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DYNAMICS OF TRAVELING WAVES FOR PREDATOR-PREY SYSTEMS WITH ALLEE EFFECT AND TIME DELAY

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Abstract. This article aims to establish the existence of traveling waves for a predator-prey system with Beddington-DeAngelis functional response, reproductive Allee effect, and time delay. We investigate the existence of solutions for a system with two special delay kernels by geometric singular perturbation theory, invariant manifold theory, and Fredholm orthogonality theory. In addition, we discuss the asymptotic behaviors of traveling waves with the aid of the asymptotic theory.

1. Introduction

The Allee effect was originally proposed by Warder Clyde Allee [2] to characterize the correlation between the population density and per capita growth rate of the population at low densities [27]. In recent decades, it has received considerable attention and numerous related studies have been conducted [1, 19, 23, 29, 31, 34]. To investigate the impact of the reproductive Allee effect in prey growth, Dey et al. [7] considered the predator-prey system with Beddington-DeAngelis functional response and reproductive Allee effect,

$$\begin{aligned} N_T &= d_N N_{yy} + aN^2(b-N) - dN - \frac{sNP}{p+N+qP}, \\ P_T &= d_P P_{yy} + \frac{esNP}{p+N+qP} - mP, \end{aligned} \tag{1.1}$$

where $N(T,y)$ and $P(T,y)$ denote the prey and predator densities at moment T and location y, respectively, d_N and d_P denote the diffusion coefficients of the prey and predator populations, the positive constants a, b, and d stand for the intrinsic growth rate, threshold for positive growth, and intrinsic mortality rate, the parameters s, p, and q represent the maximum predation rate, self-saturation constant and predator mutual interference, e and m are the conversation coefficient and the per capita natural death rate of the predator population.

Making the change of variables

$$u = \frac{N}{b}, \quad v = \frac{sP}{bm}, \quad t = mT, \quad x = \sqrt{\frac{m}{d_N}}\,y,$$

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system (1.1) becomes dimensionless system

$$
\begin{aligned}
u_t &= u_{xx} + \sigma u^2(1-u) - \eta u - \frac{uv}{\alpha + u + \beta v}, \\
v_t &= \delta v_{xx} + \frac{\gamma uv}{\alpha + u + \beta v} - v,
\end{aligned}
\tag{1.2}
$$

where $\alpha = \frac{p}{b}$, $\beta = \frac{qm}{s}$, $\gamma = \frac{es}{m}$, $\sigma = \frac{ab^2}{m}$, $\eta = \frac{d}{m}$, $\delta = \frac{d_P}{d_N}$ are dimensionless parameters. Combining the shooting method discussed in Huang [15] and the invariant manifold theory, Dey et al. [7] obtained the existence of traveling waves for system (1.2) connecting from the predator free homogeneous steady-state to the coexisting homogeneous steady-state.

The time delay is a kind of common nonlinearity, and numerous biological processes involve delays [21, 24, 28, 32, 33]. In ecology, the prey cannot immediately convert food into their own energy [3, 26]. Furthermore, the predator requires time for digesting prey before further activities [13], and the reproduction of predator is not instantaneous but mediated by some time delay required for gestation. Consequently, it is meaningful and essential to consider the time delay in predator-prey interaction.However, Dey et al. [7] did not consider the asymptotic behavior of traveling waves for the system (1.2).

There are two frequently used time delays known as local delay (1.4) and nonlocal delay (1.6). Britton [4] introduced a model for a single biological population in the form of

$$
u_t = u_{xx} + u\left(1 + au - (1+a)(f * u)\right), \tag{1.3}
$$

where f is a given function and $f * u$ denotes a local convolution in the spatial variable that can be written as

$$
(f * u)(x,t) = \int_{-\infty}^{t} f(t-s)u(x,s)ds, \tag{1.4}
$$

which is a spatial average weighted according to distance from the original position. The time delay (1.4) is called the local delay. The kernel $f(t)$ is any integrable nonnegative function that satisfies

$$
\int_0^{+\infty} f(t)dt = 1 \quad \text{and} \quad tf(t) \in L^1\left((0,\infty), \mathbb{R}\right).
$$

It is notable that the normalization assumption on f ensures that the uniform nonnegative steady-state solutions are unaffected by the delay. The two classical and special kernels are defined by

$$
f(t) = \frac{1}{\tau} e^{-t/\tau} \quad \text{and} \quad f(t) = \frac{t}{\tau^2} e^{-t/\tau}, \tag{1.5}
$$

where $\tau > 0$ is a small parameter. The former is called the local weak delay kernel and the latter the local strong delay kernel.

To recognize the effect of moving time, a spatio-temporal average weighted toward the current time and position had been studied. Subsequently, Britton [5] provided a mathematical derivation of a modified model, which incorporates a nonlocal convolution in space and time, taking the form

$$
(f * u)(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} f(x-y, t-s)u(y,s)\, dy\, ds. \tag{1.6}
$$

The time delay (1.6) is called the nonlocal delay. The kernel $f(x,t)$ is any integrable non-negative function satisfying the normalization assumption

$$\int_{-\infty}^{t}\int_{-\infty}^{+\infty} f(x,t)\,dx\,dt = 1,$$

which ensures that the uniform non-negative steady-state solutions are unaffected by the delay. The two classical and special kernels are defined by

$$f(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \frac{1}{\tau} e^{-t/\tau} \quad \text{and} \quad f(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \frac{t}{\tau^2} e^{-t/\tau}, \tag{1.7}$$

where the small parameter $\tau > 0$ measures the time delay. The former is called the nonlocal weak delay kernel and the latter the nonlocal strong delay kernel.

Traveling waves are useful in understanding the interaction of multiple species, and many powerful study methods have been established. Among them, the geometric singular perturbation theory developed by Fenichel [12] is an effective method to prove the existence of traveling waves in evolution equations with small parameters. This theory has been applied to various equations, including Keller-Segel systems [10, 25], FitzHugh-Nagumo equations [14, 20, 30], nonlinear Belousov-Zhabotinskii system[11], Liénard equations [18], Camassa-Holm equations [9, 8], etc.

Motivated by the aforementioned analysis, we incorporate time delay into the system (1.2). To avoid excessive technicalities, we shall only consider the case of time delay on prey in this paper. The predator-prey system with Beddington-DeAngelis functional response, reproductive Allee effect, and time delay which we study here is given as

$$\begin{gathered} u_t = u_{xx} + \sigma u(f*u)\left(1-(f*u)\right) - \eta u - \frac{uv}{\alpha+u+\beta v}, \\ v_t = \delta v_{xx} + \frac{\gamma uv}{\alpha+u+\beta v} - v, \end{gathered} \tag{1.8}$$

where the parameters are the same as those in system (1.2), and the term $(f*u)(x,t)$ denotes time delay. When we ignore the time delay, system (1.8) is reduced to system (1.2).

The goal of this article is to establish the existence of traveling waves for system (1.8) with two cases of $(f*u)(x,t)$ as: local delay (1.4) and nonlocal delay (1.6), where the delay kernels are chosen as (1.5) and (1.7), respectively. Furthermore, the asymptotic behavior of traveling waves for system (1.2) has also been considered. It should be pointed out that the profile system of system (1.2) is four-dimensional, while the profile system of system (1.8) with local delay kernels (1.5) is six-dimensional, or eight-dimensional with nonlocal delay kernels (1.7). Consequently, the shooting method employed by Dey et al. [7] does not apply to system (1.8). Here we use the geometric singular perturbation theory and Fredholm orthogonality theory to solve this difficulty.

The article is organized as follows. Section 2 gives some preliminaries and introduces the geometric singular perturbation theory. Section 3 and Section 4 respectively focus on studying the existence of traveling waves for the system (1.8) with local delay kernels (1.5) and nonlocal delay kernels (1.7) by the geometric singular perturbation theory and the Fredholm orthogonality theory. Section 5 is to explore the asymptotic behaviors of traveling waves with the method of the asymptotic theory.

2. Preliminaries

In this section, we provide some preliminaries and introduce the geometric singular perturbation theory [16, 17]. By taking the traveling wave transformation

$$(u,v)(x,t) = (U,V)(\xi), \quad \xi = x + ct, \tag{2.1}$$

where the constant $c > 0$ is the wave speed, system (1.2) is transformed into

$$\begin{aligned} cU' &= U'' + \sigma U^2(1-U) - \eta U - \frac{UV}{\alpha + U + \beta V}, \\ cV' &= \delta V'' + \frac{\gamma UV}{\alpha + U + \beta V} - V, \end{aligned} \tag{2.2}$$

where $' = \frac{d}{d\xi}$. System (2.2) is equivalent to the system

$$\begin{aligned} U' &= X, \\ V' &= Y, \\ X' &= cX - \sigma U^2(1-U) + \eta U + \frac{UV}{\alpha + U + \beta V}, \\ Y' &= \frac{1}{\delta}\Big[cY - \frac{\gamma UV}{\alpha + U + \beta V} + V\Big]. \end{aligned} \tag{2.3}$$

According to the analysis presented in [7], for $\sigma > 4\eta$, it follows that the system (2.3) has three equilibria

$$P_0(0,0,0,0), \quad P_1(u_1,0,0,0), \quad \text{and} \quad P_2(u_2,0,0,0),$$

where

$$u_1 = \frac{\sigma + \sqrt{\sigma^2 - 4\sigma\eta}}{2\sigma} > 0, \quad u_2 = \frac{\sigma - \sqrt{\sigma^2 - 4\sigma\eta}}{2\sigma} > 0.$$

Furthermore, under certain conditions, see [7] for more details, the system (2.3) admits the fourth equilibrium

$$P_*(u_*, v_*, 0, 0),$$

where u_* is a positive root of the cubic equation

$$\sigma\gamma\beta u^3 - \sigma\gamma\beta u^2 + (\gamma\beta\eta + \gamma - 1)u - \alpha = 0, \tag{2.4}$$

and

$$v_* = \frac{\gamma u_* - u_* - \alpha}{\beta} > 0.$$

Note that the equilibrium $P_1(u_1,0,0,0)$ and $P_*(u_*,v_*,0,0)$ correspond to the predator free homogeneous steady-state $E_1(u_1,0)$ and the coexisting homogeneous steady-state $E_*(u_*,v_*)$ of the system (1.2).

For system (2.3), Dey et al. [7] obtained the existence of the heteroclinic orbit that connects the equilibrium P_1 and P_* in the case E_* is a stable homogeneous steady-state, which can be characterized by the following lemma.

Lemma 2.1 ([7]). *If $c \geq 2\sqrt{\delta(\frac{\gamma u_1}{\alpha + u_1} - 1)}$, then system (2.3) admits a heteroclinic orbit $\Phi_\xi(P) = (U(\xi), V(\xi), X(\xi), Y(\xi))$ with $\Phi_\xi(P) \to P_1(u_1,0,0,0)$ as $\xi \to -\infty$ and $\Phi_\xi(P) \to P_*(u_*,v_*,0,0)$ as $\xi \to +\infty$.*

We now introduce the geometric singular perturbation theory, which ensures the existence of invariant manifolds under certain conditions. With the use of this approach, the higher-dimensional systems can be reduced to lower-dimensional regular perturbation systems on these manifolds. This reduction greatly simplifies the analysis of traveling waves in high-dimensional systems.

Lemma 2.2 ([16, 17]). *Consider the system*

$$\begin{aligned} x'(t) &= f(x,y,\varepsilon), \\ y'(t) &= \varepsilon g(x,y,\varepsilon), \end{aligned} \tag{2.5}$$

where $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ $(m,n \geq 1)$, *and the parameter* ε *satisfies* $0 < \varepsilon \ll 1$. *The functions* f *and* g *are* C^∞ *on the set* $V \times I$, *where* $V \in \mathbb{R}^{m+n}$ *and* I *is an open interval that includes 0. If the critical manifold* $M_0 = \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n : f(x,y,0) = 0\}$ *is normally hyperbolic, i.e., the* $m \times m$ *matrix* $(D_x f)(p,0)$ *of first partial derivatives with respect to the fast variables* x *has no eigenvalues with zero real part for all* $p \in M_0$, *then for* $0 < \varepsilon \ll 1$ *and any* $0 < r < +\infty$,

- (i) *there exists a slow manifold* M_ε *that is diffeomorphic to* M_0, *and* M_ε *is locally invariant under the flow of* (2.5);
- (ii) M_ε *is* C^r *in* x, y, ε *and can be given as a graph*

$$M_\varepsilon = \{(x,y) : x = h^\varepsilon(y)\},$$

for some C^r *function* $h^\varepsilon(y)$;
- (iii) *there exists locally stable and unstable invariable manifolds* $W^s_{loc}(M_\varepsilon)$ *and* $W^u_{loc}(M_\varepsilon)$ *lying within* $\mathcal{O}(\varepsilon)$ *and being* C^r *diffeomorphic to* $W^s_{loc}(M_0)$ *and* $W^u_{loc}(M_0)$.

3. Traveling waves for the system with local delay

This section focuses on studying the existence of traveling waves for the system (1.8) with local delay (1.4) by the geometric singular perturbation theory and the Fredholm orthogonality theory. We investigate the case of local strong delay kernel, i.e., taking $f(t) = \frac{t}{\tau^2} e^{-t/\tau}$, the case of the local weak delay kernel can be discussed by the same way.

3.1. Perturbation analysis with local delay.

Substituting the traveling wave transformation (2.1) into system (1.8), we have

$$\begin{aligned} cU' &= U'' + \sigma UW(1-W) - \eta U - \frac{UV}{\alpha + U + \beta V}, \\ cV' &= \delta V'' + \frac{\gamma UV}{\alpha + U + \beta V} - V, \end{aligned} \tag{3.1}$$

where $' = \frac{d}{d\xi}$, and

$$
\begin{aligned}
W(\xi) &= \int_{-\infty}^{t} \frac{t-s}{\tau^2} e^{-\frac{t-s}{\tau}} U(x+cs)ds \\
&= -\int_{+\infty}^{0} \frac{z}{\tau^2} e^{-\frac{z}{\tau}} U(x+ct-cz)dz \\
&= \int_{0}^{+\infty} \frac{z}{\tau^2} e^{-\frac{z}{\tau}} U(\xi-cz)dz \\
&= \int_{0}^{+\infty} \frac{t}{\tau^2} e^{-t/\tau} U(\xi-ct)dt.
\end{aligned} \tag{3.2}
$$

Differentiating the equation (3.2) with respect to ξ, we obtain

$$
\begin{aligned}
\frac{dW}{d\xi} &= \int_{0}^{+\infty} \frac{t}{\tau^2} e^{-t/\tau} U_\xi(\xi-ct)dt \\
&= -\frac{1}{c}\int_{0}^{+\infty} \frac{t}{\tau^2} e^{-t/\tau} U_t(\xi-ct)dt \\
&= -\frac{1}{c}\int_{0}^{+\infty} \frac{t}{\tau^2} e^{-t/\tau} dU \\
&= -\frac{1}{c}\Big(\frac{t}{\tau^2} e^{-t/\tau} U(\xi-ct)\Big)\Big|_0^{+\infty} + \frac{1}{c}\int_{0}^{+\infty} U(\xi-ct)d(\frac{t}{\tau^2} e^{-t/\tau}) \\
&= \frac{1}{c}\int_{0}^{+\infty} U(\xi-ct)d(\frac{t}{\tau^2} e^{-t/\tau}) \\
&= \frac{1}{c\tau}\Big(\int_{0}^{+\infty} \frac{1}{\tau} e^{-t/\tau} U(\xi-ct)dt - \int_{0}^{+\infty} \frac{t}{\tau^2} e^{-t/\tau} U(\xi-ct)dt\Big) \\
&= \frac{1}{c\tau}(\zeta - W),
\end{aligned} \tag{3.3}
$$

where

$$
\zeta(\xi) = \int_{0}^{+\infty} \frac{1}{\tau} e^{-t/\tau} U(\xi-ct)dt. \tag{3.4}
$$

Differentiating equation (3.4) with respect to ξ, we obtain

$$
\begin{aligned}
\frac{d\zeta}{d\xi} &= \int_{0}^{+\infty} \frac{1}{\tau} e^{-t/\tau} U_\xi(\xi-ct)dt \\
&= -\frac{1}{c}\int_{0}^{+\infty} \frac{1}{\tau} e^{-t/\tau} U_t(\xi-ct)dt \\
&= -\frac{1}{c}\int_{0}^{+\infty} \frac{1}{\tau} e^{-t/\tau} dU \\
&= -\frac{1}{c\tau} e^{-t/\tau} U(\xi-ct)\Big|_0^{+\infty} + \frac{1}{c}\int_{0}^{+\infty} U d(\frac{1}{\tau} e^{-t/\tau}) \\
&= \frac{1}{c\tau} U(\xi) - \frac{1}{c\tau}\int_{0}^{+\infty} \frac{1}{\tau} e^{-t/\tau} U(\xi-ct)dt \\
&= \frac{1}{c\tau}(U - \zeta).
\end{aligned} \tag{3.5}
$$

By combining equations (3.3) and (3.5), system (3.1) can be reformulated as

$$\begin{aligned} cU' &= U'' + \sigma UW(1-W) - \eta U - \frac{UV}{\alpha + U + \beta V}, \\ cV' &= \delta V'' + \frac{\gamma UV}{\alpha + U + \beta V} - V, \\ c\tau W' &= \zeta - W, \\ c\tau \zeta' &= U - \zeta, \end{aligned} \tag{3.6}$$

which can be further rewritten as a system of first order ODEs in $\mathbb{R}^6$

$$\begin{aligned} U' &= X, \\ V' &= Y, \\ X' &= cX - \sigma UW(1-W) + \eta U + \frac{UV}{\alpha + U + \beta V}, \\ Y' &= \frac{1}{\delta}\Big[cY - \frac{\gamma UV}{\alpha + U + \beta V} + V\Big], \\ c\tau W' &= \zeta - W, \\ c\tau \zeta' &= U - \zeta, \end{aligned} \tag{3.7}$$

where the small parameter τ denotes the time delay in the system (1.8). For small $\tau > 0$, system (3.7) is a slow system in which (W, ζ) are the fast variables and (U, V, X, Y) are the slow variables.

Let $\tau = 0$, then the flow of the system (3.7) is constrained to the critical manifold

$$C_0 = \{(U, V, X, Y, W, \zeta) \in \mathbb{R}^6 : W = \zeta = U\}. \tag{3.8}$$

Now we study the normal hyperbolicity of the critical manifold C_0. For a point $P \in C_0$, the matrix of the first partial derivatives with respect to the fast variables (W, ζ) is

$$J_1 = \begin{pmatrix} -1/c & 1/c \\ 0 & -1/c \end{pmatrix}. \tag{3.9}$$

Thus according to Lemma 2.2, it follows that the critical manifold C_0 is normally hyperbolic. It indicates that there exists a slow manifold C_τ, $\mathcal{O}(\tau)$ close and diffeomorphic to C_0 for $0 < \tau \ll 1$, which can be expressed as

$$\begin{aligned} C_\tau = \big\{&(U, V, X, Y, W, \zeta) \in \mathbb{R}^6 : W = U + \varpi_1(U, V, X, Y)\tau + \mathcal{O}(\tau^2), \\ &\zeta = U + \varpi_2(U, V, X, Y)\tau + \mathcal{O}(\tau^2)\big\}, \end{aligned}$$

where $\varpi_i(U, V, X, Y)$ $(i = 1, 2)$ are two smooth functions defined on a compact domain.

Substituting

$$\begin{aligned} W &= U + \varpi_1(U, V, X, Y)\tau + \mathcal{O}(\tau^2), \\ \zeta &= U + \varpi_2(U, V, X, Y)\tau + \mathcal{O}(\tau^2), \end{aligned}$$

into the system (3.7), we have

$$\begin{aligned} c\tau(U' + \varpi_1'\tau) &= c\tau X + \mathcal{O}(\tau^2) = (\varpi_2 - \varpi_1)\tau + \mathcal{O}(\tau^2), \\ c\tau(U' + \varpi_2'\tau) &= c\tau X + \mathcal{O}(\tau^2) = -\varpi_2\tau + \mathcal{O}(\tau^2). \end{aligned}$$

Comparing the coefficients of τ, it follows that

$$\varpi_1(U,V,X,Y) = -2cX, \quad \text{and} \quad \varpi_2(U,V,X,Y) = -cX.$$

Then system (3.7) restricted to C_τ can be reformulated as

$$\begin{aligned} U' &= X,\\ V' &= Y,\\ X' &= cX - \sigma U(U - 2cX\tau)(1 - U + 2cX\tau) + \eta U + \frac{UV}{\alpha + U + \beta V} + \mathcal{O}(\tau^2),\\ Y' &= \frac{1}{\delta}\Big[cY - \frac{\gamma UV}{\alpha + U + \beta V} + V\Big], \end{aligned} \tag{3.10}$$

which is a regular perturbation of the system (2.3). It is evident that the system (3.10) is simplified to the system (2.3) when $\tau = 0$.

3.2. **Analysis by the Fredholm orthogonality theory.** In this section, we shall prove that the system (3.10) admits a heteroclinic orbit connecting the equilibrium $P_1(u_1,0,0,0)$ to $P_*(u_*,v_*,0,0)$ for $0 < \tau \ll 1$.

Let $(U_0,V_0,X_0,Y_0)(\xi)$ be the heteroclinic orbit of system (2.3) obtained in Lemma 2.1, connecting the equilibrium $P_1(u_1,0,0,0)$ to $P_*(u_*,v_*,0,0)$. To solve the system (3.10) for $0 < \tau \ll 1$, we set

$$\begin{aligned} U &= U_0 + U_1\tau + \mathcal{O}(\tau^2),\\ V &= V_0 + V_1\tau + \mathcal{O}(\tau^2),\\ X &= X_0 + X_1\tau + \mathcal{O}(\tau^2),\\ Y &= Y_0 + Y_1\tau + \mathcal{O}(\tau^2). \end{aligned} \tag{3.11}$$

Substituting the transformation (3.11) into the first and second equations of system (3.10), and comparing the coefficient of τ, we have

$$U_1' = X_1, \quad \text{and} \quad V_1' = Y_1. \tag{3.12}$$

Substituting the transformation (3.11) into the third equation of system (3.10), we obtain

$$\begin{aligned} X' &= X_0' + X_1'\tau + \mathcal{O}(\tau^2)\\ &= cX_0 - U_0(\sigma U_0 - \sigma U_0^2 - \eta) + \frac{U_0V_0}{\alpha + U_0 + \beta V_0} + X_1'\tau + \mathcal{O}(\tau^2)\\ &= cX_0 - U_0(\sigma U_0 - \sigma U_0^2 - \eta) + \big[(3\sigma U_0^2 - 2\sigma U_0 + \eta)U_1 + cX_1\\ &\quad + 2c\sigma U_0X_0(1 - 2U_0)\big]\tau + \frac{U_0V_0 + (U_0V_1 + V_0U_1)\tau + \mathcal{O}(\tau^2)}{\alpha + U_0 + \beta V_0 + (U_1 + \beta V_1)\tau + \mathcal{O}(\tau^2)} + \mathcal{O}(\tau^2), \end{aligned}$$

i.e.,

$$\begin{aligned} &\big[\alpha + U_0 + \beta V_0 + (U_1 + \beta V_1)\tau\big]\Big(\frac{U_0V_0}{\alpha + U_0 + \beta V_0} + X_1'\tau\Big) + \mathcal{O}(\tau^2)\\ &= \big[\alpha + U_0 + \beta V_0 + (U_1 + \beta V_1)\tau\big]\big[(3\sigma U_0^2 - 2\sigma U_0 + \eta)U_1 + cX_1\\ &\quad + 2c\sigma U_0X_0(1 - 2U_0)\big]\tau + U_0V_0 + (U_0V_1 + V_0U_1)\tau + \mathcal{O}(\tau^2). \end{aligned}$$

Comparing the coefficient of τ, one has

$$X_1' = \Big[3\sigma U_0^2 - 2\sigma U_0 + \eta + \frac{\alpha V_0 + \beta V_0^2}{(\alpha + U_0 + \beta V_0)^2}\Big] U_1 + \frac{\alpha U_0 + U_0^2}{(\alpha + U_0 + \beta V_0^2)^2} V_1 + cX_1 + 2c\sigma U_0 X_0 (1 - 2U_0). \tag{3.13}$$

Substituting the transformation (3.11) into the fourth equation of system (3.10) and comparing the coefficient of τ, we have

$$Y_1' = -\frac{\gamma}{\delta} \frac{\alpha V_0 + \beta V_0^2}{(\alpha + U_0 + \beta V_0)^2} U_1 + \Big(\frac{1}{\delta} - \frac{\gamma}{\delta} \frac{\alpha U_0 + U_0^2}{(\alpha + U_0 + \beta V_0)^2}\Big) V_1 + \frac{c}{\delta} Y_1. \tag{3.14}$$

Combining equations (3.12), (3.13) and (3.14), we obtain the following differential equation system determining U_1, V_1, X_1 and Y_1,

$$\frac{d\psi(\xi)}{d\xi} - P(\xi)\psi(\xi) = Q(\xi), \tag{3.15}$$

where

$$\psi(\xi) = \begin{pmatrix} U_1(\xi) \\ V_1(\xi) \\ X_1(\xi) \\ Y_1(\xi) \end{pmatrix},$$

$$P(\xi) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\sigma U_0^2 - 2\sigma U_0 + \eta + \frac{\alpha V_0 + \beta V_0^2}{(\alpha + U_0 + \beta V_0)^2} & \frac{\alpha U_0 + U_0^2}{(\alpha + U_0 + \beta V_0^2)^2} & c & 0 \\ -\frac{\gamma}{\delta} \frac{\alpha V_0 + \beta V_0^2}{(\alpha + U_0 + \beta V_0)^2} & \frac{1}{\delta} - \frac{\gamma}{\delta} \frac{\alpha U_0 + U_0^2}{(\alpha + U_0 + \beta V_0)^2} & 0 & \frac{c}{\delta} \end{pmatrix},$$

$$Q(\xi) = (0, 0, 2c\sigma U_0 X_0 (1 - 2U_0), 0)^T.$$

Next we proof that the system (3.15) admits a solution satisfying

$$(U_1, V_1, X_1, Y_1)(\pm\infty) = 0.$$

Let

$$l = \frac{d}{d\xi} - P(\xi). \tag{3.16}$$

Denote L^2 as the space of square integral functions, with inner production

$$\int_{-\infty}^{+\infty} (M(\xi), N(\xi)) d\xi,$$

where

$$M(\xi) = (U_1(\xi), V_1(\xi), X_1(\xi), Y_1(\xi))^T, N(\xi) = (U_1(\xi), V_1(\xi), X_1(\xi), Y_1(\xi))^T,$$

and $(\cdot, \cdot)$ being the Euclidean inner product on $\mathbb{R}^4$. According to Fredholm orthogonality theory, we have that the system (3.15) will have a solution if and only if

$$\int_{-\infty}^{+\infty} (M(\xi), Q(\xi)) d\xi = 0,$$

for all functions $M(\xi) \in \mathbb{R}^4$ in the kernel of the adjoint of the operator l. It can be readily verified that the adjoint operator l^* is

$$l^* = -\frac{d}{d\xi} - P^T(\xi),$$

where

$$P^T(\xi)=\begin{pmatrix}0 & 0 & 3\sigma U_0^2-2\sigma U_0+\eta+\frac{\alpha V_0+\beta V_0^2}{(\alpha+U_0+\beta V_0)^2} & -\frac{\gamma}{\delta}\frac{\alpha V_0+\beta V_0^2}{(\alpha+U_0+\beta V_0)^2}\\ 0 & 0 & \frac{\alpha U_0+U_0^2}{(\alpha+U_0+\beta V_0^2)^2} & \frac{1}{\delta}-\frac{\gamma}{\delta}\frac{\alpha U_0+U_0^2}{(\alpha+U_0+\beta V_0)^2}\\ 1 & 0 & c & 0\\ 0 & 1 & 0 & \frac{c}{\delta}\end{pmatrix}. \tag{3.17}$$

To compute $\ker l^*$, we need to find that all $M(\xi)$ satisfying $l^*M(\xi)=0$, i.e.,

$$\frac{dM(\xi)}{d\xi}=-P^T(\xi)M(\xi). \tag{3.18}$$

Then the persistence question reduces to the solvability of equation (3.18). It is evident that the zero solution is a solution of the equation (3.18). Since the matrix $P^T(\xi)$ is nonconstant, it is difficult to find the general solution of equation (3.18). Nevertheless, we are only focusing on solutions that satisfy $M(\pm\infty)=0$, and in fact, the sole such solution is the zero solution. Recall that $(U_0(\xi),V_0(\xi),X_0(\xi),Y_0(\xi))$ is the solution of system (2.3) obtained in Lemma 2.1. Although we have no explicit expression for it, we know that $(U_0(\xi),V_0(\xi),X_0(\xi),Y_0(\xi))$ tends to $P_1(u_1,0,0,0)$ as $\xi\to-\infty$. Letting $\xi\to-\infty$, the matrix $-P^T$ finally becomes a constant matrix

$$\begin{pmatrix}0 & 0 & 3\sigma u_1^2-2\sigma u_1+\eta & 0\\ 0 & 0 & \frac{u_1}{\alpha+u_1} & \frac{1}{\delta}(1-\frac{\gamma u_1}{\alpha+u_1})\\ 1 & 0 & c & 0\\ 0 & 1 & 0 & \frac{c}{\delta}\end{pmatrix} \tag{3.19}$$

The characteristic equation of the above matrix is

$$(\lambda^2-c\lambda-3\sigma u_1^2+2\sigma u_1-\eta)(\delta\lambda^2-c\lambda+\frac{\gamma u_1}{\alpha+u_1}-1)=0. \tag{3.20}$$

By direct computation, it can be found that the equation (3.20) has three positive real eigenvalues and one negative eigenvalue. Consequently, the sole solution that satisfies $M(-\infty)=0$ is the zero solution. This implies that the Fredholm orthogonality condition holds trivially

$$\int_{-\infty}^{+\infty}(M(\xi),Q(\xi))d\xi=\int_{-\infty}^{+\infty}(0,Q(\xi))d\xi=0,$$

and the solutions of the system (3.15) exist, which satisfies

$$(U_1,V_1,X_1,Y_1)(\pm\infty)=0.$$

Therefore, for $0<\tau\ll 1$, the system (3.10) exists one heteroclinic orbit (3.11) connecting the equilibrium $P_1(u_1,0,0,0)$ to $P_*(u_*,v_*,0,0)$. This means that the system (1.8) with local strong delay kernel admits a traveling wave, and we obtain the following theorem.

Theorem 3.1. *If $c\geq 2\sqrt{\delta(\frac{\gamma u_1}{\alpha+u_1}-1)}$, then for any sufficiently small $\tau>0$, the system* (1.8) *with local strong delay kernel admits a traveling wave $(u,v)(x,t)=(U,V)(\xi)$ connecting from the steady-state $E_1(u_1,0)$ to $E_*(u_*,v_*)$, where $\xi=x+ct$.*

Remark 3.2. The result for $\tau=0$ is presented in [7]. Theorem 3.1 indicates that the traveling wave connecting from the steady-state $E_1(u_1,0)$ to $E_*(u_*,v_*)$ still exists for any sufficiently small $\tau>0$.

4. Traveling waves for the system with nonlocal delay

This section focuses on studying the existence of traveling waves for the system (1.8) with nonlocal delay (1.6) by the geometric singular perturbation theory and the Fredholm orthogonality theory. We mainly consider the case of nonlocal strong delay kernel, i.e., taking $f(x,t)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\frac{t}{\tau^2}e^{-t/\tau}$, the case of the nonlocal weak delay kernel can be studied similarly.

4.1. **Perturbation analysis with nonlocal delay.**

We define

$$\phi(x,t)=\int_{-\infty}^{t}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi(t-s)}}e^{-\frac{(x-y)^2}{4(t-s)}}\frac{t-s}{\tau^2}e^{-\frac{t-s}{\tau}}u(y,s)\,dy\,ds.$$

By a direct calculation, we have

$$\frac{\partial\phi}{\partial t}-\frac{\partial^2\phi}{\partial x^2}=\frac{1}{\tau}(\kappa-\phi), \tag{4.1}$$

where

$$\kappa(x,t)=\int_{-\infty}^{t}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi(t-s)}}e^{-\frac{(x-y)^2}{4(t-s)}}\frac{1}{\tau}e^{-\frac{t-s}{\tau}}u(y,s)\,dy\,ds.$$

Similarly, it follows that

$$\frac{\partial\kappa}{\partial t}-\frac{\partial^2\kappa}{\partial x^2}=\frac{1}{\tau}(u-\kappa). \tag{4.2}$$

Combining equations (4.1) and (4.2), we obtain

$$\phi_{tt}=2\phi_{txx}-\phi_{xxxx}+\frac{2}{\tau}(\phi_{xx}-\phi_t)+\frac{1}{\tau^2}(u-\phi). \tag{4.3}$$

Then system (1.8) can be reformulated as

$$\begin{gathered}
u_t=u_{xx}+\sigma u\phi(1-\phi)-\eta u-\frac{uv}{\alpha+u+\beta v},\\
v_t=\delta v_{xx}+\frac{\gamma uv}{\alpha+u+\beta v}-v,\\
\phi_{tt}=2\phi_{txx}-\phi_{xxxx}+\frac{2}{\tau}(\phi_{xx}-\phi_t)+\frac{1}{\tau^2}(u-\phi).
\end{gathered} \tag{4.4}$$

Introducing the traveling wave coordinate

$$(u,v,\phi)(x,t)=(U,V,\bar\phi)(\xi),\quad \xi=x+ct,$$

where the constant $c>0$ is the wave speed, the system (4.4) is transformed into

$$\begin{gathered}
cU'=U''+\sigma U\bar\phi(1-\bar\phi)-\eta U-\frac{UV}{\alpha+U+\beta V},\\
cV'=\delta V''+\frac{\gamma UV}{\alpha+U+\beta V}-V,\\
\bar\phi''''-2c\bar\phi'''+c^2\bar\phi''-\frac{2}{\tau}(\bar\phi''-c\bar\phi')-\frac{1}{\tau^2}(U-\bar\phi)=0,
\end{gathered} \tag{4.5}$$

which can be rewritten as the eight-dimensional system

$$\begin{aligned} U' &= X, \\ V' &= Y, \\ X' &= cX - \sigma U\bar{\phi}(1-\bar{\phi}) + \eta U + \frac{UV}{\alpha + U + \beta V}, \\ Y' &= \frac{1}{\delta}\Big[cY - \frac{\gamma UV}{\alpha + U + \beta V} + V\Big], \\ \bar{\phi}' &= \bar{\phi}_1, \\ \bar{\phi}_1' &= \bar{\phi}_2, \\ \bar{\phi}_2' &= \bar{\phi}_3, \\ \bar{\phi}_3' &= 2c\bar{\phi}_3 - c^2\bar{\phi}_2 + \frac{2}{\tau}(\bar{\phi}_2 - c\bar{\phi}_1) + \frac{1}{\tau^2}(U - \bar{\phi}). \end{aligned} \tag{4.6}$$

Set $\varepsilon = \sqrt{\tau}$ and define the new variables

$$Z = \bar{\phi}, \quad Z_1 = \varepsilon\bar{\phi}_1, \quad Z_2 = \varepsilon^2\bar{\phi}_2, \quad Z_3 = \varepsilon^3\bar{\phi}_3,$$

then system (4.6) becomes

$$\begin{aligned} U' &= X, \\ V' &= Y, \\ X' &= cX - \sigma UZ(1-Z) + \eta U + \frac{UV}{\alpha + U + \beta V}, \\ Y' &= \frac{1}{\delta}\Big[cY - \frac{\gamma UV}{\alpha + U + \beta V} + V\Big], \\ \varepsilon Z' &= Z_1, \\ \varepsilon Z_1' &= Z_2, \\ \varepsilon Z_2' &= Z_3, \\ \varepsilon Z_3' &= 2c\varepsilon Z_3 + (2 - c^2\varepsilon^2)Z_2 - 2c\varepsilon Z_1 + U - Z. \end{aligned} \tag{4.7}$$

For small $\varepsilon > 0$, system (4.7) is a slow system in which (Z, Z_1, Z_2, Z_3) are the fast variables and (U, V, X, Y) are the slow variables.

Let $\varepsilon = 0$, then system (4.7) is constrained to the critical manifold

$$S_0 = \{(U, V, X, Y, Z, Z_1, Z_2, Z_3) \in \mathbb{R}^8 : Z = U, Z_1 = Z_2 = Z_3 = 0\}. \tag{4.8}$$

For a point $P \in S_0$, the matrix of the first partial derivatives with respect to the fast variables (Z, Z_1, Z_2, Z_3) is

$$J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 2 & 0 \end{pmatrix}. \tag{4.9}$$

A computation shows that the eigenvalues of the matrix J_2 are $-1, -1, 1, 1$, hence the critical manifold S_0 is normally hyperbolic. According to Lemma 2.2, we obtain that there exists a slow manifold S_ε, $\mathcal{O}(\varepsilon)$ close and diffeomorphic to S_0 for $0 <$

$\varepsilon \ll 1$, which can be expressed as

$$
\begin{aligned}
S_\varepsilon = \Big\{ &(U,V,X,Y,Z,Z_1,Z_2,Z_3) \in \mathbb{R}^8 : \\
& Z = U + f_1(U,V,X,Y)\varepsilon + f_2(U,V,X,Y)\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\
& Z_1 = g_1(U,V,X,Y)\varepsilon + g_2(U,V,X,Y)\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\
& Z_2 = h_1(U,V,X,Y)\varepsilon + h_2(U,V,X,Y)\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\
& Z_3 = r_1(U,V,X,Y)\varepsilon + r_2(U,V,X,Y)\varepsilon^2 + \mathcal{O}(\varepsilon^3) \Big\},
\end{aligned} \tag{4.10}
$$

where f_i, g_i, h_i, r_i $(i = 1,2)$ are smooth functions defined on a compact domain. Substituting equation (4.10) into system (4.7), we have

$$
\begin{gathered}
\varepsilon X + f_1'\varepsilon^2 + f_2'\varepsilon^3 + \mathcal{O}(\varepsilon^4) = g_1\varepsilon + g_2\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\
g_1'\varepsilon^2 + g_2'\varepsilon^3 + \mathcal{O}(\varepsilon^4) = h_1\varepsilon + h_2\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\
h_1'\varepsilon^2 + h_2'\varepsilon^3 + \mathcal{O}(\varepsilon^4) = r_1\varepsilon + r_2\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\
r_1'\varepsilon^2 + r_2'\varepsilon^3 + \mathcal{O}(\varepsilon^4) = (2h_1 - f_1)\varepsilon + (2cr_1 + 2h_2 - 2cg_1 - f_2)\varepsilon^2 + \mathcal{O}(\varepsilon^3).
\end{gathered}
$$

Comparing the coefficients of ε and ε^2, one has

$$
\begin{gathered}
f_1 = 0, \quad f_2 = \frac{2UV}{\alpha + U + \beta V} - 2(\sigma U - \sigma U^2 - \eta)U, \\
h_1 = 0, \quad h_2 = cX - (\sigma U - \sigma U^2 - \eta)U + \frac{UV}{\alpha + U + \beta V}, \\
g_1 = X, \quad g_2 = 0, \quad r_1 = 0, \quad r_2 = 0.
\end{gathered}
$$

Thus system (4.7) restricted to S_ε can be reformulated as

$$
\begin{gathered}
U' = X, \\
V' = Y, \\
X' = cX - \sigma U(U + f_2\varepsilon^2)(1 - U - f_2\varepsilon^2) + \eta U + \frac{UV}{\alpha + U + \beta V} + \mathcal{O}(\varepsilon^3), \\
Y' = \frac{1}{\delta}[cY - \frac{\gamma UV}{\alpha + U + \beta V} + V],
\end{gathered} \tag{4.11}
$$

which is a regular perturbation of the system (2.3). It is evident that the system (4.11) is simplified to system (2.3) when $\varepsilon = 0$.

4.2. **Analysis by the Fredholm orthogonality theory.** This section we prove that system (4.11) admits a heteroclinic orbit connecting equilibrium $P_1(u_1, 0, 0, 0)$ to $P_*(u_*, v_*, 0, 0)$ for $0 < \varepsilon \ll 1$.

Let $(U_0, V_0, X_0, Y_0)(\xi)$ be the heteroclinic orbit of system (2.3) obtained in Lemma 2.1, connecting the equilibrium $P_1(u_1, 0, 0, 0)$ to $P_*(u_*, v_*, 0, 0)$. To solve system (4.11) for $0 < \tau \ll 1$, we set

$$
\begin{aligned}
U &= U_0 + \bar{U}_1\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\
V &= V_0 + \bar{V}_1\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\
X &= X_0 + \bar{X}_1\varepsilon^2 + \mathcal{O}(\varepsilon^3), \\
Y &= Y_0 + \bar{Y}_1\varepsilon^2 + \mathcal{O}(\varepsilon^3),
\end{aligned} \tag{4.12}
$$

Substituting the transformation (4.12) into system (4.11) and comparing the coefficient of ε^2, the differential equation system determining $\bar{U}_1$, $\bar{V}_1$, $\bar{X}_1$, $\bar{Y}_1$ is

$$\frac{d\bar{\psi}(\xi)}{d\xi} - \bar{P}(\xi)\bar{\psi}(\xi) = \bar{Q}(\xi), \tag{4.13}$$

where

$$\bar{\psi}(\xi) = \begin{pmatrix} \bar{U}_1(\xi) \\ \bar{V}_1(\xi) \\ \bar{X}_1(\xi) \\ \bar{Y}_1(\xi) \end{pmatrix},$$

$$\bar{P}(\xi) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3\sigma U_0^2 - 2\sigma U_0 + \eta + \frac{\alpha V_0 + \beta V_0^2}{(\alpha + U_0 + \beta V_0)^2} & \frac{\alpha U_0 + U_0^2}{(\alpha + U_0 + \beta V_0^2)^2} & c & 0 \\ -\frac{\gamma}{\delta}\frac{\alpha V_0 + \beta V_0^2}{(\alpha + U_0 + \beta V_0)^2} & \frac{1}{\delta} - \frac{\gamma}{\delta}\frac{\alpha U_0 + U_0^2}{(\alpha + U_0 + \beta V_0)^2} & 0 & \frac{c}{\delta} \end{pmatrix},$$

$$\bar{Q}(\xi) = (0, 0, \sigma(2U_0^2 - U_0)f_2(U_0, V_0, X_0, Y_0), 0)^T.$$

By an analysis similar to the one in Section 3.2, it follows that system (4.11) admits a heteroclinic orbit (4.12) connecting the equilibrium $P_1(u_1, 0, 0, 0)$ to $P_*(u_*, v_*, 0, 0)$ for $0 < \varepsilon \ll 1$. Therefore, the system (1.8) with nonlocal strong delay kernel admits a traveling wave, and we obtain the following theorem.

Theorem 4.1. *If $c \geq 2\sqrt{\delta(\frac{\gamma u_1}{\alpha + u_1} - 1)}$, then for any sufficiently small $\tau > 0$, the system (1.8) with nonlocal strong delay kernel admits a traveling wave $(u, v)(x, t) = (U, V)(\xi)$ connecting from the steady-state $E_1(u_1, 0)$ to $E_*(u_*, v_*)$, where $\xi = x + ct$.*

5. Asymptotic behavior

This section focus on analyzing the asymptotic behaviors of traveling waves for the system (1.2) with the method of the asymptotic theory [6]. Let $\varphi(\xi) = (u, v)(x, t) = (U, V)(\xi)$ be the traveling wave of system (1.2) satisfying the boundary conditions

$$(U, V)(-\infty) = E_1(u_1, 0), \quad (U, V)(+\infty) = E_*(u_*, v_*). \tag{5.1}$$

Differentiating the system (2.2) with respect to ξ and denoting

$$\varphi'(\xi) = (U', V')(\xi) = (\varphi_1, \varphi_2)(\xi),$$

we have

$$\begin{aligned} c\varphi_1' &= \varphi_1'' + (2\sigma U - 3\sigma U^2 - \eta)\varphi_1 - \frac{(\alpha V + \beta V^2)\varphi_1 + (\alpha U + U^2)\varphi_2}{(\alpha + U + \beta V)^2}, \\ c\varphi_2' &= \delta\varphi_2'' + \gamma\frac{(\alpha V + \beta V^2)\varphi_1 + (\alpha U + U^2)\varphi_2}{(\alpha + U + \beta V)^2} - \varphi_2. \end{aligned} \tag{5.2}$$

Note that the traveling wave $\varphi(\xi)$ satisfies the boundary conditions (5.1), hence the limiting system of system (5.2) as $\xi \to -\infty$ is

$$\begin{aligned} c\varphi_{1-}' &= \varphi_{1-}'' - (\sigma u_1 - 2\eta)\varphi_{1-} - \frac{u_1}{\alpha + u_1}\varphi_{2-}, \\ c\varphi_{2-}' &= \delta\varphi_{2-}'' + (\frac{\gamma u_1}{\alpha + u_1} - 1)\varphi_{2-}, \end{aligned} \tag{5.3}$$

which is equivalent to the system

$$\begin{aligned} \varphi_{1-}' &= \varphi_{3-}, \\ \varphi_{2-}' &= \varphi_{4-}, \\ \varphi_{3-}' &= (\sigma u_1 - 2\eta)\varphi_{1-} + \frac{u_1}{\alpha+u_1}\varphi_{2-} + c\varphi_{3-}, \\ \varphi_{4-}' &= \frac{1}{\delta}\Big[(1-\frac{\gamma u_1}{\alpha+u_1})\varphi_{2-} + c\varphi_{4-}\Big]. \end{aligned} \tag{5.4}$$

The system (5.4) can be rewritten as

$$T_-' = A_1 T_-, \tag{5.5}$$

where

$$T_-(\xi) = \begin{pmatrix} \varphi_{1-}(\xi) \\ \varphi_{2-}(\xi) \\ \varphi_{3-}(\xi) \\ \varphi_{4-}(\xi) \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \sigma u_1 - 2\eta & \frac{u_1}{\alpha+u_1} & c & 0 \\ 0 & \frac{1}{\delta}(1-\frac{\gamma u_1}{\alpha+u_1}) & 0 & \frac{c}{\delta} \end{pmatrix}. \tag{5.6}$$

The eigenvalues of the matrix A_1 are

$$\begin{gathered} \Lambda_1 = \frac{c+\sqrt{c^2+4\sigma u_1-8\eta}}{2}, \quad \Lambda_2 = \frac{c+\sqrt{c^2+\Delta(u_1)}}{2\delta}, \\ \Lambda_3 = \frac{c-\sqrt{c^2+\Delta(u_1)}}{2\delta}, \quad \Lambda_4 = \frac{c-\sqrt{c^2+4\sigma u_1-8\eta}}{2}, \end{gathered}$$

where $\Delta(u_1) = 4\delta(1-\frac{\gamma u_1}{\alpha+u_1})$. It is evident that $\Lambda_i > 0$ $(i=1,2,3)$, and $\Lambda_4 < 0$ for $u_1 = \frac{\sigma+\sqrt{\sigma^2-4\sigma\eta}}{2\sigma}$, $\sigma > 4\eta$, and $c \geq 2\sqrt{\delta(\frac{\gamma u_1}{\alpha+u_1}-1)} > 0$. The general solutions of the system (5.5) can be expressed as

$$(\varphi_{1-},\varphi_{2-},\varphi_{3-},\varphi_{4-})^T(\xi) = \sum_{i=1}^{4} a_i\rho_i e^{\Lambda_i\xi},$$

where ρ_i are the corresponding eigenvectors to the eigenvalues Λ_i $(i=1,2,\cdots,4)$, and a_i $(i=1,2,\cdots,4)$ are arbitrary constants. Since

$$(\varphi_{1-},\varphi_{2-},\varphi_{3-},\varphi_{4-})^T(-\infty) = (0,0,0,0),$$

we have $a_4 = 0$ and

$$(\varphi_{1-},\varphi_{2-},\varphi_{3-},\varphi_{4-})^T(\xi) = \sum_{i=1}^{3} a_i\rho_i e^{\Lambda_i\xi}.$$

Therefore, we obtain the asymptotic behavior

$$\begin{pmatrix} \varphi_{1-} \\ \varphi_{2-} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{3}\mu_i(m_i+o(1))e^{\Lambda_i\xi} \\ \sum_{i=1}^{3}\mu_i(n_i+o(1))e^{\Lambda_i\xi} \end{pmatrix}, \quad \xi \to -\infty, \tag{5.7}$$

where m_i, n_i $(i=1,2,3)$ are constants, and μ_i $(i=1,2,3)$ can not be zero simultaneously, see [6].

On the other hand, the limiting system of system (5.2) as $\xi \to +\infty$ is

$$\begin{aligned} c\varphi_{1+}' &= \varphi_{1+}'' - (\sigma u_* - \frac{2\gamma+1}{\gamma}k^* - 2\eta)\varphi_{1+} + \frac{\beta k^*-1}{\gamma}\varphi_{2+}, \\ c\varphi_{2+}' &= \delta\varphi_{2+}'' + (\gamma k^* - k^*)\varphi_{1+} - \beta k^*\varphi_{2+}, \end{aligned} \tag{5.8}$$

where

$$k^* = \frac{v_*}{\gamma u_*} = \frac{(\gamma - 1)u_* - \alpha}{\gamma\beta u_*} > 0. \tag{5.9}$$

Then system (5.8) is equivalent to

$$\begin{gathered}
\varphi'_{1+} = \varphi_{3+}, \\
\varphi'_{2+} = \varphi_{4+}, \\
\varphi'_{3+} = (\sigma u_* - \frac{2\gamma+1}{\gamma}k^* - 2\eta)\varphi_{1+} + \frac{1-\beta k^*}{\gamma}\varphi_{2+} + c\varphi_{3+}, \\
\varphi'_{4+} = \frac{1}{\delta}\big[(k^* - \gamma k^*)\varphi_{1+} + \beta k^* \varphi_{2+} + c\varphi_{4+}\big],
\end{gathered} \tag{5.10}$$

which can be reformulated as

$$T'_+ = A_2 T_+, \tag{5.11}$$

where

$$T_+(\xi) = \begin{pmatrix} \varphi_{1+}(\xi) \\ \varphi_{2+}(\xi) \\ \varphi_{3+}(\xi) \\ \varphi_{4+}(\xi) \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \sigma u_* - \frac{2\gamma+1}{\gamma}k^* - 2\eta & \frac{1-\beta k^*}{\gamma} & c & 0 \\ \frac{1-\gamma}{\delta}k^* & \frac{\beta}{\delta}k^* & 0 & \frac{c}{\delta} \end{pmatrix}. \tag{5.12}$$

The general solutions of system (5.11) has the form

$$(\varphi_{1+}, \varphi_{2+}, \varphi_{3+}, \varphi_{4+})^T(\xi) = \sum_{i=1}^{4} \tilde{a}_i \tilde{\rho}_i e^{\tilde{\Lambda}_i \xi},$$

where $\tilde{\rho}_i$ are the corresponding eigenvectors to the eigenvalues $\tilde{\Lambda}_i$ $(i = 1, 2, \cdots, 4)$, and $\tilde{a}_i$ $(i = 1, 2, \cdots, 4)$ are arbitrary constants.

The characteristic equation of the matrix A_2 is

$$\tilde{\Lambda}^4 - (c + \frac{c}{\delta})\tilde{\Lambda}^3 + K_2\tilde{\Lambda}^2 + K_1\tilde{\Lambda} + K_0 = 0,$$

where

$$\begin{gathered}
K_0 = \big[\sigma\beta u_* - (3\beta + \frac{1}{\gamma})k^* - 2\beta\eta + 1\big]\frac{k^*}{\delta}, \\
K_1 = \big[\sigma u_* + (\beta - \frac{1}{\gamma} - 2)k^* - 2\eta\big]\frac{c}{\delta}, \\
K_2 = -\sigma u_* + (2 + \frac{1}{\gamma} - \frac{\beta}{\delta})k^* + 2\eta + \frac{c^2}{\delta}.
\end{gathered}$$

According to the Viete theorem, the equality (5.9) and u_* is a positive root of the cubic equation (2.4), we have

$$\begin{gathered}
\tilde{\Lambda}_1 + \tilde{\Lambda}_2 + \tilde{\Lambda}_3 + \tilde{\Lambda}_4 = c + \frac{c}{\delta} > 0, \\
\tilde{\Lambda}_1\tilde{\Lambda}_2\tilde{\Lambda}_3\tilde{\Lambda}_4 = K_0 = \frac{\sigma\beta k^*}{\delta u_*}\Delta(u_*),
\end{gathered}$$

where

$$\Delta(u_*) = (u_*)^2 - \frac{2\eta\gamma^2\beta^2 + 4\gamma^2\beta - 3\gamma\beta + \gamma - 1}{\sigma\gamma^2\beta^2}u_* + \frac{(3\gamma\beta + 1)\alpha}{\sigma\gamma^2\beta^2}.$$

It is evident that $\tilde{\Lambda}_1\tilde{\Lambda}_2\tilde{\Lambda}_3\tilde{\Lambda}_4$ and $\Delta(u_*)$ have the same symbol. For simplicity, we shall only consider the situation of $\Delta(u_*) < 0$ here. The situation of $\Delta(u_*) \geq 0$ can be discussed similarly.

When $\Delta(u_*) < 0$, the distribution of eigenvalues involves four cases:

Case 1: $\mathrm{Re}(\tilde{\Lambda}_1) = \mathrm{Re}(\tilde{\Lambda}_2) > 0$, $\tilde{\Lambda}_3 < 0$, $\tilde{\Lambda}_4 > 0$, or $\tilde{\Lambda}_1 \geq \tilde{\Lambda}_2 \geq \tilde{\Lambda}_4 > 0$, $\tilde{\Lambda}_3 < 0$. Since

$$(\varphi_{1+}, \varphi_{2+}, \varphi_{3+}, \varphi_{4+})^T(+\infty) = (0, 0, 0, 0), \tag{5.13}$$

we have $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_4 = 0$ and $(\varphi_{1+}, \varphi_{2+}, \varphi_{3+}, \varphi_{4+})^T(\xi) = \tilde{a}_3\tilde{\rho}_3 e^{\tilde{\Lambda}_3\xi}$. Therefore, we deduce the asymptotic behavior

$$\begin{pmatrix} \varphi_{1+} \\ \varphi_{2+} \end{pmatrix} = \begin{pmatrix} \tilde{\mu}_3(\tilde{m}_3 + o(1))e^{\tilde{\Lambda}_3\xi} \\ \tilde{\mu}_3(\tilde{n}_3 + o(1))e^{\tilde{\Lambda}_3\xi} \end{pmatrix}, \quad \xi \to +\infty, \tag{5.14}$$

where $\tilde{m}_3$, $\tilde{n}_3$ are two constants, and $\tilde{\mu}_3$ can not be zero simultaneously.

Case 2: $\mathrm{Re}(\tilde{\Lambda}_1) = \mathrm{Re}(\tilde{\Lambda}_2) < 0$, $\tilde{\Lambda}_3 < 0$, $\tilde{\Lambda}_4 > 0$, or $\tilde{\Lambda}_1 < \tilde{\Lambda}_2 < \tilde{\Lambda}_3 < 0$, $\tilde{\Lambda}_4 > 0$. From condition (5.13), we obtain $\tilde{a}_4 = 0$ and

$$(\varphi_{1+}, \varphi_{2+}, \varphi_{3+}, \varphi_{4+})^T(\xi) = \sum_{i=1}^{3} \tilde{a}_i\tilde{\rho}_i e^{\tilde{\Lambda}_i\xi}.$$

Thus, we achieve the asymptotic behavior

$$\begin{pmatrix} \varphi_{1+} \\ \varphi_{2+} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{3} \tilde{\mu}_i(\tilde{m}_i + o(1))e^{\tilde{\Lambda}_i\xi} \\ \sum_{i=1}^{3} \tilde{\mu}_i(\tilde{n}_i + o(1))e^{\tilde{\Lambda}_i\xi} \end{pmatrix}, \quad \xi \to +\infty, \tag{5.15}$$

where $\tilde{m}_i$, $\tilde{n}_i$ $(i = 1, 2, 3)$ are constants, and $\tilde{\mu}_i$ $(i = 1, 2, 3)$ can not be zero simultaneously.

Case 3: $\tilde{\Lambda}_1 = \tilde{\Lambda}_2 < 0$, $\tilde{\Lambda}_3 < 0$, $\tilde{\Lambda}_4 > 0$. From condition (5.13), it follows that $\tilde{a}_4 = 0$ and

$$(\varphi_{1+}, \varphi_{2+}, \varphi_{3+}, \varphi_{4+})^T(\xi) = \sum_{i=1}^{2} \tilde{a}_i\tilde{\rho}_i\{I + (A_2 - \tilde{\Lambda}_i I)\xi\}e^{\tilde{\Lambda}_i\xi} + \tilde{a}_3\tilde{\rho}_3 e^{\tilde{\Lambda}_3\xi}.$$

Thus, we achieve the asymptotic behavior

$$\begin{pmatrix} \varphi_{1+} \\ \varphi_{2+} \end{pmatrix} = \begin{pmatrix} \tilde{\mu}_3(\tilde{m}_3 + o(1))e^{\tilde{\Lambda}_3\xi} + \sum_{i=1}^{2} \tilde{\mu}_i\{(\tilde{m}_i + o(1)) + (\tilde{m}_{ii} + o(1))\xi\}e^{\tilde{\Lambda}_i\xi} \\ \tilde{\mu}_3(\tilde{n}_3 + o(1))e^{\tilde{\Lambda}_3\xi} + \sum_{i=1}^{2} \tilde{\mu}_i\{(\tilde{n}_i + o(1)) + (\tilde{n}_{ii} + o(1))\xi\}e^{\tilde{\Lambda}_i\xi} \end{pmatrix},$$

as $\xi \to +\infty$, where $\tilde{m}_i$, $\tilde{n}_i$, $\tilde{m}_{ii}$, $\tilde{n}_{ii}$ $(i = 1, 2, 3)$ are constants, and $\tilde{\mu}_i$ $(i = 1, 2, 3)$ can not be zero simultaneously.

Case 4: $\tilde{\Lambda}_1 = \tilde{\Lambda}_2 = \tilde{\Lambda}_3 < 0$, $\tilde{\Lambda}_4 > 0$. By condition (5.13), one has $\tilde{a}_4 = 0$ and

$$(\varphi_{1+}, \varphi_{2+}, \varphi_{3+}, \varphi_{4+})^T(\xi) = \sum_{i=1}^{3} \tilde{a}_i\tilde{\rho}_i\{I + (A_2 - \tilde{\Lambda}_i I)\xi\}e^{\tilde{\Lambda}_i\xi}.$$

Consequently, we obtain the asymptotic behavior

$$\begin{pmatrix} \varphi_{1+} \\ \varphi_{2+} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{3} \tilde{\mu}_i\{(\tilde{m}_i + o(1)) + (\tilde{m}_{ii} + o(1))\xi\}e^{\tilde{\Lambda}_i\xi} \\ \sum_{i=1}^{3} \tilde{\mu}_i\{(\tilde{n}_i + o(1)) + (\tilde{n}_{ii} + o(1))\xi\}e^{\tilde{\Lambda}_i\xi} \end{pmatrix}, \quad \xi \to +\infty, \tag{5.16}$$

where $\tilde{m}_i$, $\tilde{n}_i$, $\tilde{m}_{ii}$, $\tilde{n}_{ii}$ $(i = 1, 2, 3)$ are constants, and $\tilde{\mu}_i$ $(i = 1, 2, 3)$ can not be zero simultaneously.

From equalities (5.7) and (5.14)-(5.16), we have the following result.

Theorem 5.1. *If* $c \geq 2\sqrt{\delta(\frac{\gamma u_1}{\alpha+u_1}-1)}$ *and* $\Delta(u_*) < 0$*, then there exist constants* $R,\ \bar{R},\ L_1,\ \bar{L}_1,\ L_2,\ \bar{L}_2$ *such that system* (1.2) *admits a traveling wave* $\varphi(\xi) = (u,v)(x,t) = (U,V)(\xi)$ *with the following asymptotic properties:*

$$\varphi(\xi) = \begin{pmatrix} u_1 + (R + o(1))e^{\Lambda_i \xi} \\ (\bar{R} + o(1))e^{\Lambda_i \xi} \end{pmatrix}, \quad \xi \to -\infty, \tag{5.17}$$

where Λ_i *can be* $\Lambda_1, \Lambda_2, \Lambda_3$*.*

For cases 1 and case 2, we have

$$\varphi(\xi) = \begin{pmatrix} u_* - (L_1 + o(1))e^{\tilde{\Lambda}_j \xi} \\ v_* - (L_2 + o(1))e^{\tilde{\Lambda}_j \xi} \end{pmatrix}, \quad \xi \to +\infty, \tag{5.18}$$

where for case 1, $\tilde{\Lambda}_j$ *is* $\tilde{\Lambda}_3$*, and for case 2,* $\tilde{\Lambda}_j$ *can be one of* $\tilde{\Lambda}_1,\ \tilde{\Lambda}_2,\ \tilde{\Lambda}_3$*.*

For cases 3 and case 4, we have

$$\varphi(\xi) = \begin{pmatrix} u_* - ((L_1 + o(1))e^{\tilde{\Lambda}_j \xi} + (\bar{L}_1 + o(1))\xi e^{\tilde{\Lambda}_1 \xi}) \\ v_* - ((L_2 + o(1))e^{\tilde{\Lambda}_j \xi} + (\bar{L}_2 + o(1))\xi e^{\tilde{\Lambda}_1 \xi}) \end{pmatrix}, \quad \xi \to +\infty, \tag{5.19}$$

where for case 3, $\tilde{\Lambda}_j$ *can be one of* $\tilde{\Lambda}_1,\ \tilde{\Lambda}_3$*, and for case 4,* $\tilde{\Lambda}_j$ *is* $\tilde{\Lambda}_1$*.*

The asymptotic behaviors of traveling waves for the system (1.8) with local delay kernels (1.5) or nonlocal delay kernels (1.7) can be investigated similarly.

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