\theta\in[-\vartheta,0]$,

$$\begin{aligned}
\sum_{\|m\|>2K}|\psi_{\epsilon,m}(t+\theta)|^2_H &= \sum_{\|m\|>2K}\big(\frac{1}{\epsilon}\delta(Bu_\epsilon)^2_m(t+\theta) + \frac{1}{\epsilon}\lambda_m u^2_{\epsilon,m}(t+\theta) + v^2_{\epsilon,m}(t+\theta)\big)\\
&\le \sum_{m\in\mathbb{Z}^k}\varrho(\frac{|m|}{K})|\psi_{\epsilon,m}(t+\theta)|^2_H\\
&\le \frac{q_5(M_4,\bar\epsilon)}{\epsilon}e^{-\bar\mu s} + \frac{q_6(M_4,\bar\epsilon)}{\varepsilon_0 K} + \frac{16}{\varepsilon_0}\sup_{r\in\mathbb{R}}\sum_{\|m\|>K}g^2_m(r) + \frac{8\lambda_0}{\varepsilon_0}\sum_{|m|>K}b^2_m.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{\|m\|>2K}\big(\frac{\lambda_0}{2}u^2_{\epsilon,m}(t+\theta) + \frac{\epsilon}{2}\dot u^2_{\epsilon,m}(t+\theta)\big)\\
&\le \sum_{\|m\|>2K}\big(\lambda_m u^2_{\epsilon,m}(t+\theta) + \epsilon v^2_{\epsilon,m}(t+\theta)\big)\\
&\le \epsilon\sum_{\|m\|>2K}|\psi_{\epsilon,m}(t+\theta)|^2_H\\
&\le q_5(M_4,\bar\epsilon)e^{-\bar\mu s} + \frac{\check\mu q_6(M_4,\bar\epsilon)}{K} + 16\check\mu\sup_{r\in\mathbb{R}}\sum_{\|m\|\ge K}g^2_m(r) + 8\lambda_0\check\mu\sum_{\|m\|\ge K}b^2_m
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\|m\|>2K}u^2_{\epsilon,m}(t+\theta) &\le \frac{2}{\lambda_0}q_5(M_4,\bar\epsilon)e^{-\bar\mu s} + \frac{\lambda_0\check\mu q_6(M_4,\bar\epsilon)}{K} + \frac{32}{\lambda_0}\check\mu\sup_{r\in\mathbb{R}}\sum_{\|m\|\ge K}g^2_m(r)\\
&\quad + 16\check\mu\sum_{\|m\|\ge K}b^2_m.
\end{aligned} \tag{4.19}$$

It follows that $\forall\eta>0$, there exists $I_2(\eta) = I_2(\eta,M_4,g_0,\bar\epsilon)\in\mathbb{N}$ and $T_2(\eta) = T_2(\eta,,M_4,\bar\epsilon) = \max\big\{0,\frac{1}{\bar\mu}\ln\frac{16q_5(M_4,\bar\epsilon)}{\lambda_0\eta^2}\big\}$ (independent of (g,ϵ)) such that for any $t\in\mathbb{R}$, $K\ge I_2(\eta)$, $s\ge T_2(\eta)$, we have

$$\frac{2}{\lambda_0}q_5(M_4,\bar\epsilon)e^{-\bar\mu s}\le\frac{\eta^2}{2},\ \frac{\lambda_0\check\mu q_6(M_4,\bar\epsilon)}{K}+\frac{32}{\lambda_0}\check\mu\sup_{r\in\mathbb{R}}\sum_{\|m\|\ge K}g_m^2(r)+16\check\mu\sum_{\|m\|\ge K}b_m^2\le\frac{\eta^2}{2},$$

$$\sum_{\|m\|>2K}u_{\epsilon,m}^2(t+\theta)\le\eta^2,\quad t\in\mathbb{R},\ K\ge I_2(\eta),\ s\ge T_2(\eta).$$

In particular,

$$\sup_{t\in\mathbb{R}}\sup_{-\vartheta\le\theta\le0}\sum_{\|m\|>2I_2(\eta)}|u_{\epsilon,m}(t+\theta)|^2\le\eta^2,\quad\forall\epsilon\in(0,\bar\epsilon].$$

The proof is complete. □

Lemma 4.3. *For each* $\tilde\varphi_\epsilon(\cdot)=(u_\epsilon(\cdot),\tilde w_\epsilon(\cdot))^{\mathrm{T}}=((u_{\epsilon,m}(\cdot))_{m\in\mathbb{Z}^k},(\tilde w_{\epsilon,m}(\cdot))_{m\in\mathbb{Z}^k})^{\mathrm{T}}\in A_\epsilon^{\mathcal{H}(g_0)}$ *it holds that*

$$\|\tilde\varphi_\epsilon(\cdot)\|_{E_\vartheta}^2=\|u_\epsilon(\cdot)\|_{\ell_\vartheta^2}^2+\|\tilde w_\epsilon(\cdot)\|_{\ell_\vartheta^2}^2\le2M_4,\quad\forall\epsilon\in(0,\bar\epsilon],$$

and for any $\eta>0$ *there exists* $I_3(\eta)\in\mathbb{N}$ *(independent of* (g,ϵ)*) such that*

$$\sup_{-\vartheta\le\theta\le0}\sum_{\|m\|>2I_3(\eta)}|u_{\epsilon,m}(\theta)|^2=\sup_{-\vartheta\le\theta\le0}\sum_{\|m\|>2I_3(\eta)}|u_{\epsilon,m}(\theta)|^2\le\eta^2,\quad\forall\epsilon\in(0,\bar\epsilon].$$

Proof. From Theorem 3.4, it follows that

$$\mathcal{A}_\epsilon^{\mathcal{H}(g_0)}=\cup_{g\in\mathcal{H}(g_0)}\mathcal{A}_{\epsilon,0}^g\subseteq\tilde B_\epsilon\subset H_\vartheta.$$

Thus, for any fixed $\tilde\varphi_\epsilon(\cdot)=(u_\epsilon(\cdot),\tilde w_\epsilon(\cdot))^{\mathrm{T}}=((u_{\epsilon,m}(\cdot))_{m\in\mathbb{Z}^k},(\tilde w_{\epsilon,m}(\cdot))_{m\in\mathbb{Z}^k})^{\mathrm{T}}\in A_\epsilon^{\mathcal{H}(g_0)}$, there must exists a $g\in\mathcal{H}(g_0)$ such that $\tilde\varphi_\epsilon(\cdot)\in\mathcal{A}_{\epsilon,0}^g$. According to Lemma 4.2, the statements in Lemma 4.3 follow. □

5. Upper semicontinuity of uniform attractors

Now, we consider the upper semicontinuity of the uniform attractor $\mathcal{A}_\epsilon^{\mathcal{H}(g_0)}\subset E_\vartheta\subseteq\ell_\vartheta^2\times\ell_\vartheta^2$ for the second order delay lattice system (3.28) as $\epsilon\to0^+$. When $\epsilon=0$, (2.4) is the first order delay lattice system (2.1) with a uniform attractor $\mathcal{A}_0^{\mathcal{H}(g_0)}\subset\ell_\vartheta^2$. Notice that $\mathcal{A}_\epsilon^{\mathcal{H}(g_0)}$ and $\mathcal{A}_0^{\mathcal{H}(g_0)}$ are in different spaces, to compare the relationship between them, we should take them in the same bigger space $\ell_\vartheta^2\times\ell_\vartheta^2$. For this purpose, basing on the structure of $\mathcal{A}_\epsilon^{\mathcal{H}(g_0)}$ and $\mathcal{A}_0^{\mathcal{H}(g_0)}$, we introduce the following set in $\ell_\vartheta^2\times\ell_\vartheta^2$:

$$\begin{aligned}\mathcal{B}_{0,t}^g=\Big\{\begin{pmatrix}u_t\\ \omega_t\end{pmatrix}:u_t(\cdot)\in\mathcal{A}_{0,t}^g\text{ and }\omega_t(\theta)=(I+\gamma A)^{-1}[-Au(t)-\lambda u(t)-f(u(t))]\\ +(I+\gamma A)^{-1}[-h(u(t-\vartheta))+g(t)],\theta\in[-\vartheta,0]\Big\}\\ \subset E_\vartheta,\quad t\in\mathbb{R},\ g\in\mathcal{H}(g_0),\end{aligned}$$

where $\mathcal{A}_{0,t}^g$ is embedded into $\mathcal{B}_{0,t}^g$ as the first component, that is, $\Pi_1\mathcal{B}_{0,t}^g=\mathcal{A}_{0,t}^g$, where $\Pi_1:(u_t(\cdot),\omega_t(\cdot))\in\ell_\vartheta^2\times\ell_\vartheta^2\to u_t(\cdot)\in\ell_\vartheta^2$ is the projector from $\ell_\vartheta^2\times\ell_\vartheta^2$ to ℓ_ϑ^2. Since $(I+\gamma A)^{-1}g(\cdot)\in C_b^1(\mathbb{R},\ell^2)$ and $(I+\gamma A)^{-1}[-Au-\lambda u-f(u)-h(u(t-\vartheta))]$ is continuous in u, so for fixed $t\in\mathbb{R}$, that $\mathcal{B}_{0,t}^g$ is compact in $\ell_\vartheta^2\times\ell_\vartheta^2$. Set

$$\mathcal{B}_0^{\mathcal{H}(g_0)}=\cup_{g\in\mathcal{H}(g_0)}\mathcal{B}_{0,t}^g\subset E_\vartheta.$$

Then $\mathcal{A}_0^{\mathcal{H}(g_0)}$ is naturally embedded into $\mathcal{B}_0^{\mathcal{H}(g_0)}$ as the first component, that is, $\Pi_1\mathcal{B}_0^{\mathcal{H}(g_0)}=\mathcal{A}_0^{\mathcal{H}(g_0)}$. In the following, we show the upper semicontinuity:

$$\lim_{\epsilon\to0^+}\mathrm{d}_h(\mathcal{A}_\epsilon^{\mathcal{H}(g_0)},\mathcal{B}_0^{\mathcal{H}(g_0)})=0.$$

Lemma 5.1. *Let $\bar{\epsilon} > 0$ be a given constant, conditions* (A1)–(A6) *hold and $\{\epsilon_n\}_{n=1}^{+\infty} \subset (0, \bar{\epsilon}]$ be an arbitrary sequence of positive numbers with $\epsilon_n \to 0$ as $n \to +\infty$. Taking $\varphi^{(n)}(\cdot) = (u^{(n)}(\cdot), \omega^{(n)}(\cdot))^{\mathrm{T}} \in A_{\epsilon_n}^{\mathcal{H}(g_0)}$, then there exists a subsequence $\{n_i\}$ of $\{n\}$ such that*

$$\varphi^{(n_i)}(\cdot) = (u^{(n_i)}(\cdot), \omega^{(n_i)}(\cdot))^{\mathrm{T}} \to (\bar{u}(\cdot), \bar{v}(\cdot))^{\mathrm{T}} = \bar{\varphi}(\cdot) \in \mathcal{B}_0^{\mathcal{H}(g_0)} \ (n_i \to +\infty) \quad \textit{strongly in } E_\vartheta.$$

Proof. By (3.29), $A_{\epsilon_n}^{\mathcal{H}(g_0)} = \cup_{g \in \mathcal{H}(g_0)} A_{\epsilon_n,0}^{g} = \cup_{g \in \mathcal{H}(g_0)} A_{\epsilon_n,t}^{g} \subset E_\vartheta$, for all $t \in \mathbb{R}$, then for any $n \in \mathbb{N}$ and any $\varphi_0^{(n)}(\cdot) = (u_0^{(n)}(\cdot), \omega_0^{(n)}(\cdot))^{\mathrm{T}} \in A_{\epsilon_n}^{\mathcal{H}(g_0)}$, there exists $g^{(n)} \in \mathcal{H}(g_0)$ such that $\varphi_0^{(n)}(\cdot) = (u_0^{(n)}(\cdot), \omega_0^{(n)}(\cdot))^{\mathrm{T}} \in A_{\epsilon_n,0}^{g^{(n)}}$. Let

$$\begin{aligned}
\varphi_t^{(n)}(\cdot) &= \varphi^{(n)}(t+\cdot) = \varphi^{(n)}(t, 0, \varphi_0^{(n)}(\cdot)) \\
&= (u_t^{(n)}(\cdot), \dot{u}_t^{(n)}(\cdot))^{\mathrm{T}} = (u_{\epsilon_n,t}(\cdot), \dot{u}_{\epsilon_n,t}(\cdot))^{\mathrm{T}} \\
&= V_{\epsilon_n}^{g^{(n)}}(t, 0)\varphi_0^{(n)}(\cdot) \in E_\vartheta
\end{aligned}$$

be the solution of problem (3.28) with value $\varphi_0^{(n)}(\cdot) \in A_{\epsilon_n,0}^{g^{(n)}}$ at $t = 0$; that is, $\varphi_t^{(n)}(\cdot)$ satisfies

$$\begin{gathered}
\epsilon_n \ddot{u}^{(n)} + \dot{u}^{(n)} + \gamma A \dot{u}^{(n)} + A u^{(n)} + \lambda u^{(n)} + f(u^{(n)}) + h\left(u^{(n)}(t - \vartheta)\right) = g^{(n)}(t), \\
u_t^{(n)}(\cdot)|_{t=0} = u_0^{(n)}(\theta), \ \dot{u}_t^{(n)}(\cdot)|_{t=0} = \omega_0^{(n)}(\theta), \quad \theta \in [-\vartheta, 0].
\end{gathered}$$

By Theorem 3.4, we have

$$\varphi_t^{(n)}(\cdot) = \varphi^{(n)}(t+\cdot) = (u_t^{(n)}(\cdot), \dot{u}_t^{(n)}(\cdot))^{\mathrm{T}} \in \mathcal{A}_{\epsilon_n,t}^{g^{(n)}} \subseteq \tilde{B}_{\epsilon_n} \subset E_\vartheta, \quad \forall t \in \mathbb{R}. \tag{5.1}$$

By the compactness of $\mathcal{H}(g_0)$ in $C_b(\mathbb{R}, \ell^2)$, there exists a subsequence of $\{g^{(n)}(\cdot)\}_{n=1}^{+\infty}$ (still denoted by $\{g^{(n)(\cdot)}\}_{n=1}^{+\infty}$) such that

$$g^{(n)}(\cdot) \to \bar{g}(\cdot) \in \mathcal{H}(g_0) \ (n \to +\infty) \quad \text{strongly in } C_b(\mathbb{R}, \ell^2).$$

In what follows, we prove that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\varphi_t^{(n_i)}(\cdot) = (u^{(n_i)}(t+\cdot), \dot{u}^{(n_i)}(t+\cdot))^{\mathrm{T}} \to (\bar{u}(t+\cdot), \dot{\bar{u}}(t+\cdot))^{\mathrm{T}} \ \in \mathcal{B}_{0,t}^{\bar{g}} \ (n_i \to +\infty) \quad \text{in } E_\vartheta$$

for $t \in \mathbb{R}$, by using Arezla-Ascoli theorem and diagonal sequence method.

From (5.1) and Lemma 4.2(i), $\{\varphi_t^{(n)}(\theta) = \varphi_{\epsilon_n}^{(n)}(t+\theta)\}_{n=1}^{+\infty}$ is uniformly bounded in $\ell^2 \times \ell^2$ with respect to $\theta \in [-\vartheta, 0]$ and $t \in \mathbb{R}$:

$$\sup_{t \in \mathbb{R}} \sup_{1 \le n < +\infty} \|\varphi_t^{(n)}(\cdot)\|_{E_\vartheta}^2 = \sup_{t \in \mathbb{R}} \sup_{1 \le n < +\infty} \sup_{-\vartheta \le \theta \le 0} (\|\dot{u}_{\epsilon_n}(t+\theta)\|^2 + \|u_{\epsilon_n}(t+\theta)\|^2) \le 2M_4.$$

In particlualr,

$$\sup_{t \in \mathbb{R}} \sup_{1 \le n < +\infty} (\|\dot{u}^{(n)}(t)\|^2 + \|u^{(n)}(t)\|^2) \le 2M_4. \tag{5.2}$$

Let $J_i = [-i, i]$, $i \in \mathbb{Z}_+$, be a sequence of closed interval of $\mathbb{R}$ such that $J_i \subset J_{i+1}$, $\bigcup_{i \in \mathbb{Z}_+} J_i = \mathbb{R}$. Taking t_1, $t_2 \in J_i$, by mean value theorem and (5.2), we have

$$\|u^{(n)}(t_1) - u^{(n)}(t_2)\| \le \sqrt{2M_4}|t_1 - t_2|,$$

which implies the equicontinuity of $\{u^{(n)}(\cdot)\}_{n=1}^{+\infty} \subset C^1(\mathbb{R}, \ell^2)$ in $C(J_i, \ell^2)$. Since E is a Hilbert space, by (5.2), there exists a subsequence of $\{(u^{(n)}(t), \dot{u}^{(n)}(t))^{\mathrm{T}}\}$ (denoted still by $\{(u^{(n)}(t), \dot{u}^{(n)}(t))^{\mathrm{T}}\}$) and $(\bar{u}(t), \tilde{u}(t))^{\mathrm{T}} \in E$ such that

$$\begin{gathered}
(u^{(n)}(t), \dot{u}^{(n)}(t))^{\mathrm{T}} \to (\bar{u}(t), \tilde{u}(t))^{\mathrm{T}} \quad (n \to +\infty) \quad \text{weakly in } \ell^2 \times \ell^2, \ \forall t \in \mathbb{R}, \\
\sup_{t \in \mathbb{R}} \|(\bar{u}(t), \tilde{u}(t))^{\mathrm{T}}\|_{\ell^2 \times \ell^2}^2 \le 2M_4.
\end{gathered}$$

By Lemma 4.3, for any $\eta > 0$, there exists $I_4(\eta) \in \mathbb{N}$ (independent of ϵ_n and n) such that for $u^{(n)}(t) = (u_m^{(n)}(t))_{m \in \mathbb{Z}}$,

$$\sup_{t \in \mathbb{R}} \sum_{\|m\| > I_4(\eta)} \|u_m^{(n)}(t)\|^2 \le \eta^2.$$

It obtain from the characteristics of a precompact set in ℓ^2 that $\{u^{(n)}(t)\}_{n=1}^{\infty}$ is precompact in ℓ^2, i.e. for any fixed $t\in\mathbb{R}$, $\{u^{(n)}(t)\}_{n=1}^{\infty}$ has a subsequence $u^{(n_i)}(t)$ strongly convergent to $\bar{u}(t)$ in ℓ^2.

By Arezla-Ascoli theorem, $\{u^{(n)}(\cdot)\}$ has a subsequence $\{u^{(n,1)}(\cdot)\}_{n=1}^{+\infty}$ such that

$$u^{(n,1)}(\cdot)\to\bar{u}(\cdot)\quad(n\to+\infty)\quad\text{strongly in } C(J_1,\ell^2)$$

and for any $k\in\mathbb{N}$, $\{u^{(n,i)}(\cdot)\}$ has a subsequence $\{u^{(n,i+1)}(\cdot)\}$ such that

$$u^{(n,i+1)}(\cdot)\to\bar{u}(\cdot)\quad(n\to+\infty)\quad\text{strongly in } C(J_{i+1},\ell^2).$$

Taking the diagonal sequence of $\{u^{(n,i)}(\cdot)\}$, we obtain a subsequence $\{u^{(i,i)}(\cdot)\}=\{u^{(i_i)}(\cdot)\}$, where $i_i\to+\infty$ as $i\to+\infty$ and the corresponding subsequence $\epsilon_{(i_i)}\to0$ as $i\to+\infty$, such that for any compact subset $J\subseteq J_i\subset\mathbb{R}$,

$$\begin{gathered}u^{(i_i)}(\cdot)\to\bar{u}(\cdot)\quad(i\to+\infty)\quad\text{strongly in } C(J,\ell^2),\\ u^{(i_i)}(\cdot-\vartheta)\to\bar{u}(\cdot-\vartheta)\quad(i\to+\infty)\quad\text{strongly in } C(J,\ell^2),\\ \dot{u}^{(i_i)}(\cdot)\to\dot{\bar{u}}(\cdot)\ (i\to+\infty)\quad\text{weak star in } L^{\infty}(J,\ell^2).\end{gathered}\tag{5.3}$$

By (2.4),

$$\begin{aligned}\dot{u}^{(i_i)}(t)&=(I+\gamma A)^{-1}[-\epsilon_{(k_k)}\ddot{u}^{(i_i)}(t)-Au^{(i_i)}(t)-\lambda u^{(i_i)}(t)-f(u^{(i_i)}(t))]\\&\quad+(I+\gamma A)^{-1}[-h\left(u^{(i_i)}(t-\vartheta)\right)+g^{(i_i)}(t)],\quad t\in\mathbb{R}.\end{aligned}\tag{5.4}$$

By (4.17) and $0<\epsilon_{i_i}\to0^+$ $(i\to+\infty)$, we have

$$\sup_{t\in\mathbb{R}}\sqrt{\epsilon_{i_i}}\|\ddot{u}^{(i_i)}(t)\|\le\sqrt{2M_4}<\infty,\quad\lim_{i\to+\infty}\sup_{t\in\mathbb{R}}(\epsilon_{i_i}\|\ddot{u}^{(i_i)}(t)\|)=0.\tag{5.5}$$

By (5.4), (5.5), the continuity of f and h, the bounded linearity of A and $(I+\gamma A)^{-1}$, for any compact subset $J\subset\mathbb{R}$, $t\in\mathbb{R}$, it follows that

$$\dot{u}^{(i_i)}(t)\to(I+\gamma A)^{-1}[-A\bar{u}(t)-\lambda\bar{u}(t)-f(\bar{u}(t))-h\left(\bar{u}(t-\vartheta)\right)+\bar{g}(t)]\tag{5.6}$$

as $i\to+\infty$. By the uniqueness of the limit, it follows that

$$\tilde{u}(\cdot)=\dot{\bar{u}}(\cdot)=(I+\gamma A)^{-1}[-A\bar{u}(\cdot)-\lambda\bar{u}(\cdot)-f(\bar{u}(\cdot))-h\left(\bar{u}\left(\cdot-\vartheta\right)\right)+\bar{g}(\cdot)]\in C_b(\mathbb{R},\ell^2).\tag{5.7}$$

Thus, $\bar{u}(t)$, $t\in\mathbb{R}$, is a global bounded solution for the system (2.1) defined on $\mathbb{R}$. By (5.2), we have

$$\sup_{t\in\mathbb{R}}\sup_{-\vartheta\le\theta\le0}(\|\dot{\bar{u}}\left(t+\theta\right)\|^2+\|\bar{u}\left(t+\theta\right)\|^2)\le2M_4.$$

By the structure of $\mathcal{A}_{0,t}^{\bar{g}}$ and $\mathcal{B}_{0,t}^{\bar{g}}$, $(\bar{u}(t+\cdot),\dot{\bar{u}}(t+\cdot))^{\mathrm{T}}\in\mathcal{B}_{0,t}^{\bar{g}}$ for any $t\in\mathbb{R}$. From (5.6), (5.7), we have

$$(u^{(i_i)}(t+\cdot),\dot{u}^{(i_i)}(t+\cdot))^{\mathrm{T}}\to(\bar{u}(t+\cdot),\dot{\bar{u}}(t+\cdot))^{\mathrm{T}}\quad(i\to+\infty))\quad\text{in } E_\vartheta\text{ for } t\in\mathbb{R}.$$

Specially,

$$\begin{aligned}\varphi_0^{(i_i)}(\cdot)&=(u_0^{(i_i)}(\cdot),\omega_0^{(i_i)}(\cdot))^{\mathrm{T}}=(u_0^{(i_i)}(\cdot),\dot{u}_0^{(i_i)}(\cdot))^{\mathrm{T}}\\&\to(\bar{u}_0(\cdot),\dot{\bar{u}}_0(\cdot))^{\mathrm{T}}=\varphi_{0,0}(\cdot)\in\mathcal{B}_{0,0}^{\bar{g}}\subset\mathcal{B}_0^{\mathcal{H}(g_0)}\ (i\to+\infty)\quad\text{in } E_\vartheta.\end{aligned}$$

The proof is complete. □

According to Lemma 5.1 and the contradiction, we obtain the following upper semicontinuity of $\mathcal{A}_\epsilon^{\mathcal{H}(g_0)}$.

Theorem 5.2. *Let conditions* (A1)–(A6) *hold. Then*

$$\lim_{\epsilon\to0^+}\mathrm{d}_h(\mathcal{A}_\epsilon^{\mathcal{H}(g_0)},\mathcal{B}_0^{\mathcal{H}(g_0)})=0\quad\text{and}\quad lim_{\epsilon\to0^+}\mathrm{d}_h(\Pi_1\mathcal{A}_\epsilon^{\mathcal{H}(g_0)},\mathcal{A}_0^{\mathcal{H}(g_0)})=0.$$

Proof. If $\lim_{\epsilon\to 0^+} \mathrm{d}_h(\mathcal{A}_\epsilon^{\mathcal{H}(g_0)}, \mathcal{B}_0^{\mathcal{H}(g_0)}) \neq 0$, then there exist $\eta_0 > 0$ and $\{\epsilon_n\}_{n=1}^{+\infty} \subset (0,1]$ with $\epsilon_n \to 0$ as $n \to +\infty$, and $(u^{(n)}(\cdot), \omega^{(n)}(\cdot))^{\mathrm{T}} \in \mathcal{A}_{\epsilon_n}^{\mathcal{H}(g_0)}$ such that

$$\mathrm{d}_h((u^{(n)}(\cdot), \omega^{(n)}(\cdot))^{\mathrm{T}}, \mathcal{B}_0^{\mathcal{H}(g_0)}) \geq \eta_0, \quad n \in \mathbb{Z}_+. \tag{5.8}$$

From Lemma 5.1, we obtain that $\{(u^{(n)}(\cdot), \omega^{(n)}(\cdot))^{\mathrm{T}}\}_{n\in\mathbb{Z}_+}$ has a subsequence converging to a point in $\mathcal{B}_0^{\mathcal{H}(g_0)}$, which contradicts with (5.8). □

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