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UNCONDITIONAL WELL-POSEDNESS FOR THE NONLINEAR SCHRÖDINGER EQUATION IN BESSEL POTENTIAL SPACES

RYOSUKE HYAKUNA

Abstract. The Cauchy problem for the nonlinear Schrödinger equation is called unconditionally well posed in a data space E if it is well posed in the usual sense and the solution is unique in the space $C([0,T];E)$. In this paper, this notion of the unconditional well-posedness is redefined so that it covers L^p-based Sobolev spaces as data space E and it is equivalent to the usual one when E is an L^2-based Sobolev space H^s. Based on this definition, it is shown that the Cauchy problem for the 1D cubic NLS is unconditionally well posed in Bessel potential spaces H^s_p for $4/3 < p \le 2$ under certain assumptions on s.

1. Introduction

We consider the Cauchy problem for the one dimensional cubic nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2u = 0, \quad u(0) = \phi \in E, \tag{1.1}$$

where E is a Banach space of complex valued functions on $\mathbb{R}$. Recall that Cauchy problem (1.1) is called locally well posed in the data space E if, for any $\phi \in E$ there are a $T = T(\phi) > 0$ and a unique solution u of (1.1) in the space $C([0,T];E) \cap Y_T \triangleq Z_T$ and the map $\phi \mapsto u$ is locally Lipschitz from E to Z_T, where Y_T is a space of functions on $[0,T] \times \mathbb{R}$. For example, (1.1) is locally (and globally) well posed in H^s for $s \ge 0$. The space Y_T is called an auxiliary space and in some cases the well-posedness can be shown without using this space. In fact, if $s > 1/2$ the solution can be obtained directly in a closed subset of $L^\infty([0,T];H^s)$ via the standard fixed point argument and the local well-posedness holds with $Z_T = C([0,T];H^s)$. This is possible because the space H^s forms an algebra if $s > 1/2$. On the other hand, when $s < 1/2$, the solution of (1.1) is usually constructed in the space $L^\infty([0,T];H^s) \cap L^q([0,T];H^s_r)$ for a suitable choice of (q,r) with the aid of the so-called Strichartz estimate (see e.g. [2]). In this case, $Y_T = L^q([0,T];H^s_r)$, and the uniquness is shown in this space, not in $C([0,T];H^s)$. Therefore, it is natural to ask if the well-posedness holds with $Z_T = C([0,T];E)$ for $s < 1/2$. This problem of whether or not the auxiliary condition is removable was first proposed by Kato [11]. He said (1.1) is *unconditionally well posed* in E if it is well posed with $Z_T = C([0,T];E)$, or equivalently, the uniqueness of solutions holds in the space $C([0,T];E)$ in addition to its well-posedness. In [11] he showed that the trivial sufficient condition $s > 1/2$ for unconditional well-posedness in H^s can be pushed down to $s = 1/6$. It is known that this is the minimal Sobolev regularity for the unconditional well-posedness of the 1D cubic NLS (1.1). Thus, the fact that $u \in L^q([0,T];H^s_r)$ is simply an additional regularity property of the solution which is removable in the uniqueness assertion for $s \ge 1/6$, while it is required to ensure the well-posedness in H^s for $s < 1/6$. Since the pioneering work [11], the problem of the unconditional well-posedness for nonlinear Schrödinger equations and other nonlinear dispersive equations has been extensively studied. We refer to [13] and references therein for earliear results in this direction.

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The aim of this article is to extend the unconditional well-posedness results for the L^2-based Sobolev spaces H^s to the Bessel potential spaces H^s_p. One might think discussing such a problem is meaningless, since if $p \neq 2$ the Schrödinger equations are not well posed in H^s_p even in the linear case (see [1, 7]). In particular, we cannot expect the persistence property $u(t) \in H^s_p$ for data $u(0) \in H^s_p$ unless $p = 2$ let alone the uniqueness issue in the space $C([0,T]; H^s_p)$. Nevertheless, we stress that the problem of the unconditional well-posedness can be generalized to the L^p-setting in a natural manner. Our idea here is motivated by Zhou. In [16] he considered the "twisted" variable $v(t) \triangleq e^{-it\partial_x^2}u(t)$ and rewrite the integral equation corresponding to (1.1) as

$$v(t) = \phi + i\int_0^t e^{-i\tau\partial_x^2}\big[(e^{i\tau\partial_x^2}v(\tau))(e^{i\tau\partial_x^2}v(\tau))(\overline{e^{i\tau\partial_x^2}v(\tau)})\big]\,d\tau. \tag{1.2}$$

Note that (1.2) is well-known as the integral equation and v as a state both in the interaction picture. Then he showed that a local solution v of (1.2) exists in the space $C([0,T]\,;\,L^p)$ for $\phi \in L^p$, $1 < p < 2$. This result implies the existence of a solution of the original Cauchy problem (1.1) such that $e^{-it\partial_x^2}u(t) \in C([0,T]\,;L^p)$. Especially, note that if $p = 2$, this is equivalent to the usual persistence property of the solution, that is, $u \in C([0,T]; L^2)$. Thus, a suitable space to discuss the problem of the unconditional uniqueness in the L^p-space is

$$\{u : e^{-it\partial_x^2}u(t) \in C([0,T]; L^p)\}. \tag{1.3}$$

In this article, we formulate the notion of the unconditional well-posedness and uniqueness in the L^p, and more generally, the Bessel potential spaces H^s_p based on this idea. Then, we show that (1.1) is unconditionally locally well posed in H^s_p under some assumptions on the Sobolev regularity. Finally, for the unconditional well-posedenss in other non-L^2-based spaces, we refer to [14], where the authors discuss the unconditional uniqueness of the periodic NLS in the Fourier-Lebesgue spaces $\mathcal{F}L^p$ as data space. Note also that for the Fourier-Lebesgue spaces, one does not need to consider the twisted variable as in the case of H^s_p, since the solution is expected to have the usual persistence property $u \in C([0,T]; \mathcal{F}L^p)$. See [6, 10].

Notation. For $1 \le p \le \infty$ and $I \subset \mathbb{R}$, $\|f\|_{L^p(I)}$ denotes the usual L^p-norm. p' is the conjugate exponent of p: $1/p + 1/p' = 1$. The Fourier transform of ϕ is denoted by $\hat{\phi}$. We denote by C a positive constant which may vary from line to line. The Bessel potential spaces are defined by

$$H^s_p(\mathbb{R}) \triangleq \{\phi \in \mathscr{S}'(\mathbb{R}) : \langle D\rangle^s\phi \in L^p(\mathbb{R})\},$$

equipped with the norm

$$\|\phi\|_{H^s_p(\mathbb{R})} \triangleq \|\langle D\rangle^s\phi\|_{L^p(\mathbb{R})}\,,$$

where $\widehat{\langle D\rangle^s\phi}(\xi) = (1+|\xi|^2)^{s/2}\hat{\phi}(\xi)$. In particular, we write $H^s = H^s_2$ as usual. For simplicity we often write L^p, H^s_p to denote $L^p(\mathbb{R}), H^s_p(\mathbb{R})$ respectively. For a function $u : I\times\mathbb{R} \to \mathbb{C}$, we set

$$\|u\|_{L^q(I;H^s_p)} \triangleq \Big(\int_I \|u(t,\cdot)\|^q_{H^s_p}\,dt\Big)^{1/q}.$$

We define function spaces $\mathfrak{C}(I;E)$ and $\mathfrak{L}^\infty(I;E)$ to introduce a concept of well-posedness of (1.1) in L^p.

Definition 1.1. Let $E \subset \mathcal{S}'(\mathbb{R})$ and $I \subset \mathbb{R}$. The space $\mathfrak{C}(I;E)$ is defined by

$$\mathfrak{C}(I;E) \triangleq \{u = e^{it\partial_x^2}v(t) : v \in C(I;E)\}$$

equipped with the norm

$$\|u\|_{\mathfrak{C}(I;E)} \triangleq \sup_{t\in I}\|e^{-it\partial_x^2}u\|_E.$$

Similarly, the space $\mathfrak{L}^\infty(I;E)$ is defined by

$$\mathfrak{L}^\infty(I;E) \triangleq \{u = e^{it\partial_x^2}v(t) : v \in L^\infty(I;E)\}$$

equipped with the norm

$$\|u\|_{\mathfrak{L}^\infty(I;E)} \triangleq \|e^{-it\partial_x^2}u\|_{L^\infty(I;E)}.$$

We first give the precise definition of "twisted" local well-posedness for (1.1) which was introduced by Zhou.

For a normed space E, $\mathcal{B}_E(r)$ denotes the closed ball with the radius $r > 0$, namely $\mathcal{B}_E(r) \triangleq \{u \in E : \|u\|_E \le r\}$.

Definition 1.2. Let E be a normed space of functions on $\mathbb{R}$. Cauchy problem (1.1) is locally well posed in E if, for any $M > 0$ there exist a $T_M > 0$ and a space Y_{T_M} of functions on $[0, T_M] \times \mathbb{R}$ such that: for any $\phi \in \mathcal{B}_E(M)$ there is a unique solution in $u \in Z_{T_M} \triangleq \mathfrak{C}([0, T_M]; E) \cap Y_{T_M}$. Moreover, the map $\phi \mapsto u$ is Lipschitz from $\mathcal{B}_E(M)$ to Z_{T_M}.

Now we give the definition of the unconditional well-poseness for (1.1).

Definition 1.3. Let E be the same as Definition 1.2. Cauchy problem (1.1) is unconditionally locally well posed in E if it is locally well posed in the sense of Definition 1.2 with $Z_{T_M} = \mathfrak{C}([0, T_M]; E)$, namely the uniqueness holds in the space $\mathfrak{C}([0, T_M]; E)$.

Remark 1.4. Note that in the case of $E = H^s$, (1.1) is locally well posed in the sense of Definition 1.2 if and only if it is locally well posed in the usual sense, since $C([0, T]; H^s) = \mathfrak{C}([0, T]; H^s)$. Similarly, it is unconditionally locally well posed in the sense of Definition 1.3 if and only if it is unconditionally locally well posed in the usual sense.

Our results in this paper are as follows. We begin with the "conditional" well-posedness result.

Proposition 1.5. *Let $4/3 < p \le 2$ and $0 < s < 3/2 - 2/p$. Then (1.1) is locally well posed in $H_p^s(\mathbb{R})$ in the sense of Definition 1.2 with $Y_{T_M} = L^q([0, T_M]; H_r^s)$ for suitable q, r.*

The conditional well-posedness results in H_p^s can be proved arguing similarly as in the proof of [9, Theorem 1.1]. We prove this proposition at the end of this paper. The main results of this paper is the unconditional well-posedness for the Cauchy problem under additional regularity assumptions on the data.

Theorem 1.6. *Let $4/3 < p \le 2$ and $s < 3/2 - 2/p$. Then (1.1) is unconditionally locally well posed in $H_p^s(\mathbb{R})$ in the sense of Definition 1.3 if:*

(1) *$4/3 < p \le 3/2$ and $s > 0$;*
(2) *$3/2 < p \le 2$ and $s \ge 2/3 - 1/p$.*

Remark 1.7. Let $s_c(p) \triangleq \max(0, 2/3 - 1/p)$ for $1 \le p \le 2$. Then Theorem 1.6 asserts that (1.1) is unconditionally locally well posed in H_p^s for $s > s_c(p)$ when $4/3 < p \le 3/2$ and for $s \ge s_c(p)$ when $3/2 < p \le 2$. The exponent $s_c(p)$ is considered as a natural threshold for the unconditional well-posedness in the following sense. In the case of the cubic NLS, one need $|u|^2 u \in L^1_{loc}$ so that the nonlinear part makes sense in the distributional framework(see e.g. [13]). Thus we require $u \in L^3_{loc}$. Now we assume $e^{-it\partial_x^2} u(t) \in H_p^s$. Then, by the well-known time decay property of the evolution group $e^{it\partial_x^2}$, we have $u(t) \in H_{p'}^s$, $t \neq 0$. When $3/2 < p \le 2$, we see that $u(t) \in L^3$ if $s \ge s_c(p) = 2/3 - 1/p$ by Sobolev's embedding. When $1 \le p \le 3/2$, we have $u(t) \in H_{p'}^s \subset L^{p'}$ for $s \ge s_c(p) = 0$ and thus we see that $u(t) \in L^3_{loc}$ since $3 \le p' \le \infty$.

2. Proof of main results

2.1. **Key lemma and proposition.** Our unconditional well-posedness results are proved by a basic embedding theorem for $\mathfrak{L}^\infty(I; H_p^s)$ and Strichartz type estimates.

Lemma 2.1. *Let $I \subset \mathbb{R}$ be a finite interval. Let $2 \le q, r \le \infty$, $1 \le p \le 2, s \ge 0$ satisfy*

$$s \ge 1 - \frac{1}{p} - \frac{1}{r}, \quad q\Big(\frac{1}{p} - \frac{1}{2}\Big) < 1 \quad (\text{with the convention that } \infty \cdot 0 = 0).$$

Then

$$\|f\|_{L^q(I; L^r(\mathbb{R}))} \le C_I \|f\|_{\mathfrak{L}^\infty(I; H_p^s(\mathbb{R}))}.$$

Proof. Let $2 \le r \le \infty$. We first fix $t \in I$. Then, by Sobolev's embedding and the well-known decay property of the free group $e^{it\partial_x^2}$, we have

$$\|e^{it\partial_x^2} f(t)\|_{L^r} \le C\|e^{it\partial_x^2} f(t)\|_{\dot{H}^{s(r)}_{p'}} \le C(4\pi|t|)^{-(1/p-1/2)}\|f(t)\|_{\dot{H}^{s(r)}_{p}},$$

where $s(r) = 1 - 1/p - 1/r$. Replacing f with $e^{-it\partial_x^2} f$, we obtain

$$\|f(t)\|_{L^r} \le C(4\pi|t|)^{-(\frac{1}{p}-\frac{1}{2})}\|e^{-it\partial_x^2} f(t)\|_{H^s_p}, \tag{2.1}$$

for any $s \ge s(r)$ and $t \in I$. The desired embedding is obtained after taking $\|\cdot\|_{L^q(I)}$-norm of both sides and applying Hölder's inequality in the right hand side. □

The next key estimates are the inhomogeneous Strichartz inequalities for not necessarily admissible pairs.

Proposition 2.2 (See [5, 12, 15])**.** *Let $2 \le q, r < \infty$, $1 \le \rho \le 2$, $1 < \gamma \le 2$ be such that*

$$2 + \frac{2}{q} + \frac{1}{r} = \frac{2}{\gamma} + \frac{1}{\rho}, \quad \frac{1}{q} + \frac{1}{r} < \frac{1}{2}, \quad \frac{3}{2} - \frac{1}{\rho} < \frac{1}{\gamma} < 1.$$

Then

$$\Big\|\int_0^t e^{i(t-\tau)\partial_x^2} F(\tau)d\tau\Big\|_{L^q(I;L^r(\mathbb{R}))} \le C\|F\|_{L^\gamma(I;L^\rho(\mathbb{R}))}. \tag{2.2}$$

Proof of Theorem 1.6. We prove the unconditional well-posedness results. We assume that the conditional local well-posedness result (Proposition 1.5) holds, which will be proved in the last section. Then it is enough to show the uniqueness of the solution in the space $\mathfrak{C}([0,T]; H^s_p)$ to conclude the unconditional well-posedness. We first consider case (i) and we let $4/3 < p \le 3/2$ and $s > 0$. We may assume s is sufficiently small, since the uniqueness of the solution in a function space also implies the uniqueness in any smaller spaces. Let $\delta > 0$ and we set

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{2} + \frac{\delta}{2}, \quad \frac{1}{r} = 1 - \frac{1}{p} - \delta, \quad \frac{1}{\rho} = 3 - \frac{3}{p} - 3\delta (= \frac{3}{r}), \quad \frac{1}{\gamma} = \frac{2}{p} - \frac{1}{2} + \frac{3}{2}\delta.$$

Then the pair (q, r) satisfies the assumption of Lemma 2.1 with $s = \delta$ and the quadruple (q, r, γ, ρ) satisfies the assumption of Proposition 2.2 if δ is sufficiently small. Now we consider two solutions $u, v \in \mathfrak{C}([0,T]; H^\delta_p(\mathbb{R}))$ of (1.1) with $u(0) = v(0) \in H^\delta_p(\mathbb{R})$. We want to show that $u(t) = v(t)$ for all $t \in [0,T]$. By Lemma 2.1 we see that $u, v \in L^q([0,T]; L^r)$. Let $0 < T_0 \le T$. We estimate the difference $\|u - v\|_{L^q([0,T_0];L^r)}$. By Duhamel's formula and Proposition 2.2,

$$\begin{aligned}\|u - v\|_{L^q([0,T_0];L^r)} &= \Big\|\int_0^t e^{i(t-\tau)\partial_x^2}(|u|^2u - |v|^2v)d\tau\Big\|_{L^q([0,T_0];L^r)} \\ &\le C\||u|^2u - |v|^2v\|_{L^\gamma([0,T_0];L^\rho)}.\end{aligned}$$

By Hölder's inequality in the time variable and the triangle inequality, the norm in the right hand side is estimated by

$$\begin{aligned}&\|u^2(\bar{u} - \bar{v})\|_{L^\gamma([0,T_0];L^\rho)} + \|u\bar{v}(u - v)\|_{L^\gamma([0,T];L^\rho)} + \|v\bar{v}(u - v)\|_{L^\gamma([0,T_0];L^\rho)} \\ &\le CT_0^{1-\frac{1}{p}}\Big(\|u\|^2_{L^q([0,T];L^r)} + \|u\|_{L^q([0,T];L^r)}\|v\|_{L^q([0,T];L^r)} + \|v\|^2_{L^q([0,T];L^r)}\Big)\|u - v\|_{L^q([0,T_0];L^r)}.\end{aligned}$$

We put $\eta_T \triangleq \max\big(\|u\|_{L^q([0,T];L^r)}, \|v\|_{L^q([0,T];L^r)}\big)$. Then we have

$$\|u - v\|_{L^q([0,T_0];L^r)} \le 3CT_0^{1-\frac{1}{p}}\eta_T^2\|u - v\|_{L^q([0,T_0];L^r)}. \tag{2.3}$$

Now we choose T_0 so that

$$3CT_0^{1-\frac{1}{p}}\eta_T^2 \le \frac{1}{2}.$$

Then $\|u - v\|_{L^q([0,T_0];L^r)} = 0$ and thus $u(t) = v(t)$ for all $t \in [0,T_0]$. In a similar manner, we can see that $u(t) = v(t)$ for all $t \in [T_0, 2T_0]$, since the time interval T_0 can be determined depending only on η_T. Repeating this argument, we finally get $u = v$ on $[0,T]$, which concludes the uniqueness in the space $\mathfrak{C}([0,T]; H^\delta_p)$.

The uniqueness assertion for case (ii) of Theorem 1.3 can be treated in the same manner. We let $3/2 < p \le 2$ and $s = s_c(p) = 2/3 - 1/p$. For a sufficiently small $\delta > 0$ we set

$$\frac{1}{q} = \frac{1}{6} - \delta, \quad \frac{1}{r} = \frac{1}{3}, \quad \frac{1}{\rho} = 1 (= \frac{3}{r}), \quad \frac{1}{\gamma} = \frac{5}{6} - \delta.$$

Then it is easy to check that q, r, γ, ρ satisfy the assumption of Proposition 2.2 if δ is small enough. Moreover, s, q, r satisfy the condition of Lemma 2.1 if $\delta < 2/3 - 1/p$. Thus

$$\mathfrak{L}^\infty([0,T]; H^s_p) \hookrightarrow L^q([0,T]; L^r).$$

Then we get the desired uniqueness in $\mathfrak{C}([0,T]; H^s_p)$ by considering two solutions $u, v \in \mathfrak{C}([0,T]; H^s_p)$ with $u(0) = v(0)$ and estimating the difference $u - v$ in $L^q([0,T]; L^r)$ as in case (i). □

3. Proof of the conditional well-posedness result

3.1. **Strichartz estimates.** In the last section we prove the conditional well-posedness for (1.1) in H^s_p. We exploit the standard Strichartz technique here. This approach is well known in the case where the initial data lie in the L^2 and H^s spaces ($s \ge 0$). It is shown in [9] that it can also work well for the case where data are in L^p, $4/3 < p < 2$. We begin with the estimates for the homogeneous equation.

Proposition 3.1 ([12, Theorem 3.2]). *Let $1 < p \le 2$ and let $q, r \in [2, \infty]$ be such that*

$$\frac{2}{q} + \frac{1}{r} = \frac{1}{p}, \quad \frac{1}{q} + \frac{1}{r} < \frac{1}{2}.$$

Then

$$\|e^{it\partial_x^2}\phi\|_{L^q(\mathbb{R}; L^r(\mathbb{R}))} \le C\|\phi\|_{L^p(\mathbb{R})}. \tag{3.1}$$

In particular, for any $s \in \mathbb{R}$ we have

$$\|e^{it\partial_x^2}\phi\|_{L^q(\mathbb{R}; H^s_r(\mathbb{R}))} \le C\|\phi\|_{H^s_p(\mathbb{R})}. \tag{3.2}$$

Inequalitites (3.1) and (3.2) are used for estimating the linear part of the corresponding integral equation. Moreover, when $3/4 < p \le 2$, stronger estimates are known:

Proposition 3.2 ([10, Theorem 5]). *Let $4/3 < p \le 2$ and let $q, r \in [2, \infty]$ be such that*

$$\frac{2}{q} + \frac{1}{r} = \frac{1}{p}, \quad 0 < \frac{1}{q} < \min\Big(\frac{1}{2} - \frac{1}{r}, \frac{1}{4}\Big).$$

Then

$$\|e^{it\partial_x^2}\phi\|_{L^q(\mathbb{R}; L^r(\mathbb{R}))} \le C\|\hat{\phi}\|_{L^{p'}(\mathbb{R})}. \tag{3.3}$$

The diagonal case $q = r = 3p$ of (3.3) goes back to Fefferman [4] and is used to obtain the well-posedness results for (1.1) in [6, 3, 10]. In this paper we essentially use (3.3) to deal with the nonlinear term. Indeed, the following equivalent form of Proposition 3.2 is useful in the estimates of the Duhamel term of the integral equation.

Corollary 3.3 ([9, Proposition 2.6]). *Let $4/3 < p \le 2$ and let σ, ρ be such that $(q, r) \triangleq (\sigma', \rho')$ satisfies the assumption of Proposition 3.2. Then*

$$\Big\|\int_0^t e^{i(t-\tau)\partial_x^2}F(\tau)d\tau\Big\|_{\mathfrak{L}^\infty([0,T]; L^p(\mathbb{R}))} \le C\|\tau^{\frac{1}{p}-\frac{1}{2}}F(\tau)\|_{L^\sigma([0,T]; L^\rho(\mathbb{R}))}.$$

In particular, for any $s \in \mathbb{R}$ we have

$$\Big\|\int_0^t e^{i(t-\tau)\partial_x^2}F(\tau)d\tau\Big\|_{\mathfrak{L}^\infty([0,T]; H^s_p(\mathbb{R}))} \le C\|\tau^{\frac{1}{p}-\frac{1}{2}}F(\tau)\|_{L^\sigma([0,T]; H^s_\rho(\mathbb{R}))}. \tag{3.4}$$

Proof of Proposition 1.5. We first introduce several particular exponents. We set

$$\frac{1}{q} = \frac{1}{2p} - \frac{1}{8} - \frac{s}{4}, \quad \frac{1}{r} = \frac{1+2s}{4}, \quad \frac{1}{\rho} = \frac{3}{4} - \frac{s}{2}\ \Big(= \frac{3}{r} - 2s\Big),$$

$$\frac{1}{\gamma} = \frac{5}{8} + \frac{s}{4} + \frac{1}{2p}, \quad \frac{1}{\sigma} = \frac{9}{8} - \frac{1}{2p} + \frac{s}{4}\ \Big(= \frac{1}{q'}\Big).$$

Then it is easy to check that (q,r,γ,σ), (q,r), and (ρ,σ) satisfy the assmption of Proposition 2.2, Proposition 3.1, and Corollary 3.3 if $4/3 < p \le 2$ and $0 < s < 3/2 - 2/p$. Let $T > 0$. We claim that

$$\|\int_0^t e^{i(t-\tau)\partial_x^2}(u_1u_2\overline{u_3})d\tau\|_{L^q([0,T];H_r^s(\mathbb{R}))} \le CT^{1+s-\frac{1}{p}} \prod_{j=1}^{3} \|u_j\|_{L^q([0,T];H_r^s(\mathbb{R}))}, \tag{3.5}$$

and

$$\|\int_0^t e^{i(t-\tau)\partial_x^2}(u_1u_2\overline{u_3})d\tau\|_{\mathfrak{L}^\infty([0,T];H_p^s(\mathbb{R}))} \le CT^{1+s-\frac{1}{p}} \prod_{j=1}^{3} \|u_j\|_{L^q([0,T];H_r^s(\mathbb{R}))}, \tag{3.6}$$

for all $u_j \in L^q([0,T];H_r^s(\mathbb{R}))$, $j = 1,2,3$, where C is independent of T. We first prove (3.5). Applying (2.2) with $F = \langle D\rangle^s(u_1u_2u_3)$, we have

$$\|\int_0^t e^{i(t-\tau)\partial_x^2}(u_1u_2\overline{u_3})d\tau\|_{L^q([0,T];H_r^s)} \le C\|u_1u_2\overline{u_3}\|_{L^\gamma([0,T];H_\rho^s(\mathbb{R}))}.$$

We estimate the right hand side. By the fractional Leibniz rule (see e.g. [8])

$$\|u_1u_2\overline{u_3}\|_{H_\rho^s(\mathbb{R})} \le C\left(\|u_1\|_{H_r^s(\mathbb{R})}\|u_2\overline{u_3}\|_{L^\kappa(\mathbb{R})} + \|u_1\|_{L^{2\kappa}(\mathbb{R})}\|u_2\overline{u_3}\|_{H^s(\mathbb{R})}\right),$$

where

$$\frac{1}{\kappa} = \frac{1}{\rho} - \frac{1}{r} = \frac{1}{2} - s.$$

For the first term in the right hand side we have by Schwartz's inequality and Sobolev's embedding

$$\|u_1\|_{H_r^s(\mathbb{R})}\|u_2\overline{u_3}\|_{L^\kappa(\mathbb{R})} \le \|u_1\|_{H_r^s(\mathbb{R})}\|u_2\|_{L^{2\kappa}(\mathbb{R})}\|u_3\|_{L^{2\kappa}(\mathbb{R})} \le C\prod_{j=1}^{3}\|u_j\|_{H_r^s(\mathbb{R})}.$$

Noting the relation

$$\frac{1}{2} = (\frac{1}{4} - \frac{s}{2}) + (\frac{1}{4} + \frac{s}{2}) = \frac{1}{2\kappa} + \frac{1}{r},$$

the second term can be estimated as

$$\|u_1\|_{L^{2\kappa}(\mathbb{R})}\|u_2\overline{u_3}\|_{H^s(\mathbb{R})} \le C\|u_1\|_{L^{2\kappa}(\mathbb{R})}\left(\|u_2\|_{H_r^s(\mathbb{R})}\|u_3\|_{L^{2\kappa}(\mathbb{R})} + \|u_2\|_{L^{2\kappa}(\mathbb{R})}\|\overline{u_3}\|_{H_r^s(\mathbb{R})}\right),$$

the right hand side of which can be controlled by $\|u_1\|_{H_r^s}\|u_2\|_{H_r^s}\|u_3\|_{H_r^s}$ by Sobolev's embedding. Consequently,

$$\|u_1u_2\overline{u_3}\|_{L^\gamma([0,T];H_\rho^s(\mathbb{R}))} \le C\|\prod_{j=1}^{3}\|u_j\|_{H_r^s(\mathbb{R})}\|_{L^\gamma([0,T])} \le CT^{1+s-\frac{1}{p}}\prod_{j=1}^{3}\|u_j\|_{L^q([0,T];H_r^s(\mathbb{R}))}$$

by Hölder's inequality in the time variable. This concludes the proof of (3.5). To prove (3.6) we use (3.4) to obtain

$$\|\int_0^t e^{i(t-\tau)\partial_x^2}(u_1u_2\overline{u_3})d\tau\|_{\mathfrak{L}^\infty([0,T];H_p^s)} \le C\|\tau^{\frac{1}{p}-\frac{1}{2}}u_1(\tau)u_2(\tau)\overline{u_3(\tau)}\|_{L^\sigma([0,T];H_\rho^s(\mathbb{R}))}.$$

As observed in the proof of (3.5), the norm $\|\tau^{\frac{1}{p}-\frac{1}{2}}u_1(\tau)u_2(\tau)\overline{u_3(\tau)}\|_{H_\rho^s}$ can be controlled by $C\tau^{\frac{1}{p}-\frac{1}{2}}\|u_1(\tau)\|_{H_r^s}\|u_2(\tau)\|_{H_r^s}\|u_3(\tau)\|_{H_r^s}$. Thus the right hand side of the above inequality is bounded from above by

$$\begin{aligned} C\|\tau^{\frac{1}{p}-\frac{1}{2}}\prod_{j=1}^{3}\|u_j\|_{L^q([0,T];H_r^s(\mathbb{R}))}\|_{L^\sigma([0,T])} &\le C\|\tau^{\frac{1}{p}-\frac{1}{2}}\|_{L^{\frac{\sigma q}{q-3\sigma}}([0,T])}\prod_{j=1}^{3}\|u_j\|_{L^q([0,T];H_r^s(\mathbb{R}))} \\ &\le CT^{1+s-\frac{1}{p}}\prod_{j=1}^{3}\|u_j\|_{L^q([0,T];H_r^s(\mathbb{R}))}, \end{aligned}$$

where we have used Hölder's inequality in the time variable. This proves (3.6).

Now we establish a local solution of the corresponding integral equation

$$u(t) = e^{it\partial_x^2}\phi + i\int_0^t e^{i(t-\tau)\partial_x^2}(|u|^2u)d\tau.$$

We find a fixed point of the operator

$$Su \triangleq e^{it\partial_x^2}\phi + i\int_0^t e^{i(t-\tau)\partial_x^2}(|u|^2u)d\tau$$

in a suitable closed subset of $\mathfrak{L}^\infty([0,T];H_p^s)\cap L^q([0,T];H_r^s)$. For $T,R>0$ we define

$$\mathcal{V}_{T,R} \triangleq \{u\in\mathfrak{L}^\infty([0,T];H_p^s)\cap L^q([0,T];H_r^s) \mid \|u\|_{\mathfrak{L}^\infty([0,T];H_p^s)} + \|u\|_{L^q([0,T];H_r^s)} \le R\}$$

equipped with the distance

$$d(u,v) \triangleq \|u-v\|_{\mathfrak{L}^\infty([0,T];H_p^s)} + \|u-v\|_{L^q([0,T];H_r^s)}.$$

By (3.2), (3.5), (3.6), for any $u\in\mathcal{V}_{T,R}$ we have

$$\begin{aligned}\|Su\|_{\mathfrak{L}^\infty([0,T];H_p^s)} + \|Su\|_{L^q([0,T];H_r^s)} &\le C\left(\|\phi\|_{H_p^s} + T^{1+s-\frac{1}{p}}\|u\|^3_{L^q([0,T];H_r^s)}\right)\\ &\le C\left(\|\phi\|_{H_p^s} + T^{1+s-\frac{1}{p}}R^3\right).\end{aligned}$$

Now we take T,R so that

$$R = 2C\|\phi\|_{H_p^s}, \quad T^{1+s-\frac{1}{p}} \le \frac{R^{-2}}{4C}.$$

Then the right-hand side is smaller than $R/2+R/4\le R$ and thus $S:\mathcal{V}_{T,R}\to\mathcal{V}_{T,R}$ is well defined. Similarly, for $u,v\in\mathcal{V}_{T,R}$

$$\begin{aligned}d(Su,Sv) &\le CT^{1+s-\frac{1}{p}}(\|u\|^2_{L^q([0,T];H_r^s)} + \|u\|_{L^q([0,T];H_r^s)}\|v\|_{L^q([0,T];H_r^s)} + \|v\|^2_{L^q([0,T];H_r^s)})\\ &\quad\times\|u-v\|_{L^q([0,T];H_r^s)}\\ &\le C\frac{R^{-2}}{4C}3R^2\|u-v\|_{L^q([0,T];H_r^s)}\\ &\le \frac{3}{4}d(u,v),\end{aligned}$$

from which we see that $S:\mathcal{V}_{T,R}\to\mathcal{V}_{T,R}$ is a contraction mapping. Consequently, a local solution $v\in\mathfrak{L}^\infty([0,T];H_p^s)\cap L^q([0,T];H_r^s)$ with $T\sim\|\phi\|_{H_p^s}^{-\frac{2}{1+s-\frac{1}{p}}}$ can be constructed by the standard fixed point theorem. Finally a difference estimate similar to the proof of Theorem 1.3 gives the uniqueness of the solution in the space $L^q([0,T];H_r^s)$ and the continuous dependence on data. $\square$

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Ryosuke Hyakuna
Polytechnic University of Japan, Tokyo, Japan
Email address: 107r107r@gmail.com