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COMPLETE CLASSIFICATION OF SELF-SIMILAR SOLUTIONS FOR SINGULAR POLYTROPIC FILTRATION EQUATIONS

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Abstract. This article concerns the complete classification of self-similar solutions to the singular polytropic filtration equation. We establish the existence and uniqueness of self-similar solutions of the form $u(x,t)=(\beta t)^{-\alpha/\beta}w((\beta t)^{-\frac{1}{\beta}}|x|)$, and the regularity or singularity at $x=0$, with $\alpha,\beta\in\mathbb{R}$ and $\beta=p-\alpha(1-mp+m)$. The asymptotic behaviors of the solutions near 0 or ∞ are also described. Specifically, when $\beta<0$, there always exist blow up solutions or oscillatory solutions. When $\beta>0$, oscillatory solutions appear if $\alpha>N$, $0<m<1$ and $1<p<2$. The main technical issue for the proof is to overcome the difficulty arising from the doubly nonlinear non-Newtonian polytropic filtration diffusion $\operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)$.

1. Introduction

The aim of this article is to study the singular polytropic filtration equation

$$u_t-\operatorname{div}\Big(|\nabla(|u|^{m-1}u)|^{p-2}\nabla(|u|^{m-1}u)\Big)=0, \tag{1.1}$$

in $\mathbb{R}^N$ with $N\ge 1$, $m>0$, $p>1$, and $m(p-1)<1$. We are concerned with the self-similar solutions for the equation (1.1). The equation (1.1) with $m(p-1)\le 1$ corresponds to the infinite diffusion property while the case with $m(p-1)>1$ corresponds to the finite propagation speed property. The equation (1.1) has possible singularity or degeneracy for both cases depending on both m and p. Two typical cases of the equation (1.1) with $m(p-1)<1$ are the well-known singular porous medium equation (with $m<1$ and $p=2$) and the singular p-Laplace equation (with $m=1$ and $p<2$). There exist blow-up solutions and oscillatory solutions in the singular polytropic filtration equation, which is not presented in the case $m(p-1)>1$. For the positive solutions of the equation (1.1), it can be simplified into

$$u_t-\operatorname{div}\Big(|\nabla u^m|^{p-2}\nabla u^m\Big)=0. \tag{1.2}$$

The equation (1.2) has a distinctive of practical applications in physics, chemistry, biology and other fields. Nonlinear diffusion equations, as an important area in the direction of partial differential equations, have received much attention in recent years. Many scholars study the properties of the equation (1.2), including the existence and uniqueness, the regularity and the asymptotic behavior of the solutions.

The interior and boundary Hölder continuity for bounded weak solutions for quasi-linear equation $u_t=\operatorname{div}(|u|^{m-1}|\nabla u|^{p-2}\nabla u)$ has been proved by Porzio and Vespri [28]. The existence and asymptotic behavior of the traveling wave solutions for the doubly nonlinear diffusion equation with Fisher-KPP reaction in $\mathbb{R}^{\mathbb{N}}\times(0,\infty)$ have been addressed by Audrito and Vázquez [3]. The global stability of sharp traveling waves for the degenerate porous medium equation with Fisher-KPP reaction was proved in [35]. Qualitative theory of non-linear degenerate parabolic equations related to the polytropic filtration equation (1.1) was established in [24, 25, 29]. For the fast

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diffusion case, asymptotic behaviors of the solutions were studied in [2, 11, 14, 18, 19]. Systematic analysis of qualitative theory for nonlinear parabolic equations can be found in [12, 31, 34].

Regarding the polytropic filtration equation (1.1), the extinction and nonnegative solutions for initial data has been shown by Yuan et al[37], which provided the existence of extinction solution in the degenerate case and nonnegative solution in the singular case. Jin and Yin discussed the critical exponents and non-extinction property for a nonlinear boundary value problem of the above equation (1.1) in singular case where p has a lower bound with $p \geq 1 + 1/(m+1)$ [22]. Then Li and Mu extended their results to $p > 1$ and gave the blow-up set and upper bound of the blow-up rate for the non-global solutions in [26]. From the above articles, it is clear that the smoothness of the solution of the polytropic filtration equation is not as good as that of the solution of the p-Laplace equation and the classification of the solution is more complicated.

In addition to the study of the related properties of the diffusion equation, some people have also considered the diffusion equations from the perspectives of source term, convection, etc (see for example [21]). The reason is that after introducing the source term or the convection, their interactions with diffusive phenomena may lead to more diversified asymptotic behaviors of the solutions, such as instantaneous shrinking of the supports of solution, extinction and other phenomena. Moreover, a large amount of scholars are interested in diffusion equations with sources or nonlocal sources, and studied the existence of global solutions, critical extinction and blow-up exponents and etc, see for example [16, 32, 38, 39]. Jin et al. considered for the fast diffusive polytropic filtration equation with sources, proving its critical extinction and blow-up exponents in [23]. Propagation profile for a multi-dimensional non-Newtonian polytropic filtration equation with orientated convection was investigated by Ye and Yin in [36].

It was Bidaut-Véron who first studied the complete classification of self-similar solutions to the singular p-Laplace equation in 2006 [5]. Moreover, he also considered the degenerate case in 2009 [6], and the large time behavior of the self-similar solutions for the equation with a source in the same year [7]. Classification of self-similar solutions of the degenerate polytropic filtration equations (with $m(p-1) > 1$) was investigated in [27]. Since the self-similar solutions make a great important in describing the regularity and stability of the solutions, the existence and further properties of the self-similar solutions has been well-studied in [1, 8, 9, 13, 20, 33].

Motivated by the previous works, we are looking for self-similar solutions of the singular polytropic filtration equation (1.1). We try to extend the classification results of the self-similar solutions for singular p-Laplace heat equation by Bidaut-Véron [5] to the singular polytropic filtration equation (1.1) with $m(p-1) < 1$. We note that the doubly nonlinear diffusion $\operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)$ is more complicated than the single porous medium diffusion $\operatorname{div}(\nabla u^m)$ and the single p-Laplace diffusion $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The variables related to u and ∇u are not separable in the analysis of phase portrait, particularly there is a singular or degenerate factor $|y|^{\frac{1}{m}-1}$ in the corresponding differential system (see (2.14) in Section 2). Two main difficulties arise in the classification of self-similar solutions of the singular polytropic filtration equation (1.1):

- The construction of energy functionals related only to Y and Y' is inapplicable, therefore we combine the energy functional and the Bendixson-Dulac criterion to show the non-existence result of periodic orbit or limit cycle.
- The discussion of monotonicity of Y is more subtle. Different from the previous works, we cannot determine the monotonicity property according to the comparison principle based on the sign of Y''.

The aim of this article is to present a complete description of the self-similar solutions to the singular polytropic filtration equation (1.1). We state the main results and related differential systems in Section 2. The local and global solutions together with their asymptotic behavior to the differential equations and the differential systems are formulated in Section 3. The existence, monotonicity and the asymptotic behavior of the self-similar solutions are classified in Section 4.

2. Main results and related systems

In this article, we are concerned with the self-similar solutions for the singular polytropic filtration equation (1.1) in $\mathbb{R}^N$ with $N \geq 1$, $m > 0$, $p > 1$, such that $m(p-1) < 1$. Note that for the singular p-Laplace equation with $p \in (1,2)$ (i.e., the case $m = 1$ for the polytropic filtration equation (1.1)), the classification of self-similar solutions was investigated by Bidaut-Véron [5]. Here for the singular polytropic filtration equation (1.1) with $m(p-1) < 1$, both m and p have influence on the asymptotic behavior and classification of self-similar solutions.

We define

$$\delta = \frac{mp}{1-mp+m}, \quad \eta = \frac{N-p}{p-1};$$

therefore

$$\frac{\delta - mN}{m(p-1)} = \delta - \eta = \frac{mN - \eta}{1-mp+m}. \tag{2.1}$$

Two critical values P_1, P_2 are involved in the problem

$$P_1 = \frac{N+mN}{1+mN}, \quad P_2 = \frac{N+mN}{1+m+mN};$$

they are connected to δ by the relations

$$p > P_1 \Leftrightarrow \delta > mN, \quad p > P_2 \Leftrightarrow \delta + m\delta - mN > 0.$$

If $u(x,t)$ is a solution, then for any $\alpha, \beta \in \mathbb{R}$, $u_\lambda(x,t) = \lambda^\alpha u(\lambda x, \lambda^\beta t)$ is a solution of (1.1) if and only if

$$\beta = p - \alpha(1-mp+m) = (1-mp+m)\Big(\frac{\delta}{m} - \alpha\Big),$$

noticing that $\beta > 0 \Leftrightarrow \alpha < \frac{\delta}{m}$. Given $\alpha \in \mathbb{R}$ such that $\alpha \neq \frac{\delta}{m}$, we search for self-similar solutions, radially symmetric in x, of the form

$$u = u(x,t) = (\beta t)^{-\alpha/\beta} w(r), r = (\beta t)^{-\frac{1}{\beta}}|x|. \tag{2.2}$$

The time variable t in (2.2) can be replaced by $t - T$. Hence, for any real time T, we obtain solutions defined for any $t > T$ when $\beta > 0$, or $t < T$ when $\beta < 0$. The following second order non-autonomous differential equation arises:

$$m^{p-1}\Big[(\big||w|^{m-1}w'\big|^{p-2}|w|^{m-1}w')' + \frac{N-1}{r}\big||w|^{m-1}w'\big|^{p-2}|w|^{m-1}w'\Big] + \alpha w + rw' = 0, \tag{E_w}$$

with $r \in (0,\infty)$. We will show a complete description of all the possible solutions of (1.1) with constant or changing sign.

2.1. **Explicit solutions. Solution U.** The simplest positive solutions of equation (E_w), existing for any α such that $(\delta - mN)(\delta - m\alpha) > 0$, are given by

$$w(r) = \ell^{1/m} r^{-\delta/m},$$

where

$$\ell = \Big(\delta^{p-1}\frac{\delta - mN}{\delta - m\alpha}\Big)^{\frac{m}{1-mp+m}} > 0. \tag{2.3}$$

They are associated to a unique solution u of (1.1) called U, singular at $x = 0$, for any $|t| > 0$:

$$U(x,t) = \Big(\frac{Ct}{|x|^p}\Big)^{\frac{1}{1-mp+m}}, \quad C = p\delta^{p-2}(\delta - mN). \tag{2.4}$$

Case $\alpha = N$. Equation (E_w) has a first integral

$$w + m^{p-1}r^{-1}||w|^{m-1}w'|^{p-2}|w|^{m-1}w' = Cr^{-N}. \tag{2.5}$$

All the solutions for $C = 0$ are given by

$$w = w_{K,1}(r) = (\delta^{-1}r^{p'} + K)_+^{-\frac{\delta}{mp'}},$$

such that

$$u = u_{K,1}(x,t) = (\beta_N t)^{-N/\beta_N}(\delta^{-1}(\beta t)^{-p'/\beta_N}|x|^{p'} + K)_+^{-\frac{\delta}{mp'}}, \quad K \in \mathbb{R}, \tag{2.6}$$

with $\beta = \beta_N = (\frac{\delta}{m} - N)(1-mp+m)$. Set $p > N(1-mp+m)$ and $K > 0$. The solutions are usually called *Barenblatt solutions* [4]. For every $c > 0$, the function $u_{K,1}$, defined on $\mathbb{R}^N \times (0,\infty)$, admits a unique solution of equation (1.1) with initial data $u(0) = c\delta_0$, where δ_0 is the Dirac mass at 0, and K is determined by $\int_{\mathbb{R}^N} u_K(x,t)\mathrm{d}x = c$.

Case $\alpha = \frac{\eta}{m}$. Equation (E_w) can be explicitly solved:

$$w(r) = Cr^{-\eta/m}, \quad u(t,x) = C|x|^{-\eta/m}, \quad C \neq 0. \tag{2.7}$$

The solutions u are independent of t.

Case $N = 1$ and $\alpha = \frac{p-1}{1-m(p-1)} > 0$. The equation (E_w) admits the explicit solutions in the form:

$$w(r) = \pm(Kr - m^{p-1}\alpha^{p-1}|K|^p)^{-\alpha}, \quad \text{where } K \text{is a constant.}$$

They correspond to the unique solutions u of (1.1) is

$$u(x,t) = \pm(K|x| - m^{p-1}\alpha^{p-1}|K|^p t)^{-\alpha}.$$

2.2. **Main results.** Before introducing our main results for equation (1.1), we give some definitions of different kinds of solutions in terms of regular solutions, singular solutions and solutions with a reduced domain.

Considering equation (E_w). The solution w is defined on a maximal interval (R_w, ∞) with $R_w \geq 0$, whose local existence and uniqueness near any point $r_1 > 0$ are easy to be proved. Returning to solution u of equation (1.1) corresponding to w by (2.2), it is defined on a subset of $(\mathbb{R}^N \setminus \{0\}) \times (0, \pm\infty)$:

$$D_w = \big\{(x,t) \mid x \in \mathbb{R}^N, \beta t > 0, |x| > (\beta t)^{\frac{1}{\beta}} R_w\big\}.$$

When w is defined on $(0,\infty)$, then u is defined on $(\mathbb{R}^N \setminus \{0\}) \times (0, \pm\infty)$.

(i) Regular solutions. Among the solutions of (E_w) defined near 0, we also show the existence and uniqueness of solutions $w = w(\cdot; a) \in C^2([R_w, \infty))$ with $R_w = 0$, such that for some $a \in \mathbb{R}$,

$$w(0) = a, \quad w'(0) = 0,$$

called *regular solutions*. If w is regular, then $D_w = \mathbb{R}^N \times (0, \pm\infty)$, and $u(\cdot,t) \in C^1(\mathbb{R}^N)$ for $t \neq 0$, u is called regular. This does not mean the regularity up to $t = 0$: indeed u presents a singularity at time $t = 0$ if and only if $0 < \alpha < \frac{\delta}{m}$. In the paper we would not consider the trivial solution $w \equiv 0$, associated to $a = 0$.

(ii) Singular solutions. If $R_w = 0$, and w is not regular, then u presents a singularity at $x = 0$ for $t \neq 0$, called a *standing singularity*. Following [10, 30], for such a solution, $x = 0$ is said to be of weak singularity if $x \mapsto w(|x|) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, or equivalently $u(\cdot,t) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ for $t \neq 0$; and of strong singularity if not. If u has a strong (resp. weak) singularity, and $\lim_{t\to 0} u(t,x) = 0$ for any $x \neq 0$, u is called a *strong* (resp. *weak*) *razor blade*. If $u(\cdot,t) \in L^1(\mathbb{R}^N)$ for $t \neq 0$, u is called *integrable*.

(iii) Solutions with a reduced domain. If $R_w > 0$, we will say that u and w have a reduced domain. Then D_w has a lateral boundary of the form $\Sigma_w = \{|x| = C(\beta t)^{\frac{1}{\beta}}\}$, of parabolic type if $\beta > 0$, of hyperbolic type if $\beta < 0$, and u has an explosion near Σ_w. We calculate the blow-up rate, which is of the order of $d(x,t)^{-\frac{p-1}{1-mp+m}}$, where $d(x,t)$ is the distance to Σ_w, by Proposition 3.13.

Next, let us give the main results with respect to the function u. For simplicity, we avoid the particular case (for example $N = 1$, $m\alpha = \delta$, or $p = P_1$) and do not mention the existence of solutions with a reduced domain. All of them and the detailed results in terms of the function w can be found in section 4. An important critical value of α is involved:

$$\alpha^* = \frac{\delta}{m} + \frac{\delta(mN - \delta)}{m(p-1)(m\delta + \delta - mN)},$$

which appears when $p > P_2$, such that $\alpha^* > 0$.

Remark 2.1. To return from w to u, consider any solution w of (2.28) defined on $(0,\infty)$, such that for some $\lambda \geq 0$ and $\mu \in \mathbb{R}$, $\lim_{r\to 0} r^\lambda w = c \neq 0$ and $\lim_{r\to\infty} r^\mu w = c' \neq 0$. Then

(i) For fixed t, u has a singularity at $x=0$, and $u(x)\sim O(|x|^{-\lambda})$ as $|x|\to 0$, $u(x)\sim O(|x|^{-\mu})$ as $|x|\to\infty$. Thus $x=0$ is of weak singularity if and only if $\lambda<N$, and u is integrable if and only if $\lambda<N<\mu$.

(ii) For fixed $x=0$, the behaviour of u near $t=0$, depends on the sign of β:

$$\lim_{t\to 0}|x|^{\mu}|t|^{(\alpha-\mu)/\beta}u(x,t)=C\neq 0,\quad \text{if } m\alpha<\delta,$$
$$\lim_{t\to 0}|x|^{\lambda}|t|^{(\alpha-\lambda)/\beta}u(x,t)=C\neq 0,\quad \text{if } m\alpha>\delta.$$

Our main results of the classification of self-similar solutions to the polytropic filtration equation (1.1) are listed for the cases $\beta>0$ and $\beta<0$ separately. We follow the same line as the self-similar solutions for p-Laplace equation in [5], which is the special case $m=1$ for the polytropic filtration equation (1.1). Compared with the single p-Laplace equation in [5], we note that for non-Newtonian polytropic filtration equation (1.1), there is a singular or degenerate factor $|y|^{\frac{1}{m}-1}$ in the corresponding autonomous system via logarithmic change of the variable (see the system (2.14)), which is the main cause of the difficulties of this paper.

(i) Solutions defined for $\beta>0$. Here we consider the solutions u of (1.1) on $(\mathbb{R}^N\setminus\{0\})\times(0,\infty)$ of the form (2.2), which means $t>0$, or equivalently $m\alpha<\delta$. Now we discuss with respect to the sign of $p-P_1$. For the proofs, we refer the reader to Theorems 4.2, 4.4, and 4.6.

Theorem 2.2. *Assume $-\infty<m\alpha<\delta$, and $p>P_1(N\leq 2)$. Then U is a solution on $(\mathbb{R}^N\setminus\{0\})\times(0,\infty)$, which is a strong razor blade. There exist also positive solutions with a strong singularity, and $u\sim O(|x|^{-\delta/m})$ as $|x|\to 0$, and $\lim_{t\to 0}|x|^{\alpha}u=L>0$ (for $x\neq 0$). For $\alpha\geq N$, there exists at most one zero in any function $u(\cdot,t)$ at time t.*

(1) For $\alpha<N$, the regular solutions on $\mathbb{R}^N\times(0,\infty)$ have a constant sign, are not integrable, and they are solutions of (1.1) with initial data $L|x|^{-\alpha}\in L^1_{\rm loc}(\mathbb{R}^N)$. There exist positive integrable razor blades, with $u\sim O(|x|^{-\eta/m})$ as $|x|\to 0$. There exist also positive solutions with a weak regularity, and $u\sim O(|x|^{-\eta/m})$ as $|x|\to 0$, and $\lim_{t\to 0}|x|^{\alpha}u=L$ (in particular if $\alpha=\frac{\eta}{m}$, then $u\equiv C|x|^{-\eta/m}$). There exist solutions with one zero and a weak or a strong singularity.

(2) For $\alpha=N$, the regular Barenblatt solutions are integrable with a constant sign. There exist solutions with one zero and of weak singularity.

(3) For $\alpha>N$ and $0<m<1$, $1<p<2$, the regular solutions have at least one zero. If $\alpha<\alpha^*$, then any solution has a finite number of zeros. If $N<\alpha^*$, there exist $\hat\alpha\in(\alpha,\delta)$ such that if $\hat\alpha<\alpha$, the regular solutions are oscillating around 0 for large $|x|$, and $r^{\delta}|w|^{m-1}w$ is asymptotically periodic in $\ln r$; and there exists exactly a solution u such that $r^{\delta}|w|^{m-1}w$ is periodic in $\ln r$.

Theorem 2.3. *Assume $-\infty<m\alpha<\delta$, and $p<P_1$. Then the regular solutions on $\mathbb{R}^N\times(0,\infty)$ have a constant sign, are not integrable, and are solutions of (1.1) with initial data $L|x|^{-\alpha}\in L^1_{\rm loc}(\mathbb{R}^N)$. The non-regular solutions are not defined on $(\mathbb{R}^N\setminus\{0\})\times(0,\infty)$.*

Observe that if $\alpha>0$, all the solutions w tend to 0 at ∞, while if $\alpha<0$, some of the solutions are unbounded near ∞.

(ii) Solutions defined for $\beta<0$. Next we search for the solutions of (1.1) on $(\mathbb{R}^N\setminus\{0\})\times(0,\infty)$. They are associated to $m\alpha>\delta$, equivalently $t<0$. One of our most important results is the existence of a critical value $\alpha_{crit}>0$ for $P_2<p<P_1$, for which the regular solutions u_{crit} are positive, integrable, and vanish identically at time 0. Another new phenomena is the existence of positive solutions such that $C_1U\leq u\leq C_2U$ for some $C_1,C_2>0$, with a periodic property, according to Theorem 2.5. From Theorems 4.7 when $p>P_1$ and 4.10, 4.12 and 4.14 when $p<P_1$, we deduce the following results.

Theorem 2.4. *Assume $\delta<m\alpha$, $p>P_1$ $(N\geq 2)$. Then any solution u on $(\mathbb{R}^N\setminus\{0\})\times(0,-\infty)$, in particular the regular ones, is oscillating around 0 for fixed $t<0$ and large $|x|$, and $r^{\delta}|w|^{m-1}w$ is asymptotically periodic in $\ln r$. There exist weak integrable razor blades, with $u\sim O(|x|^{-\eta/m})$ as $|x|\to 0$.*

Theorem 2.5. *Assume $\delta < m\alpha$, $p < P_1$. Then U is a solution on $(\mathbb{R}^N \setminus \{0\}) \times (0, -\infty)$, it is a weak razor blade. Moreover*

(1) If $p < P_2$, the regular solutions on $\mathbb{R}^N \times (0, -\infty)$ have a constant sign, are not integrable, and vanish identically at $t = 0$, with $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \le C|t|^{\frac{\alpha}{|\beta|}}$. All the solutions have a finite number of zeros.

(2) For $m\alpha < \eta$, the regular solutions have a constant sign, with the same behaviour (given by (2.6) if $\alpha = N$). There exists a positive solution u, which is not integrable, with $u \sim O(|x|^{-\alpha})$ as $|x| \to 0$ (a strong singularity if and only if $\alpha \ge N$), and $\lim_{t\to 0} |x|^\alpha u = L$. If $\alpha = \eta$, then $u(x, t) = C|x|^{-\eta/m}$ is a solution with a strong singularity.

(3) If $p > P_2$ and $1 < p < 2$, there exists a critical value α_{crit} such that $\frac{\eta}{m} < \alpha_{crit} < \alpha^*$ and the regular solutions u_{crit} have a constant sign, are integrable, and vanish identically at $t = 0$, with $\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \le C|t|^{\frac{\alpha}{|\beta|}}$.

(4) If $\alpha \in (\alpha_{crit}, \alpha^*)$ and $1 < p < 2$, there exist positive solutions u such that $r^\delta |w|^{m-1} w$ is periodic in $\ln r$, thus

$$C_1 U \le u \le C_2 U \text{ for some } C_1, C_2 > 0.$$

There exist positive solutions u, with the same bounds, such that $r^\delta|w|^{m-1}w$ is asymptotically periodic in $\ln r$ near 0. There exist positive integrable solutions u such that $r^\delta |w|^{m-1} w$ is asymptotically periodic near 0.

(5) If $\alpha_{crit} < \alpha$ and $1 < p < 2$, the regular solutions are oscillating around 0 for fixed $t < 0$ and large $|x|$, and $r^\delta|w|^{m-1}w$ is asymptotically periodic in $\ln r$. There exist solutions oscillating around 0, such that $r^\delta |w|^{m-1} w$ is periodic. If $\alpha^* < \alpha$, there exist positive integrable razor blades, with $u \sim O(|x|^{-\delta/m})$ as $|x| \to 0$.

2.3. **Different formulations of the problem.** Defining

$$J_N(r) = r^N (w + m^{p-1} r^{-1} ||w|^{m-1} w'|^{p-2} |w|^{m-1} w'), \quad J_\alpha(r) = r^{\alpha - N} J_N(r), \tag{2.8}$$

equation (E_w) is equivalent to the form

$$J_N'(r) = r^{N-1}(N - \alpha) w, \quad J_\alpha'(r) = -(N-\alpha) m^{p-1} r^{\alpha-2} ||w|^{m-1} w'|^{p-2} |w|^{m-1} w'. \tag{2.9}$$

If $\alpha = N$, then J_N is constant, which leads to (2.5).

Let us use the following logarithmic substitution: given $d \in \mathbb{R}$, and defined

$$y_d(\tau) = r^d |w|^{m-1} w, \quad Y_d = -m^{p-1} r^{(d+1)(p-1)} ||w|^{m-1} w'|^{p-2} |w|^{m-1} w', \quad \tau = \ln r. \tag{2.10}$$

Then the equation (E_w) is rewritten in the form

$$\begin{gathered} {y_d}' = d y_d - |Y_d|^{\frac{2-p}{p-1}} Y_d, \\ {Y_d}' = (p-1)(d-\eta) Y_d + e^{(p + d(p-1) - \frac{d}{m})\tau} |y_d|^{\frac{1}{m} - 1} (\alpha y_d - \frac{1}{m} |Y_d|^{\frac{2-p}{p-1}} Y_d). \end{gathered} \tag{2.11}$$

Thus y_d satisfies

$$\begin{aligned} & y_d'' + (\eta - 2d) y_d' - d(\eta - d) y_d \\ & + \frac{e^{(p + d(p-1) - \frac{d}{m})\tau}}{m(p-1)} |y_d' - d y_d|^{2-p} |y_d|^{\frac{1}{m} - 1} (y_d' + (m\alpha - d y_d)) = 0. \end{aligned} \tag{2.12}$$

The most important case is the special value $d = \delta$: setting $y = y_\delta$,

$$y(\tau) = r^\delta |w|^{m-1} w, \quad Y = -m^{p-1} r^{(\delta+1)(p-1)} ||w|^{m-1} w'|^{p-2} |w|^{m-1} w', \tag{2.13}$$

we are led to the autonomous system

$$\begin{gathered} y' = \delta y - |Y|^{\frac{2-p}{p-1}} Y, \\ Y' = (\frac{\delta}{m} - N) Y + |y|^{\frac{1}{m} - 1} (\alpha y - \frac{1}{m} |Y|^{\frac{2-p}{p-1}} Y). \end{gathered} \tag{2.14}$$

Hence equation (2.12) takes the form

$$\begin{aligned} &y'' - (\delta + \frac{\delta - mN}{m(p-1)})y' + \frac{\delta(\delta - mN)}{m(p-1)}y \\ &+ \frac{1}{m(p-1)}|\delta y - y'|^{2-p}|y|^{\frac{1}{m}-1}(y' + (m\alpha - \delta)y) = 0. \end{aligned} \tag{2.15}$$

Remark 2.6. Since (2.14) is autonomous, for any solution w of the problem (E_w), all the functions $w_\xi(r) = \xi^{-\gamma}w(\xi r), \xi > 0$, are also solutions.

System (2.14) will be considered by applying the phase plane analysis method, and shows our main results. Notice that the set of trajectories of system (2.14) in the phase plane (y, Y) is symmetric with respect to $(0, 0)$. In the phase plane (y, Y) we define

$$\mathcal{M} = \left\{(y, Y) \in \mathbb{R}^2 : |Y|^{\frac{2-p}{p-1}}Y = \delta y\right\}, \tag{2.16}$$

which is the set of the extremal points of y. We denote the four quadrants by

$$\mathcal{Q}_1 = (0, \infty) \times (0, \infty), \quad \mathcal{Q}_2 = (-\infty, 0) \times (0, \infty), \quad \mathcal{Q}_3 = -\mathcal{Q}_1, \quad \mathcal{Q}_4 = -\mathcal{Q}_2.$$

Remark 2.7. The vector field at any point $(0, \xi)$, $\xi > 0$ satisfies $y' = -\xi^{\frac{1}{p-1}} < 0$, thus points to $\mathcal{Q}_2$; moreover, $Y' = (\frac{\delta}{m} - N)\xi$, if $\delta > mN$, then $Y' > 0$, otherwise $Y' < 0$. The field at any point $(\varphi, 0)$, $\varphi > 0$ satisfies $Y' = \alpha\varphi^{1/m}$, thus points to $\mathcal{Q}_1$ if $\alpha > 0$ and to $\mathcal{Q}_4$ if $\alpha < 0$; moreover $y' = \delta\varphi > 0$.

Remark 2.8. The couple (y, Y) is related to J_N by the identity

$$J_N(r) = r^{N-\frac{\delta}{m}}(|y|^{\frac{1}{m}-1}y - Y), \quad \tau = \ln r, \tag{2.17}$$

and the formulae (2.8) can be rewritten again corresponding to the relations

$$\begin{aligned} (|y|^{\frac{1}{m}-1}y - Y)' &= (\frac{\delta}{m} - \alpha)(|y|^{\frac{1}{m}-1}y - Y) + (N - \alpha)Y \\ &= (\frac{\delta}{m} - N)(|y|^{\frac{1}{m}-1}y - Y) + (N - \alpha)|y|^{\frac{1}{m}-1}y. \end{aligned} \tag{2.18}$$

2.4. **Stationary points and energy functionals of system** (2.14)**.** In this subsection, we first discuss stationary points of system (2.14) and research their local behaviour in order to facilitate the following study of the behavior of trajectories in phase plane.

If $\delta = mN = m\alpha$, then system (2.14) admits an infinity many of stationary points, given by $\pm(k, (\delta k)^{p-1})$, $k > 0$. Apart from this case, if $(\delta - mN)(\delta - m\alpha) \le 0$, there exists a unique stationary point $(0, 0)$ in system (2.14). If $(\delta - mN)(\delta - m\alpha) > 0$, there exist three stationary points:

$$(0, 0), \quad M_\ell = (\ell, (\delta\ell)^{p-1}) \in \mathcal{Q}_1, \quad M_\ell{}' = -M_\ell \in \mathcal{Q}_3,$$

where $\ell = (\delta^{p-1}\frac{\delta - mN}{\delta - m\alpha})^{\frac{m}{1-mp+m}}$.

(i) Local behaviour at $(0, 0)$**.** The linearized problem at $(0, 0)$ is given by

$$y' = \delta y, \quad Y' = (\frac{\delta}{m} - N)Y,$$

and has the eigenvalues $\lambda_1 = \delta$ and $\lambda_2 = \frac{\delta}{m} - N$. Thus $(0, 0)$ is a saddle point when $\delta - mN < 0$, and a source when $\delta - mN > 0$.

(ii) Local behaviour at M_ℓ**.** Setting

$$y = \bar{y} + \ell, \quad Y = \bar{Y} + (\delta\ell)^{p-1}, \tag{2.19}$$

system (2.14) is equivalent in $\mathcal{Q}_1$ to

$$\begin{aligned} \bar{y}' &= \delta\bar{y} - \frac{(\delta\ell)^{2-p}}{p-1}\bar{Y} + \Phi(\bar{y}, \bar{Y}), \\ \bar{Y}' &= (\frac{\delta}{m} - N)\bar{Y} + \frac{1}{m^2}(m\alpha + (m-1)\delta)\ell^{\frac{1}{m}-1}\bar{y} - \frac{1}{m(p-1)}\delta^{2-p}\ell^{1+\frac{1}{m}-p}\bar{Y} + \Psi(\bar{y}, \bar{Y}), \end{aligned} \tag{2.20}$$

where

$$\Phi(\bar{y},\bar{Y}) = -(\bar{Y}+(\delta\ell)^{p-1})^{\frac{1}{p-1}} + \delta\ell + \frac{(\delta\ell)^{2-p}\bar{Y}}{p-1}, \tag{2.21}$$

and

$$\begin{aligned}\Psi(\bar{y},\bar{Y}) = & \frac{1}{m}(\bar{y}+\ell)^{\frac{1}{m}-1}[m\alpha(\bar{y}+\ell) - (\bar{Y}+(\delta\ell)^{p-1})^{\frac{1}{p-1}}] + (\frac{\delta}{m}-N)(\delta\ell)^{p-1} \\ & - \frac{1}{m^2}(m\alpha+(m-1)\delta)\ell^{\frac{1}{m}-1}\bar{y} + \frac{1}{m(p-1)}\delta^{2-p}\ell^{1+\frac{1}{m}-p}\bar{Y}, \quad \bar{Y} > -(\delta\ell)^{p-1}.\end{aligned} \tag{2.22}$$

The linearized problem is

$$\begin{gathered}\bar{y}' = \delta\bar{y} - \frac{1}{p-1}(\delta\ell)^{2-p}\bar{Y}, \\ \bar{Y}' = \frac{1}{m^2}(m\alpha+(m-1)\delta)\ell^{\frac{1}{m}-1}\bar{y} + (\frac{\delta}{m} - N - \frac{1}{m(p-1)}\delta^{2-p}\ell^{1+\frac{1}{m}-p})\bar{Y}.\end{gathered}$$

Its eigenvalues $\lambda_1 \le \lambda_2$ are the solutions of equation

$$\det(\lambda I - A) = \lambda^2 - (\delta + \frac{\delta}{m} - N - \frac{\delta}{m(p-1)}\cdot\frac{mN-\delta}{m\alpha-\delta})\lambda - \frac{p}{m(p-1)}(\delta - mN) = 0. \tag{2.23}$$

The discriminant Δ of the equation (2.23) is

$$\begin{aligned}\Delta &= (\delta + \frac{\delta}{m} - N - \frac{\delta}{m(p-1)}\frac{mN-\delta}{m\alpha-\delta})^2 + \frac{4p}{m(p-1)}(\delta - mN) \\ &= [\delta - \frac{\delta}{m} + N + \frac{\delta(mN-\delta)}{m(p-1)(m\alpha-\delta)}]^2 - \frac{4\delta(mN-\delta)}{m(p-1)(m\alpha-\delta)}\frac{m\alpha+(m-1)\delta}{m}.\end{aligned} \tag{2.24}$$

The critical value α^* of α from the discriminant Δ is

$$\alpha^* = \frac{\delta}{m} + \frac{\delta(mN-\delta)}{m(p-1)(m\delta+\delta-mN)}, \tag{2.25}$$

it appears when $\delta > \frac{m}{m+1}N$:

$$\alpha = \alpha^* \Leftrightarrow \lambda_1 + \lambda_2 = 0.$$

When $\delta < mN$, then $\delta < m\alpha$, M_ℓ is sink when $\delta \le \frac{mN}{1+m}$ or $\delta > \frac{mN}{1+m}$ and $\alpha < \alpha^*$; M_ℓ is source when $\delta > \frac{mN}{1+m}$ and $\alpha > \alpha^*$. When $\delta > mN$, then M_ℓ is always a saddle point, but as we will find the value α^* also plays an important role in the sequel.

Moreover, the sign of α^* and its position with respect to N or η are important in the sequel. By calculations,

$$\begin{aligned}\alpha^* &= \frac{p'(\delta^2 + (mN - N - m - 2)m\delta + 2m^2N)}{m(1+m)(m\delta+\delta-mN)} \\ &= \frac{\eta}{m} + \frac{(\delta-mN)^2}{m^2(p-1)(m\delta+\delta-mN)} \\ &= N + \frac{(\delta-mN)(\delta^2 - m(m+N+2)\delta + m^2N)}{m(\delta-m)(m\delta+\delta-mN)}.\end{aligned} \tag{2.26}$$

Thus $\alpha^* > \frac{\eta}{m} > 0$ if $N > p$ and $m\delta + \delta - mN > 0$, if $N = 1, \alpha^* > 0$ if $p > \frac{2+2m}{2m+1}$.

Also, when $\Delta > 0$ one can choose a basis of eigenvectors $u_1 = (-\frac{(\delta\ell)^{2-p}}{p-1}, \lambda_1 - \delta)$ and $u_2 = (\frac{(\delta\ell)^{2-p}}{p-1}, \delta - \lambda_2)$.

We can obtain the stability at the stationary points by researching their local behaviour, and then we can know the motion of the trajectories near stationary points. In order to obtain a global distribution of the trajectories, we need to construct energy functionals to analyse the properties of the solutions in the whole space.

A classical energy function is associated with equation (E_w) is

$$E(r) = \frac{1}{p'}||w|^{m-1}w'|^p + \frac{m^{1-p}\alpha}{m+1}|w|^{m-1}w^2, \tag{2.27}$$

which is non-increasing, since $E'(r) = -(N-1)r^{-1}||w|^{m-1}w'|^p - m^{1-p}r|w|^{m-1}w'^2$. It is not sufficient in the study, we construct a new energy function adapted to y and Y following [5]. Define

$$W(\tau) = W(y(\tau), Y(\tau)),$$

where

$$W(y,Y) = \frac{(m\delta+\delta-mN)\delta^{p-1}}{p}|y|^p + \frac{m|Y|^{p'}}{p'} - m\delta yY + \frac{m(m\alpha-\delta)}{m+1}|y|^{\frac{1}{m}-1}y^2. \tag{2.28}$$

It satisfies

$$\begin{aligned} W'(\tau) &= (\delta y - |Y|^{\frac{2-p}{p-1}}Y)(|\delta y|^{p-2}\delta y - Y)(m\delta+\delta-mN) - |y|^{\frac{1}{m}-1}(\delta y - |Y|^{\frac{2-p}{p-1}}Y)^2 \\ &= (\delta y - |Y|^{\frac{2-p}{p-1}}Y)(|\delta y|^{p-2}\delta y - Y)\Big(m\delta+\delta-mN - |y|^{\frac{1}{m}-1}\frac{\delta y - |Y|^{\frac{2-p}{p-1}}Y}{|\delta y|^{p-2}\delta y - Y}\Big). \end{aligned}$$

When $m\delta+\delta-mN \le 0$, W is non-increasing. When $m\delta+\delta-mN > 0$, we consider the curve

$$\mathcal{L} = \Big\{(y,Y)\in\mathbb{R}^2 : H(y,Y) = |y|^{\frac{1}{m}-1}\frac{\delta y - |Y|^{\frac{2-p}{p-1}}Y}{|\delta y|^{p-2}\delta y - Y} = m\delta+\delta-mN\Big\},$$

where by convention the quotient takes the value $\frac{\delta^{2-p}}{p-1}y^{\frac{1-mp+m}{m}}$ if $|\delta y|^{p-2}\delta y = Y$. It is a closed curve surrounding $(0,0)$, symmetric in reference to $(0,0)$. Let $S_{\mathcal{L}}$ be the domain with boundary $\mathcal{L}$ and containing $(0,0)$:

$$S_{\mathcal{L}} = \big\{(y,Y)\in\mathbb{R}^2 : H(y,Y) \le m\delta+\delta-mN\big\}. \tag{2.29}$$

Then $W'(\tau) \ge 0$ if $(y(\tau),Y(\tau)) \in S_{\mathcal{L}}$ and $W'(\tau) \le 0$ if $(y(\tau),Y(\tau)) \notin S_{\mathcal{L}}$. We notice that $S_{\mathcal{L}}$ is bounded if $1<p<2$ and $0<m<1$: indeed for any $(y,Y)\in\mathbb{R}^2$,

$$H(y,Y) \ge \frac{1}{2}((\delta y)^{2-p} + |Y|^{\frac{2-p}{p-1}})|y|^{\frac{1}{m}-1}.$$

Also $S_{\mathcal{L}}$ is connected, more precisely for any $(y,Y)\in S_{\mathcal{L}}$ and any $\theta\in[0,1]$, $(\theta y, \theta^{p-1}Y)\in S_{\mathcal{L}}$.

2.5. **Other systems for positive solutions.** When w has a constant sign, we set two functions corresponding to (y,Y):

$$\zeta(\tau) = \frac{|Y|^{\frac{2-p}{p-1}}Y}{y}(\tau) = -\frac{mrw'}{w}, \quad \sigma(\tau) = \frac{Y}{|y|^{\frac{1}{m}-1}y} = -\frac{m^{p-1}||w|^{m-1}w'|^{p-2}|w|^{m-1}w'}{rw}. \tag{2.30}$$

Thus ζ describes the behaviour of $\frac{w'}{w}$. They convert (2.14) into system

$$\begin{aligned} \zeta' &= \zeta(\zeta-\eta) + \frac{m\alpha-\zeta}{m(p-1)}|\zeta|^{2-p}|y|^{1-p+\frac{1}{m}} = \zeta\Big(\zeta-\eta+\frac{m\alpha-\zeta}{m(p-1)\sigma}\Big), \\ \sigma' &= (\alpha-N) + \Big(\frac{1}{m}|\sigma|^{\frac{2-p}{p-1}}\sigma|y|^{\frac{1-mp+m}{m(p-1)}}N\Big)(\sigma-1) = (\alpha-N) + \Big(\frac{\zeta}{m}-N\Big)(\sigma-1). \end{aligned} \tag{2.31}$$

In particular, System (2.31) provides a short proof of the local existence and uniqueness of the regular solutions, corresponding to its stationary point $(0,\frac{\alpha}{N})$.

Also, if w and w' have a strict constant sign, we can define a new function in any quadrant $\mathcal{Q}_i$: setting

$$\psi = \frac{1}{\sigma} = \frac{|y|^{\frac{1}{m}-1}y}{Y}.$$

then we obtain a new system on (ζ,ψ):

$$\begin{aligned} \zeta' &= \zeta\Big(\zeta-\eta+\frac{(m\alpha-\zeta)\psi}{m(p-1)}\Big), \\ \psi' &= \psi\Big(N-\frac{\zeta}{m}+\psi(\frac{\zeta}{m}-\alpha)\Big). \end{aligned} \tag{2.32}$$

That means we can convert system (2.31) into a polynomial system (2.32) without singularity. We get two stationary points $(\eta,0)$ and $(0,\pm\infty)$ from system (2.32).

We also consider another system in any quadrant $\mathcal{Q}_i$: setting

$$\zeta = -\frac{1}{g}, \quad \sigma = -s, \quad \mathrm{d}\tau = gs\,\mathrm{d}\nu = |y|^{\frac{1-m}{m}}|Y|^{\frac{p-2}{p-1}}\,\mathrm{d}\nu,$$

then we are led to the new system

$$\begin{aligned} \frac{\mathrm{d}g}{\mathrm{d}\nu} &= g\Big(s(1+\eta g) + \frac{1+m\alpha g}{m(p-1)}\Big), \\ \frac{\mathrm{d}s}{\mathrm{d}\nu} &= -s\Big(\frac{1}{m} + \alpha g + s(\frac{1}{m} + Ng)\Big). \end{aligned} \tag{2.33}$$

We obtain stationary points $(m\alpha, 0)$, $(\infty, 1)$ and $(\pm\infty, 0)$ from system (2.33).

3. Global solutions

We study the differential equation (E_w) and the related differential systems in Section 2. Most of the proofs of this section are parallel and similar to that of self-similar solution for p-Laplace equations in [5], except that the non-Newtonian polytropic filtration diffusion $\operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)$ in (1.1) changes the asymptotic behaviors, such that there is a singular or degenerate factor $|y|^{\frac{1}{m}-1}$ in the corresponding differential system (2.14), which causes the main difficulties of this paper.

3.1. **Existence of solutions of equation** (E_w).

Proposition 3.1. *For any $r_1 > 0$ ($r_1 \geq 0$, if $N = 1$), and $a, b \in \mathbb{R}$. There exists a unique solution w to the equation (E_w) in a neighborhood $\mathcal{V}$ of r_1, such that $w \in C^2(\mathcal{V})$ and $w(r_1) = a$, $w'(r_1) = b$. And w can extend to a maximal interval of the form (R_w, ∞), where $R_w \geq 0$.*

Moreover if $R_w > 0$, then w is monotone near R_w with an infinite limit.

Proof. We first prove the local existence and uniqueness. If $0 < m < 1$, and $1 < p < 2$, local existence and uniqueness near $r_1 > 0$ follow directly from the existence and uniqueness theorem applied to equation (2.14), since the map $\xi \mapsto f_p(\xi) = |\xi|^{\frac{2-p}{p-1}}\xi$ and $\eta \mapsto g_m(\eta) = |\eta|^{\frac{1}{m}-1}$ are of class C^1. If N=1, setting $r_1 = 0$, we can get a local solution in a neighborhood of 0 in $\mathbb{R}$ and extend it to $[0, \infty)$.

Consider the case of $1 < p < 2$ and $1 < m < \frac{1}{p-1}$. For $a = 0$, the system is a priori singular on the line $y = 0$ since $m > 1$. Notice that it is only singular at $(0, 0)$. In fact, near any point $(0, \xi)$ with $\xi \neq 0$, we can consider y as a variable, and

$$\frac{\mathrm{d}Y}{\mathrm{d}y} = F(y, Y), \quad F(Y, y) = \frac{(\frac{\delta}{m} - N)Y + |y|^{\frac{1}{m}-1}(\alpha y - \frac{1}{m}|Y|^{\frac{2-p}{p-1}}Y)}{\delta y - |Y|^{\frac{2-p}{p-1}}Y},$$

where F is continuous in y and C^1 in Y, hence local existence and uniqueness hold.

If $0 < m < 1$, and $2 < p < \frac{1}{m} + 1$, and if $b = 0$, the system is only singular at $(0, 0)$. In fact, near any point $(\xi, 0)$ with $\xi \neq 0$, we can consider Y as a variable, and

$$\frac{\mathrm{d}y}{\mathrm{d}Y} = G(Y, y), \quad G(Y, y) = \frac{\delta y - |Y|^{\frac{2-p}{p-1}}Y}{(\frac{\delta}{m} - N)Y + |y|^{\frac{1}{m}-1}(\alpha y - m^{-1}|Y|^{\frac{2-p}{p-1}}Y)},$$

in the same way, where G is continuous in Y and C^1 in y, thus local existence and uniqueness hold.

Next, we show that any local solution around r_1 can extend uniquely to a maximal interval of the form (R_w, ∞) with $R_w \geq 0$ and the solution is monotone near R_w with an infinite limit. These are proved in the same way as Theorem 2.2(i) in [5]. Here we omit the details for the sake of simplicity. $\square$

Notation For any point $P_0 = (y_0, Y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $\mathcal{T}_{[P_0]}$ denotes the unique trajectory in the phase plane (y, Y) of system (2.14) running through P_0. By symmetry, $\mathcal{T}_{[-P_0]} = -\mathcal{T}_{[P_0]}$.

Theorem 3.2. *(i) For any $a \in \mathbb{R}$, $a \neq 0$, the equation (E_w) admits a unique solution $w = w(\cdot, a)$ in an interval $[0, r_0)$, such that $w \in C^1([0, r_0))$ and $||w|^{m-1}w'|^{p-2}|w|^{m-1}w' \in C^1([0, r_0))$ and*

$$w(0) = a, \quad w'(0) = 0;$$

and then

$$\lim_{r\to 0} \frac{||w|^{m-1}w'|^{p-2}|w|^{m-1}w'}{rw} = -\frac{m^{1-p}\alpha}{N}. \tag{3.1}$$

It implies that in the phase plane (y, Y), a unique trajectory corresponding to the solution is denoted by $\mathcal{T}_r$, satisfying $\lim_{\tau\to-\infty} y = 0$ and $\lim_{\tau\to-\infty} \frac{Y}{|y|^{\frac{1}{m}-1}y} = \frac{\alpha}{N}$.

(ii) If $N \geq 2$, any solution defined near 0 and bounded is regular. If $N = 1$, it satisfies $\lim_{r\to 0} w' = b \in \mathbb{R}$, and $\lim_{r\to 0} w = a \in \mathbb{R}$.

Proof. The idea of the proof is from [7], we show the process of the proof here for the sake of completeness.

(i) Assume $a > 0, w > 0$. Let $\rho > 0$. From (2.8) and (2.9), any regular solution $w \in C^1([0, \rho])$ of the problem, such that $\big||w|^{m-1}w'\big|^{p-2}|w|^{m-1}w' \in C^1([0, \rho])$, satisfies $w = T(w)$, where

$$T(w(r)) = \Big(a^m + m\int_0^r |H(w)|^{\frac{2-p}{p-1}} H(w)\, \mathrm{d}s\Big)^{1/m},$$

and

$$H(w(r)) = -m^{1-p}(rw - r^{1-N}J_N(r)) = -m^{1-p}\Big(rw - r^{1-N}\int_0^r s^{N-1}j(w(s))\, \mathrm{d}s\Big), \tag{3.2}$$

with $j(w) = (N - \alpha)w$. Conversely, the mapping T is well defined from $C([0, \rho])$ into itself. If $w \in C([0, \rho])$, and $w = T(w)$, then $w \in C^1((0, \rho])$, and $\big||w|^{m-1}w'\big|^{p-2}|w|^{m-1}w' = H(w)$; then $|w^{m-1}w'|^{p-2}w^{m-1}w' \in C^1((0, \rho])$ and w satisfies (E_w) in $(0, \rho]$. Furthermore, $\lim_{r\to 0} j(w(r)) = (N - \alpha)a$. Then $||w|^{m-1}w'|^{p-2}|w|^{m-1}w' = -\frac{m^{1-p}\alpha a}{N} r(1 + o(1))$. In particular, $\lim_{r\to 0} w'(r) = 0$, and

$$\big||w|^{m-1}w'\big|^{p-2}|w|^{m-1}w' \in C^1([0, \rho]).$$

From (E_w),

$$\lim_{r\to 0} \frac{||w|^{m-1}w'|^{p-2}|w|^{m-1}w'}{rw} = -\frac{m^{1-p}\alpha}{N}.$$

We consider the ball

$$\mathcal{B}_{R,M} = \big\{w \in C([0, \rho]) : \|w - a\|_{C([0,R])} \leq M\big\},$$

where M is a parameter such that $0 < M < \frac{a}{2}$. Notice that j is continuous, hence T is a strict contraction from $\mathcal{B}_{\rho,M}$ into itself for ρ and M small enough. For any $w \in \mathcal{B}_{\rho,M}$, and any $r \in [0, \rho]$ $(\rho \leq R)$, from (3.2),

$$-m^{1-p}\Big(a + M - \frac{N - \alpha}{N}\,(a - M)\Big)r \leq H(w(r)) \leq -m^{1-p}\Big(a - M - \frac{N - \alpha}{N}(a + M)\Big)r.$$

We define $\mu(a) = \Big(1 + \frac{|N-\alpha|}{N}\Big)\,(a + M) > 0$, then

$$|H(w(r))| < m^{1-p}\mu(a)r.$$

Thus we obtain

$$\|T(w) - a\|_{C([0,R])} \leq \mu(a)^{\frac{1}{m(p-1)}}\rho^{\frac{p}{m(p-1)}},$$

and $T(w) \in \mathcal{B}_{\rho,M}$ for $\rho = \rho(a)$ small enough.

Now for any $w_1, w_2 \in \mathcal{B}_{\rho,M}$, and any $r \in [0, \rho]$, then

$$|T^m(w_1(r)) - T^m(w_2(r))| \leq m\int_0^r ||H(w_1)|^{\frac{2-p}{p-1}} H(w_1) - |H(w_2)|^{\frac{2-p}{p-1}} H(w_2)|(s)\, \mathrm{d}s.$$

For any $s \in [0,r]$, we have

$$\begin{aligned}
&||H(w_1)|^{\frac{2-p}{p-1}}H(w_1) - |H(w_2)|^{\frac{2-p}{p-1}}H(w_2)|(s) \\
&\le m^{p-2}\mu(a)^{\frac{2-p}{p-1}}s^{\frac{2-p}{p-1}}|H(w_1) - H(w_2)|(s) \\
&\le m^{-1}\mu(a)^{\frac{2-p}{p-1}}s^{\frac{1}{p-1}}\Big(|w_1 - w_2| + |N-\alpha|s^{-N}\int_0^s \sigma^{N-1}|w_1 - w_2|\mathrm{d}\sigma\Big) \\
&\le m^{-1}C(a)s^{\frac{1}{p-1}}\|w_1 - w_2\|_{C[0,\rho]}
\end{aligned} \tag{3.3}$$

with $C(a) = \mu(a)^{\frac{2-p}{p-1}}(1 + \frac{|N-\alpha|}{N})$, so

$$\|T^m(w_1(r)) - T^m(w_2(r))\|_{C([0,\rho])} \le C(a)\rho^{p'}\|w_1 - w_2\|_{C([0,\rho])} \le \frac{1}{2}\|w_1 - w_2\|_{C([0,\rho])},$$

if $\rho(a)$ is small enough. Then T^m is strict contradiction from $\mathcal{B}_{\rho,M}$ into itself, thus T is strict contradiction from $\mathcal{B}_{\rho,M}$ into itself. Hence existence and uniqueness hold in $[0,\rho]$.

(ii) If w is defined in $(0,\rho)$ and bounded, then $J_N' = r^N(N-\alpha)w$ is integrable; suppose $\lim_{r\to 0} J_N(r) = l$, then $||w|^{m-1}w'|^{p-2}|w|^{m-1}w' = m^{1-p}lr^{1-N}(1+o(1))$. If $N \ge 2$, it implies $l = 0$. This shows w is regular. Indeed if $l \ne 0$, either l is finite or $l = \infty$, then w' converge to ∞ and w converge to ∞, which contradicts to that w is bounded. If $N = 1$, then it admits $\lim_{r\to 0} w' = b$, and $\lim_{r\to 0} w = a \in \mathbb{R}$. □

Definition 3.3. In the plane (y,Y), the trajectory $\mathcal{T}_r$ starting from $(0,0)$ at $-\infty$ and its opposite $-\mathcal{T}_r$ are called regular trajectories. We shall say that y is regular. Notice that $\mathcal{T}_r$ starts in $\mathcal{Q}_1$ if $\alpha > 0$, and in $\mathcal{Q}_4$ if $\alpha < 0$.

Remark 3.4. From Theorem 3.2 and Remark **??**, all regular solutions are obtained from one of them: $w(r,a) = aw(a^{\frac{1}{\delta}}r, 1)$. Thus they have the same behaviour near ∞.

3.2. **Sign properties.** Next we will present more details on the zeros of w or w', by applying the monotonicity properties of the functions y_d and Y_d, in particular y, Y, ζ and σ. At any extremal point τ, they satisfy the following differential equations respectively

$$m(p-1)y_d''(\tau) = (md(p-1)(\eta - d) + (d - m\alpha)e^{(p+d(p-1)-\frac{d}{m})\tau}|dy_d|^{2-p}|y_d|^{\frac{1-m}{m}})y_d, \tag{3.4}$$

$$m(p-1)y''(\tau) = (-\delta(\delta - mN) + (\delta - m\alpha)|\delta y|^{2-p}|y|^{\frac{1-m}{m}})y = -m|Y|^{\frac{2-p}{p-1}}Y', \tag{3.5}$$

$$m^2Y''(\tau) = m\alpha|y|^{\frac{1-m}{m}}y' + (m-1)|y|^{\frac{1-m}{m}}y'\zeta, \tag{3.6}$$

$$\begin{aligned}
m^2(p-1)\zeta''(\tau) &= (1 - mp + m)(m\alpha - \zeta)|\zeta|^{2-p}|y|^{1-p+\frac{1}{m}}y^{-1}y' \\
&= (1 - mp + m)(m\alpha - \zeta)(\delta - \zeta)|\zeta|^{2-p}|y|^{\frac{1-mp+m}{m}},
\end{aligned} \tag{3.7}$$

$$m^2(p-1)\sigma''(\tau) = (1 - mp + m)(\sigma - 1)|\sigma|^{\frac{2-p}{p-1}}Y|y|^{\frac{2-2mp+2m-p}{m(p-1)}}y' = m(p-1)(\sigma - 1)\zeta'. \tag{3.8}$$

Proposition 3.5. *Let $w \not\equiv 0$ be any solution of (E_w).*

(i) *If $\alpha \le \max(N, \frac{\eta}{m})$, then w has at most one zero, and no zero if w is regular.*
(ii) *If $N < \max(\frac{\delta}{m}, \alpha)$ and w is regular, it has at least one zero.*

Proof. (i) Let us consider two consecutive zeros $\rho_0 < \rho_1$ of w, with $w > 0$ on (ρ_0, ρ_1), thus $w'(\rho_1) < 0 < w'(\rho_0)$. If $\alpha \le N$, then

$$\begin{aligned}
J_N(\rho_1) - J_N(\rho_0) &= -m^{p-1}(\rho_1^{N-1}||w(\rho_1)|^{m-1}w'(\rho_1)|^{p-1} + \rho_2^{N-1}||w(\rho_2)|^{m-1}w'(\rho_2)|^{p-1}) \\
&= (N-\alpha)\int_{\rho_0}^{\rho_1} s^{N-1}w\,\mathrm{d}s,
\end{aligned}$$

which is contradictory; hence w has at most one zero. If w is regular with $w(0) > 0$, and ρ_1 is a first zero, then

$${}_N(\rho_1) = (N-\alpha)\int_0^{\rho_1} s^{N-1}w\,\mathrm{d}s \ge 0,$$

we also get a contradiction. Next suppose $0 < m\alpha < \eta$ and apply the substitution (2.10), with $d = m\alpha$. Then there exists at most one zero in $y_{m\alpha}$. In fact, if $y_{m\alpha}$ has a maximal point τ where is positive, and is not constant, then following (3.4)

$$y''_{m\alpha}(\tau) = m\alpha(\eta - m\alpha)y_{m\alpha}(\tau), \tag{3.9}$$

$y''_{m\alpha} > 0$, which is a contradictory. Similarly, the regular solution satisfies $\lim_{\tau\to-\infty} y_{m\alpha} = 0$ for $\alpha > 0$, and $y_{m\alpha}$ has no maximal point, hence $y_{m\alpha}$ is positive and increasing.

(ii) If $w > 0$ on $[0,\infty)$. Consider the case $N < \alpha$, $J_N(r) = (N-\alpha)\int_0^r s^{N-1}w\,\mathrm{d}s < 0$, thus $w^{m-1-\frac{1}{p-1}}w' + \frac{1}{m}r^{\frac{1}{p-1}} < 0$. The function $r \mapsto \delta^{-1}r^{p'} - w^{m-\frac{1}{p-1}}$ is non-increasing and $w = O(r^{-\delta/m})$ at ∞, then y is bounded at ∞. For any $r \geq 1$, we have $J_N(r) \leq J_N(1) < 0$, since $J'_N(r) < 0$. Thus $y^{\frac{1}{m}}(\tau) + |J_N(1)|e^{(\frac{\delta}{m}-N)\tau} \leq Y(\tau)$ for any $\tau \geq 0$. Then $\lim_{\tau\to\infty} Y = \infty$, thus $\lim_{\tau\to\infty} y' = -\infty$ from (2.14), which is contradictory. Hence, ρ_1 is the first zero of w, which means w has at least one zero, and $J_N(\rho_1) < 0$, $w' < 0$. □

Proposition 3.6. *Let y be any non-constant solution of (2.15), the corresponding solution w is defined on the maximal interval (R_w,∞) with $R_w \geq 0$, where $(y,Y) \neq (0,0)$, and s be an extremity of the maximal interval.*

(i) *If y has a constant sign near s, then Y also has a constant sign.*
(ii) *If y is strictly monotone near s, then Y,ζ,σ are also monotone near s.*
(iii) *If y is not strictly monotone near s, then $s = \pm\infty$, $\delta < mN < m\alpha$, y oscillates around ℓ.*

Proof. (i) The function w has at most one extremal point on (R_w,∞). Indeed, if $w > 0$, r is extremal point, we find $(|w^{m-1}w'|^{p-2}w^{m-1}w')' = -m^{1-p}\alpha w, \alpha \neq 0$, then if $\alpha < 0$, we know w' is monotone increasing. Since w' is continuous, r is a unique point. From (2.13), Y has a constant sign near s.

(ii) Assume y is strictly monotone near s. Firstly, consider the function ζ, which satisfies (2.31). If there are τ_0 such that $\zeta'(\tau_0) = m\alpha(m\alpha - \eta)$. If $m\alpha \neq \eta$, then τ_0 is unique, since ζ' is continuous, thus $m\alpha - \zeta$ has a constant sign near s. Then at any extremal point τ of ζ, $\zeta''(\tau)$ also has a constant sign from (3.7). Thus ζ is strictly monotone near s. If $m\alpha = \eta$, one gets $\zeta \equiv m\alpha$. Next consider Y. At the extremal point τ of Y, then $m^2Y''(\tau) = m\alpha|y|^{\frac{1-m}{m}}y' + (m-1)|y|^{\frac{1-m}{m}}y'\zeta$ for (3.6). We know Y'' is of constant sign near τ, since y and ζ are monotone. Then τ is unique and Y is monotone near s. At last consider σ, which satisfies (2.31). If there are τ_0 satisfying $\sigma(\tau_0) = 1$, then $\sigma'(\tau) = \alpha - N$. If $\alpha \neq N$, then τ_0 is unique, and $\sigma - 1$ has a constant sign near s. Thus at the extremal point τ of σ, $\sigma''(\tau)$ has a constant sign from (3.8). Then σ is strictly monotone near s. If $\alpha = N$, then $\sigma(\tau) \equiv 1$.

(iii) Assume y is not strictly monotone near s. Then we can find a strictly monotone sequence $\{\tau_n\}_{n=1}^{\infty}$ with converging to s, such that $y'(\tau_n) = 0$, $y''(\tau_{2n+1}) < 0 < y''(\tau_{2n})$. Since $y(\tau_n) = \delta^{-1}|Y|^{\frac{2-p}{p-1}}Y$, we deduce $Y > 0$ near s from (i). From (3.5), one know

$$(\delta - m\alpha)\delta^{2-p}y(\tau_{2n+1})^{1+\frac{1}{m}-p} < \delta(\delta - mN) < (\delta - m\alpha)\delta^{2-p}y(\tau_{2n})^{1+\frac{1}{m}-p},$$

thus $\delta < \min(mN, m\alpha)$, and $y(\tau_{2n}) < \ell < y(\tau_{2n+1})$. It cannot happen if s is finite, since y tends to ∞. It is also impossible when $\alpha \leq N$. In fact there exist at least two points $\theta_1 < \theta_2$ such that $y(\theta_1) = y(\theta_2) = \ell$ and $y > \ell$ on (θ_1,θ_2), $y'(\theta_1) > 0 > y'(\theta_2)$. According to the system (2.14), $Y(\theta_1) < (\delta\ell)^{p-1} < Y(\theta_2)$. From (2.18), $(e^{(N-\frac{\delta}{m})\tau}(y^{1/m} - Y))' = (N-\alpha)e^{(N-\frac{\delta}{m})\tau}y^{1/m}$, and we know one solution of (2.14) is the constant $(\ell, (\delta\ell)^{p-1})$, hence

$$(e^{(N-\frac{\delta}{m})\tau}(y^{1/m} - \ell^{1/m} - Y + (\delta\ell)^{p-1}))' = (N-\alpha)e^{(N-\frac{\delta}{m})\tau}(y^{1/m} - \ell^{1/m}) > 0$$

on (θ_1,θ_2). It is contradictory to integrate on this interval. □

3.3. **Behaviour of w near 0 or ∞.** Here we suppose y is monotone, such that w has constant sign near 0 or ∞, we can assume $w > 0$.

Proposition 3.7. *Let (y,Y) be a solution of system(2.14), such that y is strictly monotone and $y > 0$ near $s = \pm\infty$. Then*

$$\lim_{\tau\to s}\zeta = \lambda, \quad \lambda = 0, m\alpha, \eta, \text{or } \delta.$$

Moreover, one of the following eventualities holds

(i) (y, Y) *converges to a stationary point different from* $(0,0)$*, then* $\lambda = \delta$*, and* $(\delta - mN)(\delta - m\alpha) > 0$ *or* $\delta = m\alpha = mN$.

(ii) (y, Y) *converges to* $(0,0)$*, then*
- *either* $\lambda = 0, s = -\infty$*, and* y *is regular, or N=1.*
- *or* $\lambda = \eta$*, then either* $(s = \infty, \delta < mN)$ *or* $(s = \infty, \delta = mN, \alpha - N < 0)$ *or* $(s = -\infty, \delta > mN)$ *or* $(s = -\infty, \delta = mN, \alpha - N > 0)$.

(iii) $\lim_{\tau \to s} y = \infty$*, and* $\lambda = m\alpha$*, then either* $(s = \infty, m\alpha < \delta)$ *or* $(s = \infty, \delta = m\alpha, \delta - mN < 0)$ *or* $(s = -\infty, m\alpha > \delta)$ *or* $(s = -\infty, \delta = m\alpha, \delta - mN > 0)$.

Proof. In the case, the function y is monotone, then Y, σ, ζ are also monotone from Proposition 3.6, thus ζ has a limit $\lambda \in [-\infty, \infty]$ and σ has a limit $\mu \in [-\infty, \infty]$, and (y, Y) converges to a stationary point, or $\lim y = \infty$; then $\lim |Y| = \infty$, from system (2.14). To use the L'Hospital's rule, we consider the two quotients

$$\frac{Y'}{(y^{1/m})'} = \frac{(\delta - mN)\sigma + m\alpha - \zeta}{\delta - \zeta} \tag{3.10}$$

and

$$\frac{(|Y|^{\frac{2-p}{p-1}}Y)'}{y'} = \frac{\zeta(\delta - mN + (m\alpha - \zeta)/\sigma)}{m(p-1)(\delta - \zeta)} = \frac{\zeta(\delta - mN) + (m\alpha - \zeta)|\zeta y|^{2-p}|y|^{\frac{1}{m}-1}}{m(p-1)(\delta - \zeta)}. \tag{3.11}$$

(i) If $(\delta - mN)(\delta - m\alpha) > 0$ and (y, Y) converges to $(\ell, (\delta\ell)^{p-1})$, then obviously $\lambda = \delta$. If $\delta = m\alpha = mN$ and $\lim_{\tau \to s} y = k > 0$, then $\lim_{\tau \to s} Y = (\delta k)^{p-1}$, thus $\lambda = \delta$.

(ii) (y, Y) converges to $(0,0)$. Then λ is finite. Indeed, if $\lambda = \pm\infty$, then $\mu = 0$, the quotient (3.11) converges to $\frac{mN-\delta}{m(p-1)}$, since $|\zeta y| = |Y|^{\frac{1}{p-1}} = o(y^{1/m})$ and y tends to 0, thus $\zeta = \frac{|Y|^{\frac{2-p}{p-1}}Y}{y}$ has the same limit, applying L'Hospital's rule, which is impossible.

• If $mN < \delta$, then $(0,0)$ is a source, so $s = -\infty$. If μ is finite, either $\mu = 0$, then $\lambda = m\alpha$, from the quotient (3.10). The quotient (3.11) converges to $\frac{(\delta - mN)\alpha}{(p-1)(\delta - m\alpha)}$, as mentioned above, we know $|\zeta y| = o(y^{1/m})$ and y tends to 0, following L'Hospital's rule, which is contradictory; or $\mu = \frac{\alpha}{N}$, and $|\zeta|^{p-1} = |\mu| y^{\frac{1-mp+m}{m}}(1 + o(1))$, then $\lambda = 0$ and w is regular. If $\mu = \pm\infty$, then $\lambda = \frac{\lambda(\delta - mN)}{m(p-1)(\delta - \lambda)}$ from quotient (3.11), thus $\lambda = 0$ or $\lambda = \eta$. If $\lambda = 0$, then $\frac{\zeta'}{\zeta} \to -\eta$ from system (2.31), and $s = -\infty$, thus necessarily $\eta < 0$, which means $N = 1$.

• If $mN > \delta$ (thus $N \geq 2$), we know $(0,0)$ is a saddle point, thus either $s = -\infty$ and $\mu = \frac{\alpha}{N}, \lambda = 0$, then w is regular. Or $s = \infty, \mu = \pm\infty$, then $\lambda = 0$ or $\lambda = \eta$. Now if $\lambda = 0$, the quotient (3.10) converges to $\mp\infty$, we obtain a contradiction. Thus $\lambda = \eta$.

• If $mN = \delta$ (thus $N \geq 2$), either $\lambda = 0$, thus $y' = mNy(1 + o(1)) > 0$, thus $s = -\infty$ and $\mu = \frac{\alpha}{N}$ from quotient (3.10). Or $\lambda > 0$, then $\lambda = \delta = mN = \eta$ from (3.11). Furthermore if $s = \infty$, then $\alpha - N < 0$; if $s = -\infty$, then $\alpha - N > 0$. In fact, $(y^{1/m} - Y)' = (N - \alpha)y^{1/m}$ can be rewritten as $(Y - y^{1/m})' = (\alpha - N)y^{1/m}$, and $y = m^{-1}N^{-1}Y^{\frac{1}{p-1}}(1 + o(1))$, $(Y - y^{1/m}) = Y(1 + o(1))$ since $\lambda = \eta = mN$ and $\mu = \infty$, then

$$(Y - y^{1/m})' = (\alpha - N)m^{-1/m}N^{-1/m}Y^{\frac{1}{m(p-1)}} = (\alpha - N)m^{-1/m}N^{-1/m}(Y - y^{1/m})^{\frac{1}{m(p-1)}}.$$

If $s = -\infty$, and $\alpha - N \leq 0$ or $s = \infty$, and $\alpha - N \neq 0$, which is impossible.

(iii) y tends to ∞. If $s = \infty$, then $y' > 0$, for $y' = y(\delta - \zeta) > 0$, thus $\zeta < \delta$; if $s = -\infty$, then $y' < 0$, thus $\zeta > \delta$. If $\lambda = \pm\infty$, we find the quotient (3.11) converges to ∞; thus $\lambda = \infty$ and $s = -\infty$. In any case, $\zeta' < 0$, thus $|\mu| \leq \frac{1}{m(p-1)}$ from system (2.31), then $|\mu| = 1$ from (3.10). But $Y' = -\frac{1}{m}|Y|^{\frac{1-mp+m}{m(p-1)}}Y(1 - m\alpha \cdot o(1) - (\delta - mN)|Y|^{-\frac{1-mp+m}{m(p-1)}})$ since system (2.14), a contradiction follows by integration. Thus λ is finite, and $\lambda \neq 0$. Indeed, if $\lambda = 0$, then $\mu = 0$, since $\sigma = |\zeta|^{p-2}\zeta y^{-\frac{1-mp+m}{m}}$, but $\mu = \frac{m\alpha}{\delta}$, from (3.10), which is a contradiction.

• If $m\alpha \neq \delta$, then $\lambda = m\alpha$ or $\lambda = \delta$, from (3.11). In turn $\sigma = |\lambda|^{p-2}\lambda y^{-\frac{1-mp+m}{m}}(1 + o(1))$, thus $\mu = 0$. Necessarily $\lambda = m\alpha$. Indeed, if $\lambda = \delta$, ζ converges to ∞ from (3.11), which is contradictory. And if $s = \infty$, then $y' > 0$, thus $\zeta < \delta$, hence $m\alpha < \delta$. If $s = -\infty$, then similarly $m\alpha > \delta$.

• If $m\alpha=\delta$, then $\lambda=m\alpha=\delta\neq mN$, and $(\frac{\delta}{m}-N)(\delta-\zeta)>0$ from (3.11); thus if $s=\infty$, then $(\delta-mN)<0$; if $s=-\infty$, then $\delta-mN>0$. $\square$

Next in order to show a precise behaviour of w in all cases, we improve Proposition 3.7.

Proposition 3.8. *Under the assumptions of Proposition 3.7, let w be the solution of (E_w) associated to y by* (2.13).

(i) *If $\lambda=\delta$, and $(\delta-mN)(\delta-m\alpha)>0$ or $\delta=mN=m\alpha$, then (near 0, or ∞),*
$$\lim r^{\delta}|w|^{m-1}w=\ell.$$
(ii) *If $\lambda=m\alpha\neq\delta$, then (near 0, or ∞) $\lim r^{\alpha}w=L>0$.*
(iii) *If $\lambda=\eta>0,\eta\neq mN$, then $\lim r^{\frac{\eta}{m}}w=c>0$.*
(iv) *If $\lambda=m\alpha=\delta\neq mN$, then*
$$\lim r^{\frac{\delta}{m}}(\ln r)^{-\frac{1}{1-mp+m}}w=\kappa=:\Big(\frac{1}{m}|mN-\delta|\delta^{p-1}(1-mp+m)\Big)^{\frac{1}{1-mp+m}}. \tag{3.12}$$
(v) *If $\lambda=\eta=mN=\delta\neq m\alpha$, then*
$$\lim r^{N}(\ln r)^{\frac{1+mN}{1+m}}w=\tilde{\rho}=:\Big(\frac{1}{mN}\Big)^{1/m}\Big((mN)^{1/m}\,\frac{mN-m}{(1+m)|\alpha-N|}\Big)^{\frac{1+mN}{1+m}}. \tag{3.13}$$
(vi) *If $N=1,\lambda=\eta=-1$ or $\lambda=0$ (near 0) then*
$$\lim_{r\to0}w=a\in\mathbb{R},\ \lim_{r\to0}w'=b; \tag{3.14}$$
and $b\neq0$, and $a=0$ (thus $b>0$) if $\lambda=-m$.

Proof. (i) This follows directly from (2.13) and Proposition 3.7.

(ii) Let $\lambda=m\alpha\neq\delta$. From system (2.31), $rw'=-\alpha w(1+o(1))$. Next we employ Proposition 3.7:

• Either $s=\infty$ and $m\alpha<\delta$; thus for any $\gamma>0$, $w=O(r^{-\alpha+\gamma})$, $\frac{1}{w}=O(r^{\alpha+\gamma})$ near ∞ and $w'=O(r^{-\alpha-1+\gamma})$, $J'_\alpha=O(r^{\alpha(1-mp+m)-p-1+\gamma})$, thus J'_α is integrable, hence J_α has a limit L, and $\lim r^{\alpha}w=L$, for $\mu=0$ and $J_\alpha(r)=r^{\alpha}w(1+o(1))$. If $L=0$, then $r^{\alpha}w=O(r^{\alpha(1-mp+m)-p+\gamma})$, hence $w\leq r^{\alpha(1-mp+m)-p-\alpha+\gamma}$ and $\frac{1}{w}\geq r^{-\alpha(1-mp+m)+p+\alpha+\gamma}$, we reach a contradiction by the estimate of $\frac{1}{w}=O(r^{\alpha+\gamma})$ for γ small enough. Thus $L>0$.

• Or $s=-\infty$, and $\delta<m\alpha$, and $\lim_{\tau\to s}y=\infty$, $w=O(r^{-\alpha-\gamma})$, $\frac{1}{w}=O(r^{\alpha-\gamma})$, $w'=O(r^{-\alpha-1-\gamma})$ near 0, and $J'_\alpha=O(r^{\alpha(1-mp+m)-p-1-\gamma})$, thus J'_α is still integrable; thus $\lim r^{\alpha}w=L\geq0$. If $L=0$, then $r^{\alpha}w=O(r^{\alpha(1-mp+m)-p-\gamma})$, we reach a contradiction by the estimate of $\frac{1}{w}$. Then again $L>0$.

(iii) Let $\lambda=\eta>0,\eta\neq mN$. From Proposition 3.7 either $(s=\infty,\delta<mN)$ or $(s=-\infty,\delta>mN)$. As mentioned above we obtain $w=O(r^{-\frac{\eta}{m}\pm\gamma})$ and $\frac{1}{w}=O(r^{\frac{\eta}{m}\pm\gamma})$ near ∞ or 0 for any $\gamma>0$. Here we substitute $d=\eta$ into (2.10). Thus $y_\eta=O(e^{\pm\gamma\tau})$, $\frac{1}{y_\eta}=O(e^{\pm\gamma\tau})$, $y'_\eta=O(e^{\pm\gamma\tau})$, hence $Y_\eta=O(e^{\pm\gamma\tau})$, and $Y'_\eta=O(e^{\pm\gamma\tau})$ from (2.11). From (2.11), we deduce $Y'_\eta=O(e^{\frac{1}{m}(1-mp+m)(\delta-\eta)\pm\gamma)\tau})$. When $s=\infty$, then $\delta<\eta$, when $s=-\infty$, then $\delta>\eta$ from (2.1). Thus Y'_η is integrable, and Y_η has a limit denoted by k, and $Y_\eta-k=O(e^{\frac{1}{m}(1-mp+m)(\delta-\eta)\pm\gamma)\tau})$. Now, $(e^{-\eta\tau}y_\eta)'=-e^{-\eta\tau}Y_\eta^{\frac{1}{p-1}}$, thus y_η has a limit denoted by $c=\frac{1}{\eta}k^{\frac{1}{p-1}}$, it implies $\lim r^{\eta}w^{m}=c$, which is equivalent to $\lim r^{\frac{\eta}{m}}w=\bar{c}$, $\bar{c}=c^{1/m}$. If $c=0$, for $Y_\eta=O(e^{\frac{1}{m}(1-mp+m)(\delta-\eta)\pm\gamma)\tau})$, then $y_\eta=O(e^{\frac{1}{m}(1-mp+m)(\delta-\eta)\pm\gamma)\tau})$, we reach a contradiction by $\frac{1}{y_\eta}=O(e^{\pm\gamma\tau})$ for γ small enough.

(iv) Let $\lambda=m\alpha=\delta\neq mN$, thus $(s=\infty,\delta-mN<0)$ or $(s=-\infty,\delta-mN>0)$, and $\lim_{\tau\to s}y=\infty$. Then $Y=(\delta y)^{p-1}(1+o(1))$, and $\mu=0$, thus $(y^{1/m}-Y)=y^{1/m}(1+o(1))$, and from (2.18),
$$(y^{1/m}-Y)'=(N-\alpha)Y=(N-\delta)\delta^{p-1}(y^{1/m}-Y)^{m(p-1)}(1+o(1)).$$
Then $y=(\frac{1}{m}|mN-\delta|\delta^{p-1}(1-mp+m)|\tau|)^{\frac{m}{1-mp+m}}(1+o(1))$, which is equivalent to (3.12).

(v) Let $\lambda = \eta = mN = \delta \neq m\alpha$, thus $(s = \infty, \alpha - N < 0)$ or $(s = -\infty, \alpha - N > 0)$, and $\lim_{\tau\to s} y = 0$. Then $Y = (mNy)^{p-1}(1 + o(1))$, and $\mu = \infty$, thus $Y - y^{1/m} = Y(1 + o(1))$, and from (3.12),

$$\begin{aligned}(Y - y^{1/m})' &= (\alpha - N)y^{1/m} = (\alpha - N)(mN)^{-1/m}Y^{\frac{1}{m(p-1)}} \\ &= (\alpha - N)(mN)^{-1/m}(Y - y^{1/m})^{\frac{1}{m(p-1)}}(1 + o(1)).\end{aligned}$$

Therefore $y = c|\tau|^{-\frac{m(1+mN)}{1+m}}$, $c = \frac{1}{mN}((mN)^{1/m}\frac{mN-m}{(1+m)|\alpha-N|})^{\frac{m(1+mN)}{1+m}}$ (where $p = \frac{(1+m)N}{1+mN}$), which is equivalent to (3.13).

(vi) Let $\lambda = 0$, $mrw' = o(w)$, by integrating, we know $w + r|w'| = O(r^{-k})$ for any $k > 0$. Then $J_1'(r) = (1-\alpha)w$ is integrable, thus J_1 has a limit at 0, and $\lim_{r\to 0} rw = 0$, thus $\lim_{r\to 0} w^{m-1}w' = c \in \mathbb{R}$, then $\lim_{r\to 0} w = a \geq 0$, and $\lim_{r\to 0} w' = b$. Then $b \neq 0$, since the regular solutions satisfy (3.1), and $a \neq 0$. If $a = 0$, implies $w = br(1+o(1))$, then $\zeta = -m$. If $\lambda = \eta = -1$, then from (2.30), w is increasing, thus w has a limit $a \geq 0$ at 0, $w' = -a\lambda m^{-1}r^{-1}(1 + o(1))$, and by integrating, we know $a = 0$. And $((w^{m-1}w')^{p-1})' = m^{1-p}(1 - \alpha)w(1 + o(1))$, thus $w^{m-1}w'$ is integrable, so w' has a limit $b \neq 0$. ☐

Proposition 3.9. *(i) Suppose $mN \leq \delta < m\alpha$, or $mN < \delta \leq m\alpha$. Then any solution y has a infinite number of zeros near ∞.*

(ii) Suppose that y has a infinite number of zeros near $\pm\infty$. Then either $mN < m\alpha < \delta$ and $|y| < \ell$, $|Y| < (\delta\ell)^{p-1}$ near $\pm\infty$ or $mN < \delta = m\alpha$ or $\max(\delta, mN, \eta) < m\alpha$. Besides, if $\delta < mN < m\alpha$, then $|y| > \ell$ at this extremal points.

Proof. (i) Suppose that is not the case. Then assume $y > 0$ for large τ; we know y is monotone from Proposition 3.6 (iii). Applying Proposition 3.7 with $s = \infty$, it is contradictory.

(ii) Suppose that y is oscillating around 0 near $\pm\infty$. Then from (3.5), at the extremal points,

$$|y|^{\frac{1-mp+m}{m}}(\delta - m\alpha) < \delta^{p-1}(\delta - mN). \tag{3.15}$$

We can prove that the inequality is strict: if one equality holds, then y is constant. Thus $|y| < \ell$ from (3.15) and $|Y| < (\delta\ell)^{p-1}$ for $y' = 0$. And $\max(mN, \eta) < m\alpha$, from Proposition 3.5, then y has a infinite number of zeros near $\pm\infty$. If $\delta < mN < m\alpha$, then $|y| > \ell$ at its extremal points from (3.5). ☐

Proposition 3.10. *Suppose that $m\delta + \delta - mN \leq 0$. Then any solution y has a finite number of zeros near $\ln R_w$. If it is defined near $\pm\infty$, and no monotone, then it converges to $\pm M_\ell$. There is no cycle and no homoclinic orbit in $\mathbb{R}^2$.*

Proof. (i) Suppose that y has an infinite number of zeros. Then $m\alpha > mN > 0$ from Proposition 3.9. If there exists two consecutive zeros $\rho_1 < \rho_2$ of w, and $\tau \in (\ln\rho_1, \ln\rho_2)$ is a maximal point of $|y_d|$, from (3.4), then

$$m(p-1)d(\eta - d) + (d - m\alpha)e^{(p+d(p-1)-\frac{d}{m})\tau}|dy_d|^{2-p}|y_d|^{\frac{1-m}{m}} \leq 0.$$

That means, with $\rho = e^\tau \in (\rho_1, \rho_2)$,

$$\rho^p|w|^{1-mp+m}(m\alpha - d) \geq m(p-1)d(\eta - d). \tag{3.16}$$

First, fix $d < m\alpha$, $d < \eta$. Then we consider the energy function $E(r)$ defined by (2.27). It is non-increasing. Then $E(r)$ is bounded on $(\ln R_w, \infty) \cap [\rho_1, \infty)$, so w is bounded since $\alpha > 0$. Suppose that there exists a strictly monotone sequence $\{r_n\}_{n=1}^{\infty}$ of consecutive zeros of w, converging to $\bar{r} \in [0, \infty)$. If $\bar{r} > 0$, we can find sequence $\{r_n\}_{n=1}^{\infty}$, satisfying $w > 0$ on (r_{2n-1}, r_{2n}) and $w < 0$ on (r_{2n}, r_{2n+1}). Then there exists $s_n \in (r_n, r_{n+1})$ such that $w'(s_n) = 0$, since $w \in C^1[0, \infty)$, it implies $w(\bar{r}) = w'(\bar{r}) = 0$. Since $E'(r)$ is non-increasing, then $E(r) \equiv 0, w(r) \equiv 0$, we obtain a contradiction from (3.16), because the left-hand side tends to 0, the right-hand side tends to a constant greater than 0. If $\bar{r} = 0$, that means $R_w = 0$. As above, we also reach a contradiction.

(ii) Suppose that y is positive near $\pm\infty$, and non monotone. Since $\delta < mN < m\alpha$, following Proposition 3.6, we know y oscillates around ℓ. There exists a sequence of extremal points $\{\tau_n\}_{n=1}^{\infty}$, where $y(\tau_n) < \ell$, and $|Y(\tau_n)|^{\frac{2-p}{p-1}}Y(\tau_n) = \delta y(\tau_n)$, thus $(y(\tau_n), Y(\tau_n))$ is bounded. It means that

$w(\rho_n), w'(\rho_n)$ (where $\rho_n = e^{\tau_n}$) are bounded, thus $E(\rho_n)$ is bounded, and $E(r)$ is bounded. Hence, (y, Y) is bounded. Denote the vector field of system (2.14) by (f_1, f_2) and defined $h(x, y) = 1$. Then $\operatorname{div}(hf_1, hf_2) = \delta + \frac{\delta}{m} - N - \frac{1}{m(p-1)}|y|^{\frac{1-m}{m}}|Y|^{\frac{2-p}{p-1}}$, which is negative if $m\delta + \delta - mN \le 0$. Bendixson-Dulac criterion shows that there is no periodic orbit. Thus the trajectory converges to M_ℓ. Finally, if there exists a homoclinic orbit, then $\mathcal{T}_r$ is homoclinic, $\lim_{\tau\to-\infty}(y, Y) = \lim_{\tau\to\infty}(y, Y) = (0, 0)$, thus $\lim_{r\to 0}(w, w') = \lim_{r\to\infty}(w, w') = (0, 0)$ and $\lim_{r\to 0} E = \lim_{r\to\infty} E = 0$, hence $E \equiv 0$ and $w \equiv 0$, which is contradictory. $\square$

Proposition 3.11. *Suppose that y is not monotone near ∞ (positive or changing sign), then y and Y are bounded.*

Proof. If y is changing sign and $mN < m\alpha < \delta$, then $|y|$ is bounded by ℓ from Proposition 3.9. Moreover, if $m\delta + \delta - mN \le 0$, y also is changing sign, from Proposition 3.10. Thus $\delta \le \frac{m}{m+1}N < mN$, and $\delta < m\alpha$. If y stays positive, we know $\delta < mN < m\alpha$, following Proposition 3.6. In any case $\delta < m\alpha$. Here we apply the energy function W defined by (2.28). Further $W(y, Y)$ can be rewritten under the form

$$W(y, Y) = F(y, Y) + G(y, Y)$$

with

$$\begin{aligned} F(y, Y) &= \frac{m|Y|^{p'}}{p'} - m\delta yY + \frac{m|\delta y|^p}{p}, \\ G(y, Y) &= \frac{m(m\alpha - \delta)}{m+1}|y|^{\frac{1}{m}-1}y^2 + \frac{(\delta - mN)\delta^{p-1}}{p}|y|^p. \end{aligned} \tag{3.17}$$

We find $F(y, Y) \ge 0$, from Young inequality, thus $W(y, Y) \ge G(y, Y) > 0$ for large y. Then $W'(\tau) < 0$ whenever $(y(\tau), Y(\tau)) \notin S_{\mathcal{L}}$, where $S_{\mathcal{L}}$ is given at (2.29). Denote τ_0 be arbitrary in the interval of definition of y. Then $W(\tau) \le W(\tau_0)$ for any τ such that $\tau - \tau_0 \ge 0$ and $(y(\tau), Y(\tau)) \notin S_{\mathcal{L}}$. Since $S_{\mathcal{L}}$ is bounded, there exists $k > 0$ large enough such that $W(\tau) \le k$ for any τ such that $\tau - \tau_0 \ge 0$, and $(y(\tau), Y(\tau)) \in S_{\mathcal{L}}$. And we can choose $k > W(\tau_0)$. Then $W(\tau) \le k$ for $\tau - \tau_0 \ge 0$, Thus y and Y are bounded near ∞. $\square$

Next we show a further sign property. By applying Proposition 3.6 and Proposition 3.7, we can improve Proposition 3.5.

Proposition 3.12. *Assume $-\infty < m\alpha \le \delta$ and $\alpha < N$. Then the regular solutions have a constant sign, y is strictly monotone and $\lim_{\tau\to\infty}\zeta = m\alpha$. Moreover, any solution has at most one zero, and then $\lim_{\tau\to\infty}\zeta = m\alpha$.*

The above proposition is proved in the same manner as [5, Proposition 2.13].

3.4. **Behaviour of w near $R_w > 0$.**

Proposition 3.13. *Let w be any solution of (E_w) with a reduced domain (R_w, ∞) $(R_w > 0)$. Then*

$$\lim_{r\to R_w}|r - R_w|^{\frac{p-1}{1-mp+m}} R_w^{\frac{1}{1-mp+m}} w = \Big(\frac{p-1}{1-mp+m}\Big)^{\frac{p-1}{1-mp+m}}, \quad \text{and} \quad \lim_{\tau\to\ln R_w}\sigma = 1. \tag{3.18}$$

Proof. Following Proposition 3.1, assume that w is decreasing near R_w and $\lim_{r\to R_w} w = \infty$, thus $y > 0$ and $Y > 0$ near $\ln R_w$, and $\lim_{\tau\to\ln R_w} y = \infty$. And σ also is monotone near $\ln R_w$, from Proposition 3.6. Thus σ has a limit μ such that $\mu \in [0, \infty]$. Suppose $\mu = 0$, then $Y = o(y^{1/m}) = o(y^{1/m} - Y)$, from (2.18) we have

$$(y^{1/m} - Y)' = (\frac{\delta}{m} - \alpha)(y^{1/m} - Y) + (N - \alpha)Y = \Big(\frac{\delta}{m} - \alpha + o(1)\Big)(y^{1/m} - Y),$$

hence y cannot blow up in finite time. Similarly, if $\mu = \infty$, then $y^{1/m} = o(Y) = o(Y - y^{1/m})$, and

$$(y^{1/m} - Y)' = (\frac{\delta}{m} - N)(y^{1/m} - Y) + (N - \alpha)y^{1/m} = \Big(\frac{\delta}{m} - \alpha + o(1)\Big)(y^{1/m} - Y),$$

which is also contradictory; thus $\mu \in (0,\infty)$. Moreover, if λ is finite, $\sigma = |\zeta|^{p-2}\zeta y^{-\frac{1-mp+m}{m}}$ converges to 0, which is not true. Thus $\lim_{\tau\to\ln R_w}\zeta = \infty$, hence $\mu = 1$ from equation (3.10). Therefore $\sigma(\tau) = -m^{p-1}\frac{|w^{m-1}w'|^{p-2}w^{m-1}w'}{rw} = 1+o(1)$, then $w^{-\frac{1}{p-1}}w^{m-1}w'+m^{-1}r^{\frac{1}{p-1}}(1+o(1)) = 0$, and (3.18) holds. □

3.5. **More information on the stationary points. (i) Höpf bifurcation point.** When $\delta+m\delta-mN>0$, a Höpf bifurcation appears at the critical value $\alpha=\alpha^*$. Then there exist some cycles appearing near α^*, from the Poincaré-Andronov-Hopf theorem in [15]. We find more precise results by applying the Lyapounov test for a week source; it requires an expansion up to the order 3 near M_ℓ, in a suitable basis of eigenvectors, where the linearized problem has a rotation matrix.

Theorem 3.14. *Let $\delta < mN, m\delta+\delta-mN > 0$ and $1 < p < 2$, $\alpha=\alpha^*$, then M_ℓ is a week source; moreover if $\alpha<\alpha^*$ and $\alpha^*-\alpha$ is small enough, then there exists a unique limit cycle in $\mathcal{Q}_1$, attracting at $-\infty$.*

Proof. When $\alpha=\alpha^*$, we can obtain the eigenvalues given by $\lambda_1=-ib, \lambda_2=ib$, with $b=\sqrt{\frac{p}{m(p-1)}(mN-\delta)}$. Let

$$u(\alpha)=\frac{(\delta\ell)^{2-p}}{p-1},\quad v(\alpha)=\frac{\delta}{m(p-1)}\cdot\frac{mN-\delta}{m\alpha-\delta},$$

then

$$v(\alpha^*)=\frac{\delta(mN-\delta)}{m(p-1)(m\alpha^*-\delta)}=\frac{m\delta+\delta-mN}{m}=\frac{\delta^{2-p}\ell^{\frac{1-mp+m}{m}}}{m(p-1)}.$$

First we apply the substitution (2.19), which leads to (2.20). The functions Φ and Ψ, which is defined at (2.21) and (2.22), can be expanded near $(0,0)$ of the form

$$\Phi(\bar y,\bar Y)=B_2\bar Y^2+B_3\bar Y^3+\cdots,$$

and

$$\Psi(\bar y,\bar Y)=C_{2,0}\bar y^2+C_{1,1}\bar y\bar Y+C_{0,2}\bar Y^2+C_{3,0}\bar y^3+C_{2,1}\bar y^2\bar Y+C_{1,2}\bar y\bar Y^2+C_{0,3}\bar Y^3+\cdots,$$

where

$$B_2=-\frac{2-p}{2(p-1)^2}(\delta\ell)^{3-2p},\quad B_3=-\frac{(2-p)(3-2p)}{6(p-1)^3}(\delta\ell)^{4-3p},$$

$$C_{2,0}=\frac{1-m}{2m^3}\ell^{\frac{1}{m}-2}(m\alpha-\delta+2m\delta),\quad C_{1,1}=-\frac{1-m}{m^2(p-1)}\delta^{2-p}\ell^{\frac{1}{m}-p}=-\frac{2m}{m\alpha-\delta+2m\delta}C_{2,0}u(\alpha),$$

$$C_{0,2}=-\frac{2-p}{2m(p-1)^2}\delta^{3-2p}\ell^{2-2p+\frac{1}{m}},\quad \delta C_{0,2}=-\frac{(2-p)m^2}{(1-m)(m\alpha-\delta+2m\delta)}C_{2,0}u^2(\alpha),$$

$$C_{3,0}=\frac{(1-m)(1-2m)}{6m^4}\ell^{\frac{1}{m}-3}(m\alpha-\delta+3m\delta)=\frac{2m(1-2m)(m\alpha-\delta+3m\delta)}{3(1-m)(m\alpha-\delta+2m\delta)^2}\cdot\frac{C_{2,0}^2u(\alpha)}{v(\alpha^*)},$$

$$C_{2,1}=-\frac{(1-m)(1-2m)}{2m^3(p-1)}\delta^{2-p}\ell^{-1+\frac{1}{m}-p}=-\frac{2m^2(1-2m)}{(1-m)(m\alpha-\delta+2m\delta)^2}\frac{C_{2,0}^2u^2(\alpha)}{v(\alpha)},$$

$$C_{1,2}=-\frac{(2-p)(1-m)}{2m^2(p-1)^2}\delta^{3-2p}\ell^{1+\frac{1}{m}-2p},\quad \delta C_{1,2}=-\frac{2(2-p)m^3}{(1-m)(m\alpha-\delta+2m\delta)^2}\frac{C_{2,0}^2u^3(\alpha)}{v(\alpha)},$$

$$C_{0,3}=-\frac{(2-p)(3-2p)}{6m(p-1)^3}\delta^{4-3p}\ell^{3+\frac{1}{m}-3p},\quad \delta^2C_{0,3}=-\frac{2m^4(2-p)(3-2p)}{3(1-m)^2(m\alpha-\delta+2m\delta)^2}\frac{C_{2,0}^2u^4(\alpha)}{v(\alpha)},$$

$$\delta B_2=-\frac{(2-p)m^2}{(1-m)(m\alpha-\delta+2m\delta)}\frac{C_{2,0}u^3(\alpha)}{v(\alpha)}.$$

Next we use the substitution

$$\tau=-\frac{\theta}{b},\quad \bar y(\tau)=u(\alpha)x_1(\theta),\quad \bar Y=\delta x_1(\theta)+bx_2(\theta),$$

to obtain

$$x_1'(\theta) = x_2 - \frac{1}{bu(\alpha)}\Phi(u(\alpha)x_1, \delta x_1 + bx_2),$$
$$x_2'(\theta) = -x_1 - \frac{1}{b^2}\Psi(u(\alpha)x_1, \delta x_1 + bx_2) + \frac{\delta}{b^2 u(\alpha)}\Phi(u(\alpha)x_1, \delta x_1 + bx_2).$$

They can be written the expansion of order 3 in the form

$$x_1' = x_2 + a_{2,0}x_1^2 + a_{1,1}x_1x_2 + a_{0,2}x_2^2 + a_{3,0}x_1^3 + a_{2,1}x_1^2x_2 + a_{1,2}x_1x_2^2 + a_{0,3}x_2^3 + \cdots,$$

and

$$x_2' = -x_1 + b_{2,0}x_1^2 + b_{1,1}x_1x_2 + b_{0,2}x_2^2 + b_{3,0}x_1^3 + b_{2,1}x_1^2x_2 + b_{1,2}x_1x_2^2 + b_{0,3}x_2^3 + \cdots.$$

We compute the Lyapounov coefficient

$$L = 3a_{3,0} + a_{1,2} + b_{2,1} + 3b_{0,3} - a_{2,0}a_{1,1} + b_{1,1}b_{0,2} - 2a_{0,2}b_{0,2} - a_{0,2}a_{1,1} + 2a_{2,0}b_{2,0} + b_{1,1}b_{2,0}.$$

After simplification, we obtain

$$\frac{b(1-m)^2(m\alpha - \delta + 2m\delta)^2\delta^2}{4m^5 C_{2,0}^2 u^4(\alpha)} L = \frac{1-p}{m\delta + \delta - mN}\left[\delta p + (2-p)b^2\right] < 0.$$

The property of M_ℓ follows from [17], let us notice the fact that θ has the opposite sign of τ. Moreover there exists an unstable limit cycle attracting at $-\infty$ for all α near α^* such that M_ℓ is a sink, that means $\alpha < \alpha^*$. □

(ii) Node points or spiral points. When system (2.14) has three stationary points, and if M_ℓ is a source or sink, then $\delta < mN$. It is a interesting point that M_ℓ is a node point. Moreover α^* corresponds to a spiral point, from (2.23).

We find M_ℓ is a node point when $\Delta \geq 0$ from (2.24), that means $\delta \leq \frac{mN}{1+m} - \frac{2\sqrt{mp'(mN-\delta)}}{1+m}$ or $\delta > \frac{mN}{1+m} - \frac{2\sqrt{mp'(mN-\delta)}}{1+m}$ and $\alpha \leq \alpha_1$, or $\delta > \frac{mN}{1+m} + \frac{2\sqrt{mp'(mN-\delta)}}{1+m}$ and $\alpha \geq \alpha_2$, one verifies $\alpha_1 < \alpha_2$, where

$$\begin{aligned} \alpha_1 &= \frac{\delta}{m} + \frac{\delta(mN-\delta)}{m(p-1)(m\delta + \delta - mN + 2\sqrt{mp'(mN-\delta)})}, \\ \alpha_2 &= \frac{\delta}{m} + \frac{\delta(mN-\delta)}{m(p-1)(m\delta + \delta - mN - 2\sqrt{mp'(mN-\delta)})}. \end{aligned} \tag{3.19}$$

Remark 3.15. If $\delta < mN$, then $N \leq \alpha_1$, and $N = \alpha_1 \Leftrightarrow N = \frac{\delta}{m} + p'$. Also $\alpha_1 < \frac{\eta}{m} \Leftrightarrow \delta^2 + (4m - N + 3)m\delta + m^2N > 0$.

(iii) Nonexistence of cycles. If there exists a cycle $\mathcal{O}$ in system (2.14) in $\mathbb{R}^2$, then $\mathcal{O}$ surrounds at least one stationary point. If it surrounds $(0,0)$, the associated solutions y have changing sign. If it only surrounds M_ℓ, then it stays in $\mathcal{Q}_1$, thus y stays positive. In fact, $\mathcal{O}$ cannot intersect $\{(\varphi, 0), \varphi > 0\}$ at two points, and similarly $\{(0,\xi), \xi > 0\}$, from Remark **??**.

For suitable values of $m\alpha, \delta, mN$, we can verify that cycles cannot exist by applying Bendixson-Dulac criterion. System (2.14) can be rewritten under the form

$$y' = f_1(y, Y), \quad Y' = f_2(y, Y), \tag{3.20}$$

with

$$f_1(y, Y) = \delta y - |Y|^{\frac{2-p}{p-1}}Y, \quad f_2(y, Y) = (\frac{\delta}{m} - N)Y + |y|^{\frac{1-m}{m}}(\alpha y - \frac{1}{m}|Y|^{\frac{2-p}{p-1}}Y).$$

Thus

$$\frac{\partial f_1}{\partial y}(y, Y) + \frac{\partial f_2}{\partial Y}(y, Y) = \delta + \frac{\delta}{m} - N - \frac{1}{m(p-1)}|y|^{\frac{1-m}{m}}|Y|^{\frac{2-p}{p-1}}. \tag{3.21}$$

For example, if $m\delta + \delta - mN \leq 0$, we find that there exists no periodic orbit in $\mathbb{R}^2$ by a direct consequence of Bendixson-Dulac criterion, which was proved in Proposition 3.10. Next we consider cycles in $\mathcal{Q}_1$.

Theorem 3.16. *Let $\delta < mN$, then $\delta < m\alpha$. When M_ℓ is a node point, there exists no cycle or no homoclinic orbit in $\mathcal{Q}_1$ if $m > 1$ and $p < 2 - \frac{(m-1)\delta}{m\alpha-\delta+2m\delta} < 2$.*

Proof. Let us use the linearization (2.19), (2.20). Consider the line L given by the equation $A\bar{y} + \bar{Y} = 0$, where A is a real parameter. The points of L are in $\mathcal{Q}_1$ when $-(\delta\ell)^{p-1} < \bar{Y}$ and $-\ell < \bar{y}$. We obtain:

$$\begin{aligned} A\bar{y}' + \bar{Y}' = \Big[&\frac{(\delta\ell)^{2-p}}{p-1}A^2 + (\delta - \frac{\delta}{m} + N + \frac{1}{m(p-1)}\delta^{2-p}\ell^{\frac{1-mp+m}{m}})A \\ &+ \frac{m\alpha + (m-1)\delta}{m^2}\ell^{\frac{1-m}{m}}\Big]\bar{y} + A\Phi(\bar{y},\bar{Y}) + \Psi(\bar{y},\bar{Y}). \end{aligned}$$

From (2.24), we find an A such that

$$\frac{(\delta\ell)^{2-p}}{p-1}A^2 + \Big(\delta - \frac{\delta}{m} + N + \frac{1}{m(p-1)}\delta^{2-p}\ell^{\frac{1-mp+m}{m}}\Big)A + \frac{m\alpha + (m-1)\delta}{m^2}\ell^{\frac{1-m}{m}} = 0,$$

and $A \neq 0$. Furthermore, $\Phi(\bar{y},\bar{Y}) \leq 0$ is on $L \cap \mathcal{Q}_1$. In fact $(p-1)\Phi(t)' = ((\delta\ell)^{p-1} + t)^{\frac{2-p}{p-1}} - (\delta\ell)^{2-p}$, thus Φ has a maximum 0 on $(-(\delta\ell)^{p-1}, \infty)$ at point 0, and it is non-positive on this interval. $\Psi(\bar{y},\bar{Y}) \leq 0$ is also on $L \cap \mathcal{Q}_1$. In fact,

$$\Psi_{\bar{y}} = \frac{\alpha}{m}(\bar{y}+\ell)^{\frac{1}{m}-1} - \frac{1-m}{m^2}(\bar{y}+\ell)^{\frac{1}{m}}(\bar{Y} + (\delta\ell)^{p-1})^{\frac{1}{p-1}} - \frac{(m\alpha + (m-1)\delta)}{m^2}\ell^{\frac{1}{m}-1},$$

and

$$\Psi_{\bar{Y}} = -\frac{1}{m(p-1)}(\bar{y}+\ell)^{\frac{1}{m}-1}(\bar{Y} + (\delta\ell)^{p-1})^{\frac{2-p}{p-1}} + \frac{1}{m(p-1)}\delta^{2-p}\ell^{1-p+\frac{1}{m}},$$

by a computations, we have

$$\begin{gathered} \Psi_{\bar{y}}(0,0) = 0, \quad \Psi_{\bar{Y}}(0,0) = 0, \quad \Psi_{\bar{y}\bar{y}}(0,0) = \frac{(1-m)(m\alpha - \delta + 2m\delta)}{m^3}\ell^{\frac{1}{m}-2}, \\ \Psi_{\bar{y}\bar{Y}}(0,0) = -\frac{1-m}{m^2(p-1)}\delta^{2-p}\ell^{\frac{1}{m}-p}, \quad \Psi_{\bar{Y}\bar{Y}}(0,0) = -\frac{2-p}{m(p-1)^2}\delta^{3-2p}\ell^{2-2p+\frac{1}{m}}. \end{gathered}$$

Hence, applying the extremum principle of binary function, then we know Ψ has a maximum 0 on $(-l,\infty) \cap (-(\delta\ell)^{p-1}, \infty)$ at point 0 if $m > 1$ and $p < 2 - \frac{(m-1)\delta}{m\alpha-\delta+2m\delta} < 2$, thus it is non-positive on the interval. Then the orientation of the vector field does not change along $L \cap \mathcal{Q}_1$, which means that there exists no cycle in $\mathcal{Q}_1$; and similarly no homoclinic orbit can exist. In the case $\alpha = N$, then $Y \equiv y^{1/m} \in [0,\ell)$ defines the trajectory $\mathcal{T}_r$, associated to the solutions given by (2.6) with $K > 0$, and there exists no cycle in $\mathcal{Q}_1$, otherwise it would intersect $\mathcal{T}_r$. □

Theorem 3.17. *Assume $\delta < mN$, $\delta - m\alpha < 0 < m\delta + \delta - mN$, $0 < m < 1, 1 < p < 2$ and $(p-2)(m\alpha - \delta)(m\alpha - \delta + 2m\delta) + m(1-m)\delta^2 < 0$. If $\alpha - \alpha^* \geq 0$, there exists no cycle or no homoclinic orbit in $\mathcal{Q}_1$.*

Proof. In this case, M_ℓ is a source or a weak source. Suppose there is a cycle in $\mathcal{Q}_1$ in system (2.14). Then any trajectory starting from M_ℓ has a limit cycle in $\mathcal{Q}_1$, with attracting at ∞. Such a cycle is stable; it means that the Floquet integral on the period $[0,\mathcal{P}]$ is non-positive. Thus from (3.21),

$$\int_0^{\mathcal{P}} \Big(\frac{\partial f_1}{\partial y}(y,Y) + \frac{\partial f_2}{\partial Y}(y,Y)\Big)\,\mathrm{d}\tau = \int_0^{\mathcal{P}} \Big(\delta + \frac{\delta}{m} - N - \frac{1}{m(p-1)}y^{\frac{1-m}{m}}Y^{\frac{2-p}{p-1}}\Big)\,\mathrm{d}\tau \leq 0. \quad (3.22)$$

Now from (2.20),

$$0 = \delta\int_0^{\mathcal{P}} \bar{y}\,\mathrm{d}\tau - \frac{(\delta\ell)^{2-p}}{p-1}\int_0^{\mathcal{P}} \bar{Y}\,\mathrm{d}\tau + \int_0^{\mathcal{P}} \Phi(\bar{y},\bar{Y})\,\mathrm{d}\tau,$$

and

$$0 = \frac{m\alpha + m\delta - \delta}{m^2}\ell^{\frac{1}{m}-1}\int_0^{\mathcal{P}} \bar{y}\,\mathrm{d}\tau + \Big(\frac{\delta}{m} - N - \frac{\delta^{2-p}\ell^{1+\frac{1}{m}-p}}{m(p-1)}\Big)\int_0^{\mathcal{P}} \bar{Y}\,\mathrm{d}\tau + \int_0^{\mathcal{P}} \Psi(\bar{y},\bar{Y})\,\mathrm{d}\tau;$$

then

$$
\begin{aligned}
&\int_0^{\mathcal{P}} \Psi(\bar{y},\bar{Y})\,\mathrm{d}\tau + \Big[\frac{\delta - mN}{m} - \frac{1}{m(p-1)}\delta^{2-p}\ell^{1+\frac{1}{m}-p}\Big]\frac{p-1}{(\delta\ell)^{2-p}}\int_0^{\mathcal{P}} \Phi(\bar{y},\bar{Y})\,\mathrm{d}\tau \\
&= \Big[\frac{mN-\delta}{m}(p-1)\delta^{p-1}\ell^{p-2} - \frac{m\alpha-\delta}{m^2}\ell^{\frac{1}{m}-1}\Big]\int_0^{\mathcal{P}} \bar{y}\,\mathrm{d}\tau,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^{\mathcal{P}} \Psi(\bar{y},\bar{Y})\,\mathrm{d}\tau - \frac{m\alpha + m\delta - \delta}{m^2\delta}\int_0^{\mathcal{P}} \Phi(\bar{y},\bar{Y})\,\mathrm{d}\tau \\
&= -\Big(\frac{m\alpha-\delta}{m^2}\delta^{1-p}\ell^{1+\frac{1}{m}-p} - \frac{\delta-mN}{m}\Big)\int_0^{\mathcal{P}} \bar{Y}\,\mathrm{d}\tau,
\end{aligned}
$$

Let

$$
F(\bar{y},\bar{Y}) = \Psi(\bar{y},\bar{Y}) + \Big[\frac{\delta - mN}{m} - \frac{1}{m(p-1)}\delta^{2-p}\ell^{1+\frac{1}{m}-p}\frac{p-1}{(\delta\ell)^{2-p}}\Big]\Phi(\bar{y},\bar{Y}),
$$

$$
G(\bar{y},\bar{Y}) = \Psi(\bar{y},\bar{Y}) - \frac{m\alpha + m\delta - \delta}{m^2\delta}\Phi(\bar{y},\bar{Y}).
$$

We can show that $F(\bar{y},\bar{Y})$ has a minimum at point $(0,0)$, if $0 < m < 1, 1 < p < 2$ and $(p-2)(m\alpha-\delta)(m\alpha-\delta+2m\delta) + m(1-m)\delta^2 < 0$. Simultaneously, $G(\bar{y},\bar{Y})$ also has a minimum at point $(0,0)$. Thus $F(\bar{y},\bar{Y})$ and $G(\bar{y},\bar{Y})$ are nonnegative in $\mathcal{Q}_1$. Hence,

$$
\int_0^{\mathcal{P}} \bar{y}\,\mathrm{d}\tau < 0 \ \Rightarrow y < \ell, \quad \int_0^{\mathcal{P}} \bar{Y}\,\mathrm{d}\tau < 0.
$$

Since $y' = \delta y - Y^{\frac{1}{p-1}}$, we have

$$
\int_0^{\mathcal{P}} Y^{\frac{1}{p-1}}\,\mathrm{d}\tau = \delta\int_0^{\mathcal{P}} y\,\mathrm{d}\tau < \delta\ell\mathcal{P}. \tag{3.23}
$$

From (3.22), (3.23) and the Hölder inequality, we find that

$$
\begin{aligned}
(\delta + \frac{\delta}{m} - N)\mathcal{P} &\le \frac{1}{m(p-1)}\int_0^{\mathcal{P}} y^{\frac{1-m}{m}} Y^{\frac{2-p}{p-1}}\,\mathrm{d}\tau \\
&\le \frac{1}{m(p-1)}\delta^{2-p}\ell^{1+\frac{1}{m}-p}\mathcal{P} \\
&= \frac{\delta(mN-\delta)}{m(p-1)(m\alpha-\delta)}\mathcal{P},
\end{aligned}
$$

thus $\alpha - \alpha^* < 0$, which is contradictory. Next suppose there exists an homoclinic orbit. From [17, P303 Theorem 9.3.3],the saddle connection is repelling, since the sum of the eigenvalues λ_1, λ_2 of the linearized problem at $(0,0)$ is $\delta + \frac{\delta}{m} - N (>0)$. Then the solutions just inside the cycle spiral toward the loop near $-\infty$. We know M_ℓ is a source, or a weak source, such solutions have a limit cycle attracting at ∞, which is contradictory. □

Theorem 3.18. *Assume $\delta < mN$ and $\delta - m\alpha < 0 < \delta + \frac{\delta}{m} - N$. If $m\alpha \le \eta$, there exists no cycle or no homoclinic orbit in $\mathcal{Q}_1$.*

Proof. Suppose system (2.14) admits a cycle in $\mathcal{Q}_1$.

(i) Let $m\alpha \le \eta$. M_ℓ is a sink when $\alpha < \alpha^*$, thus any trajectory, which converges to M_ℓ at ∞, has a limit cycle $\mathcal{O}$ in $\mathcal{Q}_1$, attracting at $-\infty$. Denote (y, Y) be any solution of orbit $\mathcal{O}$, of period $\mathcal{P}$. Then $\mathcal{O}$ is unstable, thus the Floquet integral is nonnegative. Following (3.21) we obtain

$$
t_0^{\mathcal{P}}\Big(\delta + \frac{\delta}{m} - N - \frac{1}{m(p-1)}y^{\frac{1-m}{m}}Y^{\frac{2-p}{p-1}}\Big)\,\mathrm{d}\tau \ge 0.
$$

Moreover y is bounded from above and below; thus $y_{m\alpha}$, defined by (2.10) with $d = m\alpha$, satisfies $\lim_{\tau\to-\infty} y_{m\alpha} = 0$, $\lim_{\tau\to\infty} y_{m\alpha} = \infty$. From (3.9), we know $y''_{m\alpha} > 0$ for $m\alpha \le \eta$, then it has only minimal points. Thus $y'_{m\alpha} > 0$ in $\mathbb{R}$. From (2.11) and (2.12) with $d = m\alpha$,

$$\frac{y''_{m\alpha}}{y'_{m\alpha}} + (\eta - 2m\alpha) + \frac{1}{m(p-1)} y_{m\alpha}^{\frac{1-m}{m}} Y_{m\alpha}^{\frac{2-p}{p-1}} = \frac{m\alpha(\eta - m\alpha)y_{m\alpha}}{y'_{m\alpha}} = \frac{m\alpha(\eta - m\alpha)y_{m\alpha}}{m\alpha y_{m\alpha} - Y_{m\alpha}^{\frac{1}{p-1}}} > \eta - m\alpha.$$

Integrating on $[0, \mathcal{P}]$ it implies $\eta - 2m\alpha + \delta + \frac{\delta}{m} - N > \eta - m\alpha$, which is contradictory to $\delta - mN + \delta - m\alpha < 0$.

(ii) Suppose system (2.14) admits an homoclinic orbit in $\mathcal{Q}_1$. Since $\delta < mN$, $(0,0)$ is a saddle point, thus $\mathcal{T}_r$ is the only trajectory starting from $(0,0)$ in $\mathcal{Q}_1$, and there exists the only one trajectory $\mathcal{T}_s$ converging to $(0,0)$ in $\mathcal{Q}_1$ for large τ, with $\lim_{\tau\to\infty} \frac{Y}{y^{1/m}} = \infty$, and $\lim_{r\to\infty} r^{\eta} w^m = c > 0$.

We know $\mathcal{T}_r$ satisfies $\lim_{\tau\to-\infty} e^{-m\alpha\tau} y_{m\alpha}(\tau) = a^m > 0$, thus $\lim_{\tau\to-\infty} y_{m\alpha} = 0$; and $y_{m\alpha}$ has only minimal points. As above, $\mathcal{T}_r$ is monotone increasing and positive, and $\mathcal{T}_s$ satisfies $\lim_{\tau\to\infty} e^{(\eta-m\alpha)\tau} y_{m\alpha} = c > 0$. If $m\alpha < \eta$, then $\lim_{\tau\to\infty} y_{m\alpha} = 0$, thus $\mathcal{T}_r \neq \mathcal{T}_s$. If $m\alpha = \eta$, $\mathcal{T}_s$ is a explicit solution given by (2.7), which means $y_{m\alpha}$ is constant, thus again $\mathcal{T}_r \neq \mathcal{T}_s$. $\square$

(iv) Boundedness of cycles. When there exist cycles, apart from a few cases, we cannot prove their uniqueness, hence we pay attention to the following properties.

Theorem 3.19. *When it is nonempty, the set $\mathcal{C}$ of all the cycles of system* (2.14) *is bounded in* $\mathbb{R}^2$.

Proof. Assume that there exists a cycle in $\mathbb{R}^2$. From Proposition 3.5, 3.6, 3.9 and 3.10, it can happen only in three cases: $mN < m\alpha < \delta$, $mN < \delta = m\alpha$, $\max(\delta, mN, \eta) < m\alpha$ and $\delta > \frac{mN}{1+m}$. In the first case, the set $\mathcal{C}$ is bounded and contained in $(-\ell, \ell) \times (-(\delta\ell)^{p-1}, (\delta\ell)^{p-1})$, from Proposition 3.9. In other cases we apply the energy function W. The trajectory $\mathcal{O}$ corresponding to the solution is denoted by (y, Y). Then W is periodic, and its extremal points are exactly the points of the curve $\mathcal{L}$. In fact, if there exists an extremal point τ_1 such that $W'(\tau_1) = 0$ and $(y(\tau_1), Y(\tau_1)) \notin \mathcal{L}$, and it is on the curve $\mathcal{M}$ defined at (2.16). Thus, $y'(\tau_1) = 0$, and $y''(\tau_1) \neq 0$, since $\mathcal{O}$ can not reduced to a stationary point. Consequently, $(\delta y - |Y|^{\frac{2-p}{p-1}} Y)(|\delta y|^{p-2}\delta y - Y) > 0$ near τ_1, then W' has a constant sign, and τ_1 is not an extremal point. By this means, we obtain estimates for W independent of the trajectory:

$$\max_{\tau\in\mathbb{R}} |W(\tau)| = M = \max_{(y,Y)\in\mathcal{L}} |W(y, Y)|.$$

At the maximal points τ of y, we obtain $|Y(\tau)|^{\frac{2-p}{p-1}} Y(\tau) = \delta y(\tau)$, thus

$$\begin{aligned} W(\tau) &= \frac{(m\delta + \delta - mN)\delta^{p-1}}{p} |y|^p + \frac{m|Y|^{p'}}{p'} - m\delta y Y + \frac{m(m\alpha - \delta)}{m+1} |y|^{\frac{1}{m}-1} y^2 \\ &= \frac{(\delta - mN)\delta^{p-1}}{p} |y|^p + \frac{m(m\alpha - \delta)}{m+1} |y|^{\frac{1}{m}-1} y^2. \end{aligned}$$

In any case, y is bounded and independent of the trajectory, and

$$\frac{m|Y|^{p'}}{p'} \le m\delta y Y + \frac{(m\delta + \delta - mN)\delta^{p-1}}{p} |y|^p + \frac{m(m\alpha - \delta)}{m+1} y^{\frac{1}{m}-1} y^2 + M,$$

Hence Y is also uniformly bounded, and $\mathcal{C}$ is bounded. $\square$

4. Classification of self-similar solutions

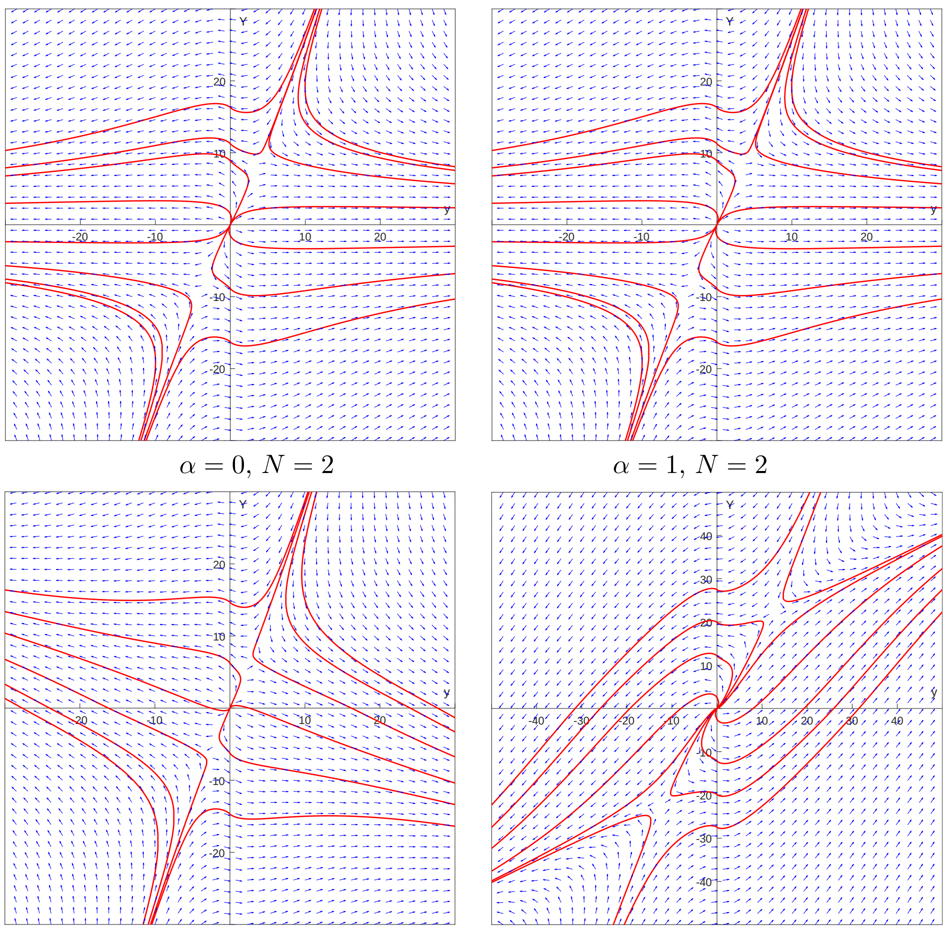
We classify the self-similar solutions to the polytropic filtration equation (1.1) and list them in a similar way as the self-similar solutions for p-Laplace equation in [5], which is the special case $m = 1$ for the polytropic filtration equation (1.1). Note that for non-Newtonian polytropic filtration equation (1.1), both m and p have influence on the asymptotic behavior and classification of self-similar solutions for the singular case $m(p-1) < 1$, which is the main cause of the difficulties of this paper.

4.1. **General properties.**

Lemma 4.1. *Assume $-\infty < \max(m\alpha, mN) < \delta$. Then in the phase plane (y, Y), there exist*

- (i) *a trajectory $\mathcal{T}_1$ converging to M_ℓ at ∞, such that y is increasing as long as it is positive;*
- (ii) *a trajectory $\mathcal{T}_2$ in $\mathcal{Q}_1 \cup \mathcal{Q}_4$ converging to M_ℓ at $-\infty$, and unbounded at ∞, with $\lim_{\tau\to\infty} \zeta = m\alpha$;*
- (iii) *a trajectory $\mathcal{T}_3$ converging to M_ℓ at $-\infty$, such that y has at least one zero;*
- (iv) *a trajectory $\mathcal{T}_4$ in $\mathcal{Q}_1$, converging to M_ℓ at ∞, with $\lim_{\tau\to\ln R_w} \frac{Y}{y^{1/m}} = 1$ $(R_w > 0)$;*
- (v) *a trajectory $\mathcal{T}_5$ in $\mathcal{Q}_1 \cup \mathcal{Q}_4$ unbounded at $\pm\infty$, with $\lim_{\tau\to\infty} \zeta = m\alpha$,$\lim_{\tau\to\ln R_w} \frac{Y}{y^{1/m}} = 1$.*

The proof is similarly to the proof of [5, Lemma 3.1]. We omit it. Next we study the various global behaviours, according to the value of α. We describe the solutions of (E_w) by phase plane analysis.



$\alpha = 0$, $N = 2$ $\quad$ $\alpha = 1$, $N = 2$

$\alpha = -1$, $N = 2$ $\quad$ $\alpha = 2$, $N = 2$

FIGURE 1. Phase planes with $m = 0.8$ and $p = 1.8$)

4.2. **Subcase $m\alpha \le mN < \delta$.**

Theorem 4.2. *Suppose $-\infty < m\alpha \le mN < \delta$. Then the regular solutions w have a constant sign, and $\lim_{r\to\infty} r^\alpha |w| = L > 0$ if $\alpha < N$, especially $\lim_{r\to\infty} |w| = L > 0$ if $\alpha = 0$, $\lim_{r\to\infty} r^\delta |w|^{m-1} w = \ell$ if $\alpha = N$. And $w(r) = \ell^{1/m} r^{-\delta/m}$ is also a solution. There exist solutions satisfying one of the following properties:*

(1) *(Only if $\alpha < N$) w is positive,* $\lim_{r\to 0} r^{\frac{\eta}{m}} w = c > 0$ *if* $N \geq 2$ *(and* (3.14) *holds with* $a > 0 > b$ *if* $N = 1$*), and* $\lim_{r\to\infty} r^{\delta}|w|^{m-1}w = \ell$*;*
(2) *w is positive,* $\lim_{r\to 0} r^{\delta}|w|^{m-1}w = \ell$, $\lim_{r\to\infty} r^{\alpha}w = L > 0$*;*
(3) *w has precisely one zero,* $\lim_{r\to 0} r^{\delta}|w|^{m-1}w = \ell$, $\lim_{r\to\infty} r^{\alpha}w = L < 0$*;*
(4) *w is positive,* $R_w > 0$, $\lim_{r\to\infty} r^{\delta}|w|^{m-1}w = \ell$*;*
(5) *w is positive,* $R_w > 0$, $\lim_{r\to\infty} r^{\alpha}w = L > 0$*;*
(6) *w has one zero,* $R_w > 0$, $\lim_{r\to\infty} r^{\alpha}w = L \neq 0$*;*
(7) *(Only if $\alpha < N$) w is positive,* $\lim_{r\to 0} r^{\frac{\eta}{m}} w = c > 0$ *if* $N \geq 2$ *(and* (3.14) *holds with* $a > 0 > b$ *if* $N = 1$*), and* $\lim_{r\to\infty} r^{\alpha}w = L > 0$*;*
(8) *w has one zero, with* $\lim_{r\to 0} r^{\frac{\eta}{m}} w = c > 0$ *if* $N \geq 2$ *(and* (3.14) *holds with* $a > 0 > b$ *if* $N = 1$*), and* $\lim_{r\to\infty} r^{\alpha}w = -L < 0$*;*
(9) $N = 1, w > 0$ *and* (3.14) *holds with* $a \geq 0, b > 0$ *and* $\lim_{r\to\infty} r^{\alpha}w = L$.

Up to symmetry, all the solutions of (E_w) are described.

The proof of the above theorem is similarly to that of [5, Theorem 3.2]. See the illustration Figure 1.

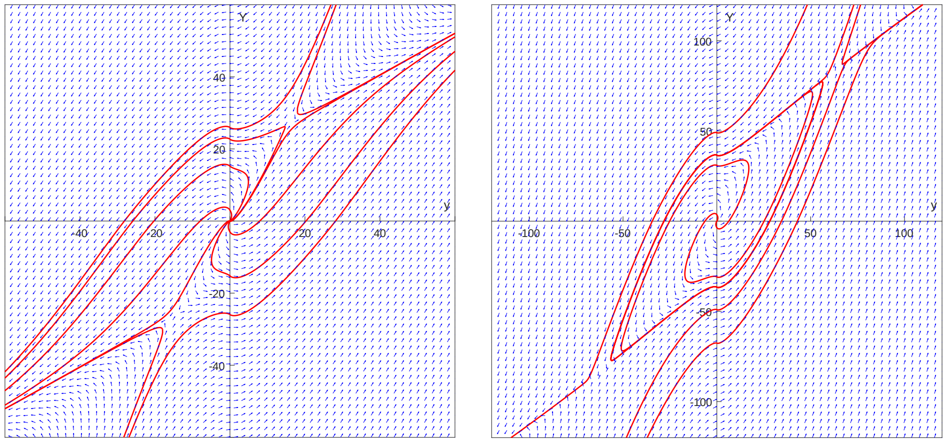
4.3. **Subcase** $mN < m\alpha < \delta$**.** In this case, we show that system (2.14) admits some periodic trajectories, according to the value of α with respect to α^*. Notice that $N < \alpha^*$ whenever $\delta^2 - (m + N + 2)m\delta + m^2N > 0$ from (2.26). Our adopted method is the Poincaré-Bendixson theorem, by applying the level curves of the energy function W.

Lemma 4.3. *Suppose $mN < m\alpha < \delta$. Consider the level curves*
$$C_k = \{(y, Y) \in \mathbb{R}^2 \mid W(y, Y) = k\}(k \in \mathbb{R})$$
of the function W defined at (2.28)*, which are symmetric with respect to* $(0, 0)$*. Let*
$$k_\ell = W(\ell, (\delta\ell)^{p-1}) = \frac{m}{m+1}(\delta - mN)\delta^{p-2}\ell^p. \tag{4.1}$$
If $k > k_\ell$, there are two unbounded connected components in C_k. If $0 < k < k_\ell$, there are three connected components in C_k, and one of them is bounded. If $k = k_\ell$, C_{k_ℓ} is connected with a double point at M_ℓ. If $k = 0$, and one of the three connected components of C_0 is $\{(0, 0)\}$. If $k < 0$, there are two unbounded connected components in C_k.

Proof. The above assertion for the special case $m = 1$ is proved in [5]. Here we can only consider the study of C_k to the set $y > 0$. Define a function $\varphi(s) = \frac{m|s|^{p'}}{p'} - ms + \frac{m}{p}$, for any $s \in \mathbb{R}$, which has the inverse function. Then we analysis the properties of the function $\varphi(s)$ and its inverse function to obtain the results. The rest proof is similar as the proof of [5, Lemma 3.3]. $\square$

Theorem 4.4. *Suppose* $0 < m < 1$, $1 < p < 2$ *and* $mN < m\alpha < \delta$*. Then* $w(r) = \ell^{1/m}r^{-\delta/m}$ *is a solution. Moreover,*

(i) *If $\alpha \leq \alpha^*$, then any solution of (E_w) has at most a finite number of zeros.*
(ii) *There exist $\hat{\alpha}$ such that* $\max(mN, m\alpha^*) < m\hat{\alpha} < \delta$*, if $\alpha > \hat{\alpha}$, there is a cycle around* $(0, 0)$ *in the phase plane (y, Y).*
(iii) *Consider any α such that there exists no such cycle. Then the regular solutions have a finite number of zeros and* $\lim_{r\to\infty} r^{\alpha}w = L_r \neq 0$ *or* $\lim_{r\to\infty} r^{\delta}|w|^{m-1}w = \ell$*. There exist solutions of type (2)-(6) of Theorem 5.2, and other solutions have one of the following properties:*
  (1') *(only if $L_r \neq 0$)* $\lim_{r\to\infty} r^{\alpha}w = L_r \neq 0$ *and* $\lim_{r\to 0} r^{\frac{\eta}{m}} w = c \neq 0$ *(or* (3.14) *holds if* $N = 1$*);*
  (7') $\lim_{r\to 0} r^{\frac{\eta}{m}} w = c \neq 0$ *(or* (3.14) *holds if* $N = 1$*) and* $\lim_{r\to\infty} r^{\alpha}w = L \neq 0$.
(iv) *Consider any α such that there exists such a cycle, thus there exists solutions w with oscillating near 0 and ∞, satisfying $r^{\delta}|w|^{m-1}w$ is periodic in $\ln r$. The regular solutions w oscillate near ∞, satisfying $r^{\delta}|w|^{m-1}w$ is asymptotically periodic in $\ln r$. There exist solutions of type (2), (4), (5) of Theorem 4.2, and other solutions have one of the following properties:*



$N = 2 < \alpha = 2.32 < \frac{\delta}{m}$ $\qquad$ $N = 2 < \alpha = 3.8 < \frac{\delta}{m}$

FIGURE 2. Phase planes with $m = 0.8$ and $p = 1.8$

(1") *with precisely one zero,* $R_w > 0$*, and* $\lim_{r\to\infty} r^{\delta}|w|^{m-1}w = \ell$*;*

(3") *with* $\lim_{r\to 0} r^{\delta}|w|^{m-1}w = \ell$*, and oscillating near* ∞*;*

(9) *with* $\lim_{r\to 0} r^{\frac{\eta}{m}} w = c \neq 0$ *(or* (3.14) *holds if* $N = 1$*) and oscillating near* ∞*.*

(10) *with precisely one zero,* $R_w > 0$*, and* $\lim_{r\to\infty} r^{\alpha} w = L \neq 0$*;*

(11) *with* $R_w > 0$ *and oscillating near* ∞*.*

Proof. First observe (E_w) always admits solutions of type (2), (4), (5) of Theorem 4.2, from Lemma 4.1.

(i) Suppose $\alpha \leq \alpha^*$ (Fig. 2). Consider any trajectory $\mathcal{T}$. If y has an infinity of zeros near $\pm\infty$, then following Proposition 3.9, $\mathcal{T}$ is contained in the set $\mathcal{D} = \{(y, Y) \in \mathbb{R}^2 \mid |y| < \ell, |Y| < (\delta\ell)^{p-1}\}$ near $\pm\infty$. It means that $\mathcal{T}$ is bounded near $\pm\infty$, hence the limit set at $\pm\infty$ is contained $\mathcal{D}$. However $M_\ell \notin \mathcal{D}$; and $(0, 0)$, which is a source, a node point, can not belong to the limit set Γ at ∞. Indeed, according to Poincaré-Bendixson theorem in [17], Γ is a closed orbit, so system (2.14) admits a cycle. Moreover, from (3.20) and (3.21),

$$\frac{\partial f_1}{\partial y}(y, Y) + \frac{\partial f_2}{\partial Y}(y, Y) = \frac{1}{m(p-1)}\big(D^{\frac{1-m}{m}}(\delta D)^{2-p} - |y|^{\frac{1-m}{m}}|Y|^{\frac{2-p}{p-1}}\big),$$

thus applying Bendixson-Dulac criterion, there exists no cycle in the set $\{|y| < D, |Y| < (\delta D)^{p-1}\}$, where $D = (\delta^{p-2}(p-1)(m\delta + \delta - mN))^{\frac{m}{1-mp+m}}$ and $0 < m < 1$, $1 < p < 2$. Now we find that

$$\alpha \leq \alpha^* \Leftrightarrow \ell \leq D, \text{ then } (\delta\ell)^{p-1} \leq (\delta D)^{p-1}. \tag{4.2}$$

Hence there exists no cycle in $\mathcal{D}$, which is contradictory.

(ii) Assume $\alpha > \max(N, \alpha^*)$. The curve $\mathcal{L}$ intersects $\mathcal{M}$ at point $(D, (\delta D)^{p-1})$. Then

$$S_{\mathcal{L}} \cap \mathcal{M} = \{(\theta D, \delta^{p-1}(\theta D)^{p-1}) : \theta \in [0, 1]\};$$

and $D < \ell$ from (4.2), thus M_ℓ is not contained in $S_{\mathcal{L}}$. We can find $k_1 > 0$ small enough satisfying $C^b_{k_1}$ is contained in $S_{\mathcal{L}}$. Next we look for $k \in (0, k_\ell)$ satisfying $\mathcal{L}$ is in the domain restricted by C^b_k. By symmetry, we only consider the points of $\mathcal{L}$ such that $y \geq 0$. By a straightforward calculation, it means that $W(y, Y) \leq K\delta^p D^p$, where $K = \max(\frac{2}{p}, \frac{2m\delta + \delta - mN}{p})$. Let $\hat{\alpha} = \hat{\alpha}(\delta, N)$ be given by $K\delta^p D^p = k_\ell$, then

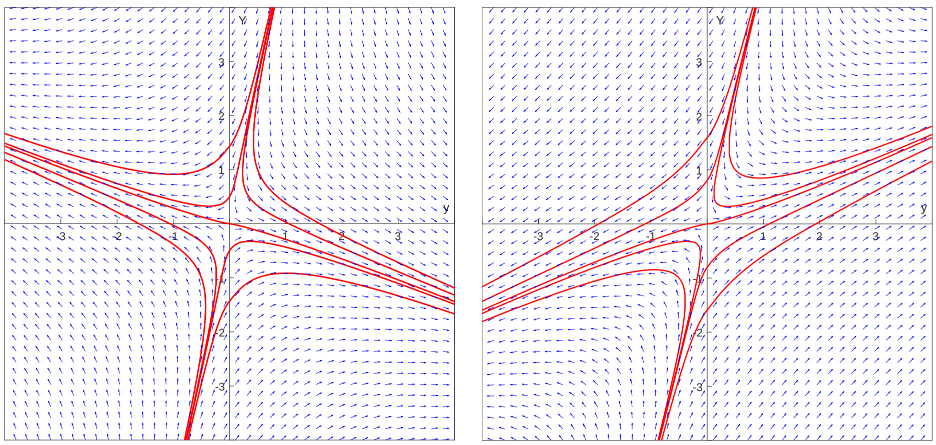
$$\delta - m\hat{\alpha} = \Big[\frac{m(\delta - mN)}{(m+1)K\delta^2}\Big]^{\frac{1-mp+m}{mp}} \frac{\delta - mN}{\delta^{p-3}(p-1)(m\delta + \delta - mN)}.$$

If $\alpha > \hat{\alpha}$, we can find $k_2 (< k_\ell)$ such that $\mathcal{L}$ is contained in the set $\{(y, Y) \in \mathbb{R}^2 \mid W(y, Y) < k_2\}$, which has three connected components. Since $S_{\mathcal{L}}$ is connected, it is contained in $C^b_{k_2}$. Hence $S_{\mathcal{L}}$ is delimitated by $C^b_{k_1}$ and $C^b_{k_2}$, it implies $S_{\mathcal{L}}$ is bounded and positively invariant. There exist no any stationary point in $S_{\mathcal{L}}$, thus it contains a cycle, from the Poincaré-Bendixson theorem (see Fig. 2).

The rest of the proof follows similarly as the proof of Theorem 3.4(iii) and (iv) in [5]. $\square$

Remark 4.5. From the numerical studies, in a similar way as the classification of self-similar solutions of singular p-Laplacian equations by Bidaut-Véron [5], here for the polytropic filtration equations we also conjecture that $\hat{\alpha}$ is unique, and the number of zeros of w is increasing with $\alpha \in (N, \hat{\alpha})$; and moreover there exists $\alpha_1 = N < \alpha_2 < \cdots < \alpha_n < \alpha_{n+1} < \cdots$, such that the regular solutions have n zeros for any $\alpha \in (\alpha_n, \alpha_{n+1})$, with $\lim_{r\to\infty} r^{\alpha} w = L_r \neq 0$, and $n+1$ zeros for $\alpha = \alpha_{n+1}$, with $\lim_{y\to\infty} r^{\delta} w^m = \pm\ell$.

4.4. **Subcase $m\alpha \le \delta \le mN$.** In this case, system (2.14) admits a unique stationary point $(0,0)$, and $N \ge 2$.



$\alpha = -2$, $N = 6$ $\qquad$ $\alpha = 2, N = 6$

FIGURE 3. Phase planes with $m = 0.8$ and $p = 1.8$

Theorem 4.6. *Assume $-\infty < m\alpha \le \delta \le mN$ and $\alpha \neq N$. Then the regular solutions, defined on $(0,\infty)$, have a constant sign, and the positive ones satisfy $\lim_{r\to\infty} r^{\alpha} w = L > 0$ if $m\alpha \neq \delta$, or (3.12) holds if $m\alpha = \delta$. All the other solutions have a reduced domain ($R_w > 0$). Among them, the solutions satisfy one of the following properties:*

(1) *w is positive, $\lim_{r\to\infty} r^{\frac{\eta}{m}} w = c \neq 0$ if $\delta < mN$, or $\lim_{r\to\infty} r^{N} (\ln r)^{\frac{1+mN}{1+m}} w = \tilde{\rho}$ defined at (3.13) if $\delta = mN$.*
(2) *w is positive, $\lim_{r\to\infty} r^{\alpha} w = L > 0$ if $m\alpha \neq \delta$, of (3.12) holds if $m\alpha = \delta$.*
(3) *w has one zero, such that $\lim_{r\to\infty} r^{\alpha} w = L \neq 0$ if $m\alpha \neq \delta$, or (3.12) holds if $m\alpha = \delta$.*

Up to symmetry, all the solutions are described.

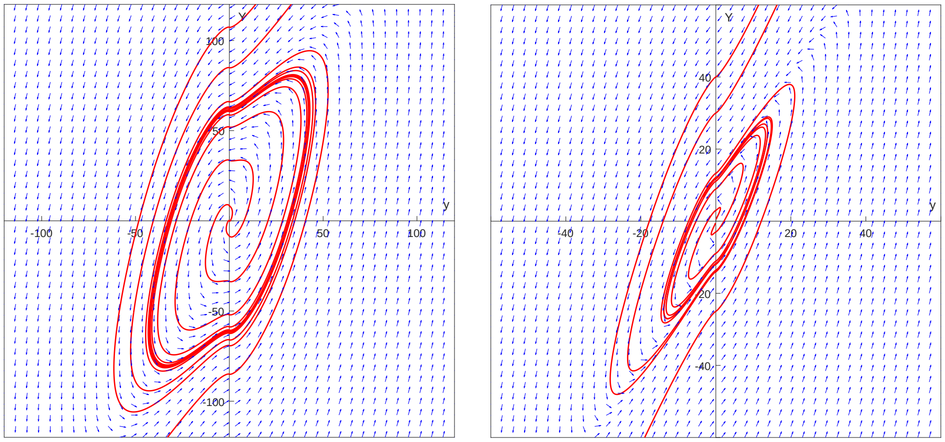
The proof is similarly as that of [5, Theorem 3.5]. See the illustration in Figure 3.

4.5. **Subcase $mN \le \delta \le m\alpha$.**

Theorem 4.7. *Suppose $mN \le \delta \le m\alpha$ and $\alpha \neq N$. Then*

(i) *There is a cycle surrounding $(0,0)$ in system (2.14), thus the corresponding solution has a changing sign such that $r^{\delta}|w|^{m-1}w$ is periodic in $\ln r$. All the other solutions w, in particular the regular ones, are oscillating near ∞, and $r^{\delta}|w|^{m-1}w$ is asymptotically periodic in $\ln r$. Equation (E_w) admits solutions w such that $\lim_{r\to 0} r^{\frac{\eta}{m}} w = c \neq 0$ if $2 \le N < \frac{\delta}{m}$ and (3.13) holds if $mN = \delta$ or (3.14) holds if $N = 1$.*
(ii) *Equation (E_w) admits solutions such that $R_w > 0$ or $\lim_{r\to 0} r^{\alpha} w = L \neq 0$ if $m\alpha \neq \delta$, or (3.12) holds if $m\alpha = \delta$.*

The proof is similarly as that of [5, Theorem 5.1]. See the illustration in Figure 4.



$\alpha = 6,\ N = 2$ $\qquad$ $\alpha = 6,\ N = 5$

FIGURE 4. Phase planes with $m = 0.8$ and $p = 1.8$

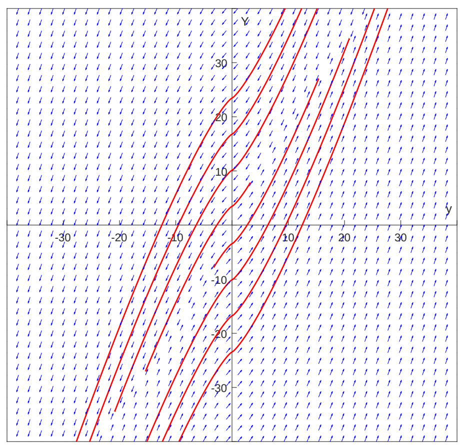


FIGURE 5. Phase plane with $\alpha = N = 5$, $m = 0.8$, and $p = 1.6$

Theorem 4.8. *Suppose $m\alpha = \delta = mN$. Then the regular solutions, given by (2.6), have a constant sign. For any $k \in \mathbb{R}$, $w(r) = |k|^{\frac{1}{m}-1}kr^{-N}$ is a solution. There are solutions satisfying one of the following properties:*

(i) *w is positive, $\lim_{r\to 0} r^N w = c_1 > 0$ and $\lim_{r\to\infty} r^N w = c_2 > 0 (c_1 < c_2)$;*
(ii) *w has one zero, $\lim_{r\to 0} r^N w = c_1 > 0$ and $\lim_{r\to\infty} r^N w = c_2 < 0$;*
(iii) *w is positive, $R_w > 0$, and $\lim_{r\to 0} r^N w = c \neq 0$;*
(iv) *w has one zero, $R_w > 0$,and $\lim_{r\to\infty} r^N w = c \neq 0$.*

Up to a symmetry, all the solutions are described.

The proof is similar to that of [5, Theorem 5.2]. See the illustration Figure 5. Note that for $\delta = mN$, (2.5) is equivalent to $Y \equiv y^{1/m} - C$, from (2.17). For any $k \in \mathbb{R}$, we find $(y, Y) \equiv (k, |mNk|^{p-2}mNk)$, on the curve $\mathcal{M}$, is a solution of system (2.14),

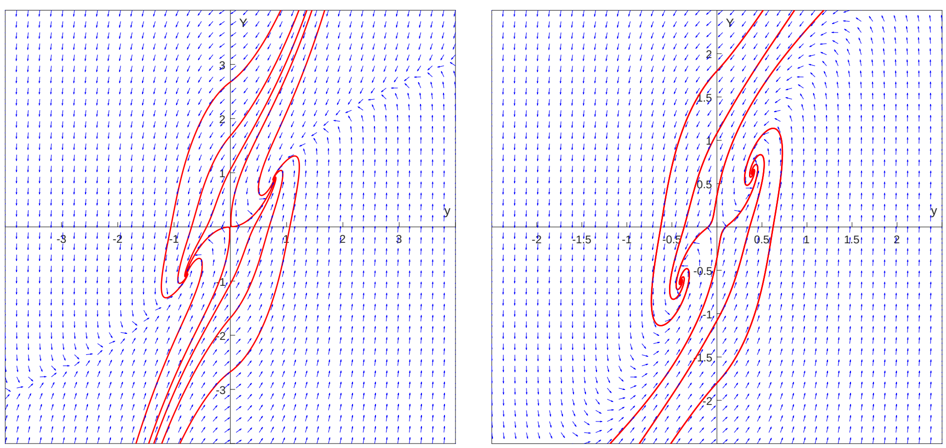
4.6. **Subcase $\delta < \min(m\alpha, mN)$.** In this case, the system admits three stationary points, $(0, 0)$, is a saddle point, and M_ℓ, M'_ℓ are sinks when $\delta \le \frac{mN}{1+m}$ or $\delta > \frac{mN}{1+m}$ and $\alpha < \alpha^*$, and sources when $\delta > \frac{mN}{1+m}$ and $\alpha > \alpha^*$, and node points whenever $\delta > \frac{mN}{1+m} - \frac{2\sqrt{mp'(mN-\delta)}}{1+m}$ and $\alpha \le \alpha_1$, or $\delta > \frac{mN}{1+m} + \frac{2\sqrt{mp'(mN-\delta)}}{1+m}$ and $\alpha \ge \alpha_2$, where α_1, α_2 are defined at (3.19). From Remark 3.15, α_1 can be greater or less than $\frac{\eta}{m}$. A very interesting point is that there exist two types of periodic trajectories in system (2.14), either surrounding $(0,0)$, coinciding with changing sign solutions, or located in $\mathcal{Q}_1$ or $\mathcal{Q}_3$, coinciding with constant sign solutions. We know that $\delta < mN$ implies $\delta < mN < \eta$ from (2.1). And $\frac{mN}{1+m} < \delta$ implies $\frac{\eta}{m} < \alpha^*$ from (2.26). In a similar way as the classification of self-similar solutions of singular p-Laplace equations by Bidaut-Véron [5], here for the polytropic filtration equations we also show some general properties of the phase plane firstly.

Remark 4.9. (i) The trajectory $\mathcal{T}_r$ starts in $\mathcal{Q}_1$. Because $(0,0)$ is a saddle point, system (2.14) admits a unique trajectory $\mathcal{T}_s$ converging to $(0,0)$, in $\mathcal{Q}_1$ for large τ, and $\lim_{r\to\infty} r^{\frac{\eta}{m}} w = c > 0$, from Proposition 3.7 and 3.8. Furthermore, if $\mathcal{T}_r$ does not stay in $\mathcal{Q}_1$, then $\mathcal{T}_s$ stays in $\mathcal{Q}_1$, and $\mathcal{T}_s$ is bounded and in the interior of the domain delimitated by $\mathcal{Q}_1 \cup \mathcal{T}_r$, from Remark **??**. If $\mathcal{T}_r$ is homoclinic, it stays in $\mathcal{Q}_1$.

(ii) Any trajectory $\mathcal{T}$, where y is not monotone near ∞, is bounded near ∞ from Proposition 3.11. According to the Poincaré-Bendixson theorem, and trajectory, which is $\mathcal{T}$ bounded at $\pm\infty$, either converges to $(0,0)$ or $\pm M_\ell$, or the limit set $\Gamma_\pm$ at $\pm\infty$ is cycle, or $\Gamma_\pm$ is homoclinic hence $\mathcal{T} = \mathcal{T}_r$, $\Gamma_\pm = \overline{\mathcal{T}_r}$.

(iii) If system (2.14) admits a limit cycle surrounding $(0,0)$, then it also surrounds the points $\pm M_\ell$ from (3.15).

First consider the case $m\alpha \le \eta$, where there exists no cycle or no homoclinic orbit in $\mathcal{Q}_1$, from Theorem 3.17.



$\alpha = 5$, $N = 4$ $\qquad\qquad$ $\alpha = 10$, $N = 4$

FIGURE 6. Phase planes with $m = 0.5$, $p = 1.5$, and $\eta = 5$

Theorem 4.10. *Suppose $\delta < \min(m\alpha, mN)$, and $m\alpha \le \eta$. Then the regular solutions have a constant sign, and $\lim_{r\to\infty} r^\delta |w|^{m-1} w = \ell$. And $w(r) = \ell^{1/m} r^{-\delta/m}$ is a solution. There are the solutions satisfying one of the following properties:*

- (i) *If $m\alpha < \eta$, there exist solutions such that*
  - (1) *w is positive, $\lim_{r\to 0} r^\alpha w = L$ and $\lim_{r\to\infty} r^\delta |w|^{m-1} w = \ell$;*
  - (2) *w is positive, $R_w > 0$, and $\lim_{r\to\infty} r^{\frac{\eta}{m}} w = c > 0$;*
  - (3) *w is positive, $R_w > 0$, and $\lim_{r\to\infty} r^\delta |w|^{m-1} w = \ell$;*
  - (4) *w has one zero, $R_w > 0$, and $\lim_{r\to\infty} r^\delta |w|^{m-1} w = \ell$;*
- (ii) *If $m\alpha = \eta$, then $w = Cr^{-\eta/m}$ is a solution. The equation (E_w) admits the solutions of type (4), but not of type (2) or (3).*

Proof. (i) For the case when $m\alpha < \eta$, the proof follows similarly as that of [5, Theorem 5.4]. See the illustration in Figure 6.

(ii) For the case when $m\alpha = \eta$ (see Figure 6), there exists no positive solution satisfying $R_w > 0$, thus no solution of type (2) or (3). Indeed all the trajectories stay under $\mathcal{T}_s$, and $\mathcal{T}_s$ is defined by the equation $\zeta \equiv \eta$, that means $w \equiv cr^{-\eta/m}$ for (2.30), or $\mathcal{T}_s$ is equivalent to

$Y_\eta \equiv c, Y'_\eta \equiv 0$. Consider any trajectory $\mathcal{T}_{[P]}$ running through some point $P = (\varphi, 0)$, $\varphi > 0$, and the solution (y, Y) starting from P at time 0. Then $Y_\eta(0) = 0$, and $Y_\eta < 0$ on $(-\infty, 0)$ thus $Y'_\eta = \frac{1}{m} e^{\frac{1-mp+m}{m}(\delta-\eta)} y_\eta^{\frac{1-m}{m}} (\eta y - |Y|^{\frac{2-p}{p-1}}) > 0$ in $\mathcal{Q}_4$. Suppose it satisfies $R_w = 0$. Then $\mathcal{T}_{[P]}$ starts from $\mathcal{Q}_3$ with $\lim_{\tau\to-\infty} \zeta = m\alpha = \eta$ and $\lim_{\tau\to-\infty} y_\eta = -L < 0$, then $\lim_{\tau\to-\infty} Y_\eta = -(m\alpha|L|)^{p-1}$. From (2.12), a straightforward calculation shows

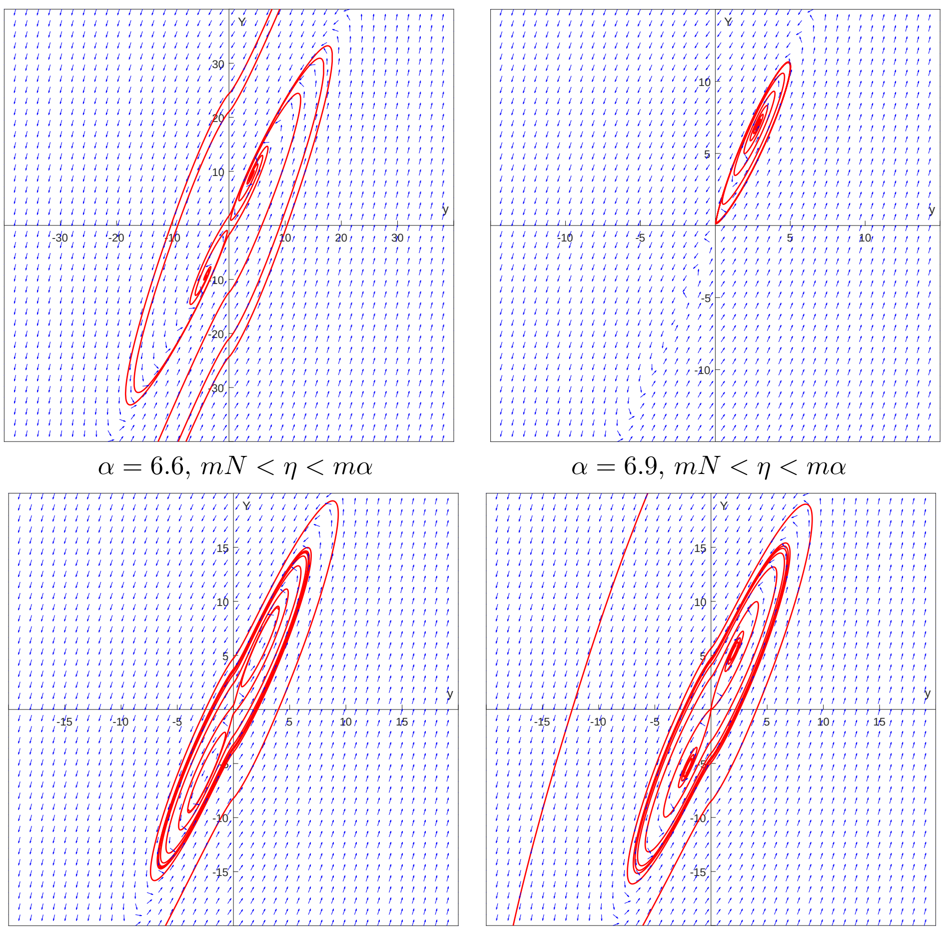
$$y''_\eta = \eta y'_\eta - \frac{e^{(p+\eta(p-1)-\frac{\eta}{m})\tau}}{m(p-1)} |y'_\eta - \eta y_\eta|^{2-p} |y_\eta|^{\frac{1}{m}-1} y'_\eta$$

Then $y''_\eta < 0$ near $-\infty$, which is contradictory. Thus $R_w > 0$ and w is of type (4). $\square$

We now pay more attention to the interesting case, where $\eta < m\alpha$.

Lemma 4.11. *Assume $\delta < \min(m\alpha, mN)$ and $\eta < m\alpha$. If $\delta > \frac{mN}{1+m}$ and $\alpha < \alpha^*$ and $\mathcal{T}_s$ stays in $\mathcal{Q}_1$, then it has a limit cycle at $-\infty$ in $\mathcal{Q}_1$, or it is homoclinic.*

The proof is to that of [5, Lemma 5.5]. Here we omit i.



$\alpha = 6.6$, $mN < \eta < m\alpha$ $\alpha = 6.9$, $mN < \eta < m\alpha$

$\alpha = 7 < \alpha^* = 7.083$ $\alpha = 7.2 > \alpha^* = 7.083$

FIGURE 7. Phase planes with $m = 0.8$, $p = 1.8$, $N = 6$, and$\eta = 5.25$

Theorem 4.12. *Suppose $\frac{mN}{1+m} \le \delta < \min(m\alpha, mN)$ and $1 < p < 2$. Then $w(r) = \ell^{1/m} r^{-\delta/m}$ is still a solution. Moreover*

(i) There exists a (maximal) critical value α_{crit} of α, such that

$$\max(\frac{\eta}{m}, \alpha_1) < \alpha_{crit} < \alpha^*,$$

and the regular trajectory is homoclinic: all the regular solutions have a constant sign and satisfy $\lim_{r\to\infty} r^{\frac{\eta}{m}} w = c \neq 0$.

(ii) For any $\alpha \in (\alpha_{crit}, \alpha^*)$*, there does exist a unique cycle in* $\mathcal{Q}_1$*, which implies that the equation* (E_w) *admits positive solutions* w *such that*$r^\delta|w|^{m-1}w$ *is asymptotically periodic in* $\ln r$ *near* 0 *and* $\lim_{r\to\infty} r^\delta|w|^{m-1}w = \ell \neq 0$*. The equation* (E_w) *also has positive solutions such that* $r^\delta|w|^{m-1}w$ *is asymptotically periodic in* $\ln r$ *near* 0 *and* $\lim_{r\to\infty} r^{\frac{\eta}{m}} w = c > 0$*.*

(iii) For any $\alpha \geq \alpha^*$*, there does not exist such a cycle in* $\mathcal{Q}_1$*, but the equation* (E_w) *admits positive solutions such that* $\lim_{r\to 0} r^\delta|w|^{m-1}w = \ell$ *and* $\lim_{r\to\infty} r^{\frac{\eta}{m}} w = c > 0$*.*

(iv) For any $\alpha > \alpha_{crit}$*, there exists also a cycle around* $(0,0)$ *and* $\pm M_\ell$*, thus* $r^\delta|w|^{m-1}w$ *is changing sign and periodic in* $\ln r$*. The regular solutions, are changing sign and oscillating at* ∞*, and* $r^\delta|w|^{m-1}w$ *is asymptotically periodic in* $\ln r$*. There are solutions in the equation* (E_w) *satisfying* $R_w > 0$ *or* $\lim_{r\to 0} r^\alpha w = L \neq 0$*, and oscillating at* ∞*, and* $r^\delta|w|^{m-1}w$ *is asymptotically periodic in* $\ln r$*.*

The proof is similarly to that of [5, Theorem 5.6]. See the illustration Figure 7. Note that for non-Newtonian polytropic filtration equation (1.1), both m and p exert critical influence on the asymptotic behavior. The result in Theorem 3.14 is valid under the condition $1 < p < 2$. And when applying Theorem 3.17, we should pay attention to the fact that its conclusion holds exclusively under the conditions $0 < m < 1$, $1 < p < 2$ and $(p-2)(m\alpha-\delta)(m\alpha-\delta+2m\delta)+m(1-m)\delta^2 < 0$.

Remark 4.13. *An open question is the uniqueness of* α_{crit}*. It can be shown that if there exist two critical values* $\alpha^1_{crit} > \alpha^2_{crit}$*, then the first orbit is contained in the second one.*

In the case $\delta \leq \frac{mN}{1+m}$, there exists no cycle in $\mathbb{R}^2$ from Proposition 3.10, and we obtain the following result.

Theorem 4.14. *Suppose* $\delta \leq \frac{mN}{1+m}$ *and* $\delta < m\alpha$*. Then the regular solutions have a constant sign, and* $\lim_{r\to\infty} r^\delta|w|^{m-1}w = \ell$*. All the solutions have a finite number of zeros. And* $w(r) = \ell^{1/m}r^{-\delta/m}$ *is a solution. Moreover, if* $m\alpha \leq \eta$*, the results were shown in Theorem 4.10. If* $\eta \leq m\alpha$*, there exist at least one zero in all the other solutions. There exist solutions, such that* $\lim_{r\to\infty} r^{\frac{\eta}{m}} w = c \neq 0$*, with a number* n *of zeros, for* $n \geq 1$*. All the other solutions satisfy* $\lim_{r\to\infty} r^\delta|w|^{m-1}w = \ell$*, with* n *or* $n+1$ *zeros, for* $n \geq 1$*.*

The iss similarly to that of [5, Theorem 5.7].

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